# Quantum and classical chaos for a single trapped ion

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In this paper we investigate the quantum and classical dynamics of a single trapped ion subject to nonlinear kicks derived from a periodic sequence of Gaussian laser pulses. We show that the classical system exhibits diffusive growth in the energy, or ''heating,'' while quantum mechanics suppresses this heating. This system may be realized in current single trapped-ion experiments with the addition of near-field optics to introduce tightly focused laser pulses into the trap.

## I. INTRODUCTION

Recent experiments on the nonlinear dynamics of cold trapped atoms have provided a remarkable verification of key theoretical ideas in the subject of quantum chaos [1,2], including dynamical localization and the effect of decoherence in restabilizing the classical limit. In all these experiments, however, the observed results are obtained from a large ensemble of single atom experiments which run in parallel but independently. Until now, there have been no experiments which investigate quantum chaos of a *single* particle monitored over a period of time. In contrast, in the related field of ion trapping, technological advances now enable a single ion to be trapped, cooled to the ground state of the trap, and monitored, almost without error [3]. The quantum dynamics of the center of mass motion of the ion is extremely well described by a three-dimensional harmonic potential. In some experiments, two degrees of freedom are very tightly bound and the interesting harmonic motion takes place in a single degree of freedom. Of course this system is integrable. However, if this degree of freedom is subject to a periodic nonlinear potential, chaos may result. In this paper we investigate the quantum and classical dynamics of a single trapped ion subject to nonlinear kicks derived from a periodic sequence of Gaussian laser pulses. This system may be realized in current single trapped-ion experiments with the addition of near-field optics to introduce tightly focussed laser pulses into the trap. Another suggestion for investigating quantum chaos in a single trapped ion has recently been suggested by Berman *et al.* [4].

The recently achieved ability to engineer dynamics for a single trapped ion has followed from the potential application of this system for quantum computational gates. As such these systems necessarily operate at the quantum level and provide the ideal experimental context to test quantum nonlinear dynamics. Indeed such experiments will ultimately involve the ability to follow the quantum dynamics of many trapped ions with complex many-body interactions introduced by externally imposed time dependent Hamiltonians. We thus believe it timely to consider tests of quantum chaos which can be made with current technology.

In Sec. II we define the classical dynamical system and give a detailed analysis of the classical motion and the transition to chaos. In Sec. III we give a quantum description of

the problem and in Sec. IV show that this system exhibits a suppression in the diffusion of momentum and position. In other words the total energy of the trapped ion is localized in contrast to classical diffusive heating of the motion. Finally in Sec. V we discuss a possible physical realization of the system.

## II. CLASSICAL MAP

The system has the Hamiltonian

$$\widetilde{H} = \frac{\widetilde{p}^2}{2m} + \frac{m\Omega^2}{2}\widetilde{x}^2 + \kappa e^{-\alpha\widetilde{x}^2} T \sum_{n=-\infty}^{\infty} \delta(\widetilde{t}-nT),$$

where $m$ is the mass of the ion trapped in a harmonic potential of frequency $\Omega$. The ion is subject to a periodic sequence of laser pulses with period $T$. The $\kappa e^{-\alpha\widetilde{x}^2}$ term in the Hamiltonian describes the potential felt by the ion due to the Gaussian structure of the laser. If we rescale time, position and momentum by letting $\widetilde{t}=tT$, $\widetilde{x}=(1/\sqrt{\alpha})x$, and $\widetilde{p}=(m\Omega/\sqrt{\alpha})p$ the Hamiltonian rescales to

$$H(x,p,t) = \frac{\alpha T}{m\Omega}\widetilde{H}(\widetilde{x},\widetilde{p},\widetilde{t})$$

$$= \frac{\omega}{2}(p^2+x^2) + ke^{-x^2}\sum_{n=-\infty}^{\infty}\delta(t-n), \quad (1)$$

where $\omega=\Omega T$ and $k=\kappa\alpha T/m\Omega$ are dimensionless parameters. The new variables $x$, $p$, and $t$ together with the Hamiltonian $H$ are also dimensionless.

Between kicks the system has the solution $(x,p) = (x_0\cos\omega t+p_0\sin\omega t, -x_0\sin\omega t+p_0\cos\omega t)$. The effect of the kick is to add a position dependent shift in the momentum of $2kxe^{-x^2}$. Denoting the mapping by $F$ we can write it as the composition of a kick $K$ and a linear rotation $W$

$$F = W\circ K,$$

where

$$K(x,p) = (x, p+2kxe^{-x^2}),$$

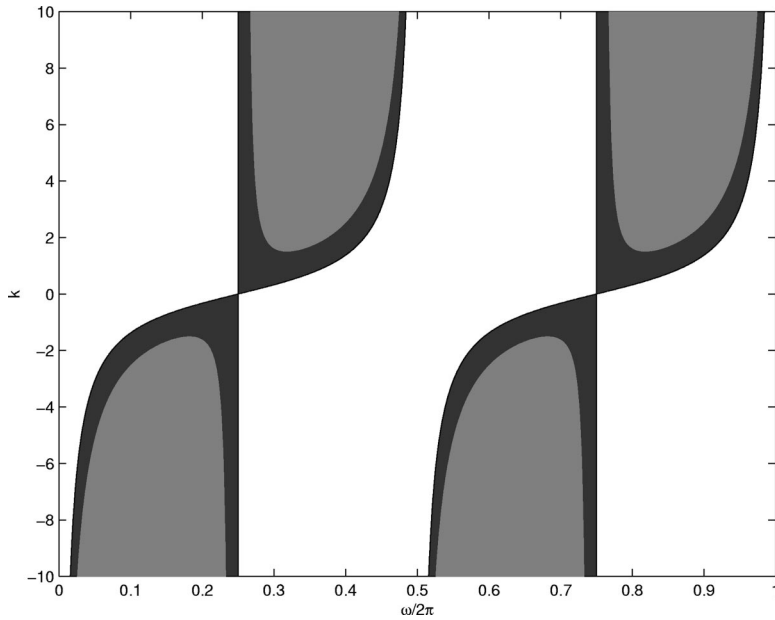$$W(x,p) = (x\cos\omega + p\sin\omega, -x\sin\omega + p\cos\omega).$$

FIG. 1. Bifurcation diagram for the period 4 orbits (2). The stable region is in dark and the unstable in light.

Hence $F$ maps $(x,p)$ from just before a kick to one period later.

The fixed points of $F$ are at

$$(x,p)=(0,0),\left[\pm\sqrt{\ln\left(k\cot\frac{\omega}{2}\right)},\mp\tan\frac{\omega}{2}\sqrt{\ln\left(k\cot\frac{\omega}{2}\right)}\right].$$

The origin is stable if $k\cot(\omega/2)<1$ and $-k\tan(\omega/2)<1$. When $k\cot(\omega/2)>1$ it becomes unstable via a pitchfork bifurcation which creates the second two fixed points. These are stable for $k\cot(\omega/2)<\exp[\frac{1}{2}\csc^2(\omega/2)]$. When $-k\tan(\omega/2)>1$ the origin becomes unstable via a period doubling bifurcation and two period 2 orbits are created at

$$(x,p)=\left[\pm\sqrt{\ln\left(-k\tan\frac{\omega}{2}\right)},\mp\cot\frac{\omega}{2}\sqrt{\ln\left(-k\tan\frac{\omega}{2}\right)}\right].$$

These are stable for $-k\tan(\omega/2)<\exp[\frac{1}{2}\sec^2(\omega/2)]$.

One can also find two sets of period 4 orbits which exist for $-k\tan\omega>1$. The first set is at

$$(x,p)=\left(\pm\chi,\pm\chi\cot\frac{\omega}{2}\right),\left(\pm\chi,\mp\chi\tan\frac{\omega}{2}\right), \quad (2)$$

where $\chi=\sqrt{\ln(-k\tan\omega)}$. They are stable if $-k\tan\omega<\exp(\frac{1}{2}|\sec\omega|)$. A second set lie in between these orbits at

$$(x,p)=(\pm\chi,\pm\chi\cot\omega),(0,\pm\chi\csc\omega).$$

These are always unstable.

It is instructive to study the process of creation and destruction of these period 4 orbits in more detail. In Fig. 1 we have drawn a bifurcation diagram. The period 4 orbits exist
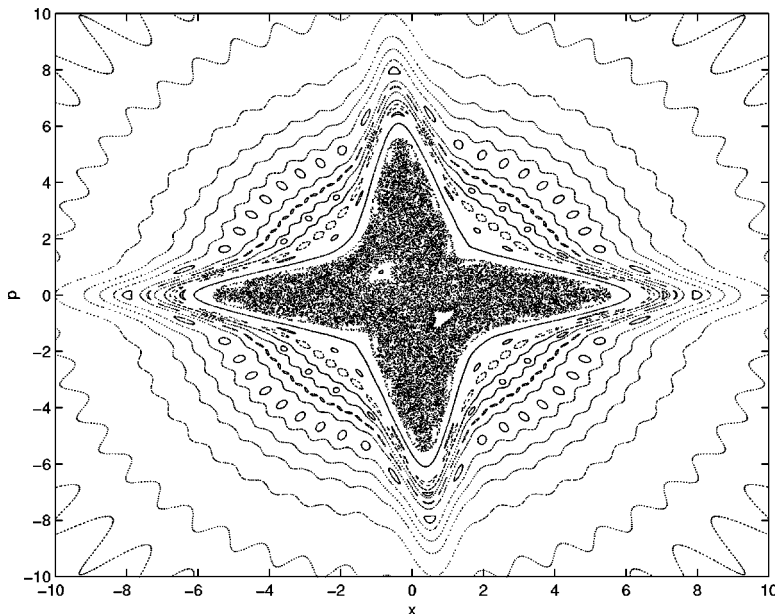
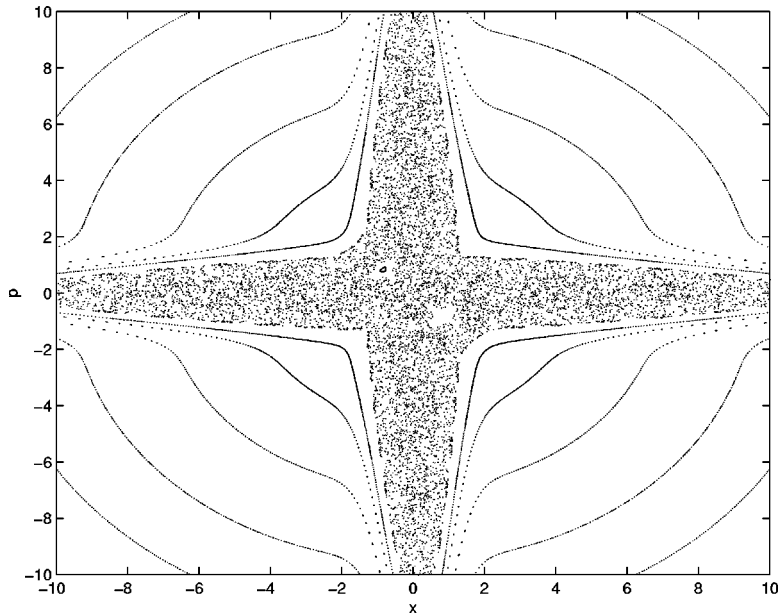

FIG. 2. Phase portrait for $k=2$ and $\omega/2\pi=0.24$.

FIG. 3. As for Fig. 2 except $\omega/2\pi = 0.248$.

for parameter values lying in the shaded regions. The regions of lighter shade show where the first set of orbits (2) are unstable. In Figs. 2 –6 a sequence of phase space pictures are drawn for values of $\omega$ along the line $k=2$. Figure 2 is for $\omega/2\pi = 0.24$ and shows typical phase space structure for the system. As $\omega$ is increased, the arms of the star-shaped chaotic region grow in size. This can be seen in Fig. 3 where $\omega/2\pi = 0.248$. At $\omega/2\pi = \frac{1}{4}$ (Fig. 4) these arms are infinitely long chaotic channels which divide the phase space into four regions, creating the period 4 orbits at infinity. Figure 5 shows the period 4 orbits just after they are created ($\omega/2\pi = 0.252$). They now move towards the origin as $\omega$ is increased further. On this journey the first set (2) shed their stability via a pitchfork bifurcation but then regain it before destroying at the origin. Figure 6 shows the orbits just before destruction ($\omega/2\pi = 0.42$).

It is simple to see how the period 4 orbits were created at infinity when $\omega = 2\pi/4$. The kick has little influence on orbits here and thus $F$ reduces to simple linear rotation with period 4. In general, it can be shown that an orbit of period $p$ is created at infinity when

$$\omega = \frac{2\pi q}{p},$$

and then destroyed at the origin when

$$\cos\omega + k\sin\omega = \cos\frac{2\pi q}{p},$$

where $q$ and $p$ are natural numbers with a greatest common divisor of one. The condition for destruction is found by
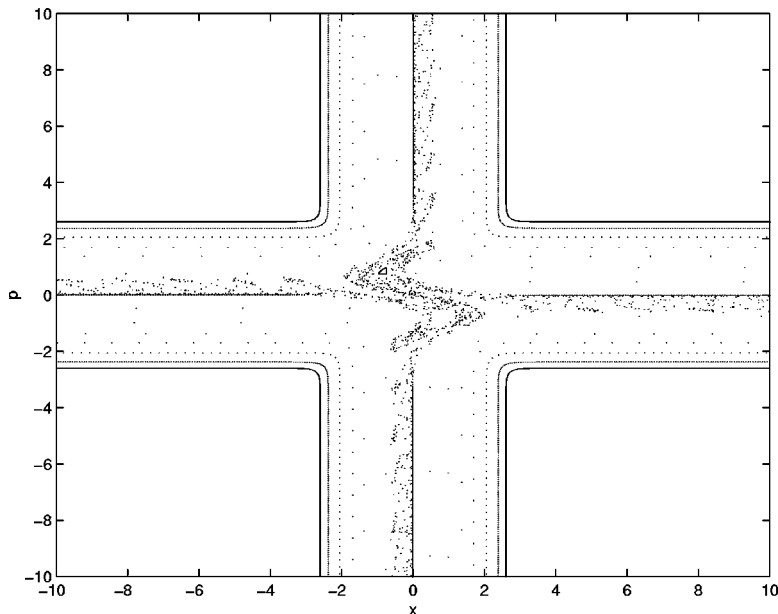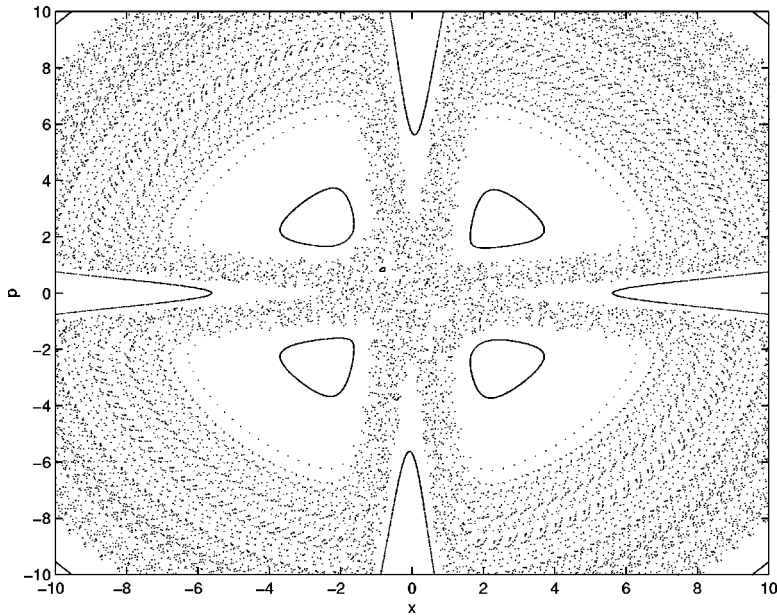


FIG. 4. As for Fig. 2 except $\omega/2\pi = \frac{1}{4}$.

FIG. 5. As for Fig. 2 except $\omega/2\pi = 0.252$.

looking at the eigenvalues of the tangent map at the origin and then equating this linear rotation with the period of the orbit.

### III. QUANTUM MAP

To construct the quantum map we start with the rescaled Hamiltonian (1) and define a dimensionless Planck's constant $k$ via the commutation relation for position and momentum

$$[x,p] = \frac{\alpha}{m\omega}[\tilde{x},\tilde{p}] = i\frac{\alpha\hbar}{m\omega} \equiv ik.$$

The time evolution of an initial state, $|\psi^n\rangle$, from just before a kick through to one period later is given by
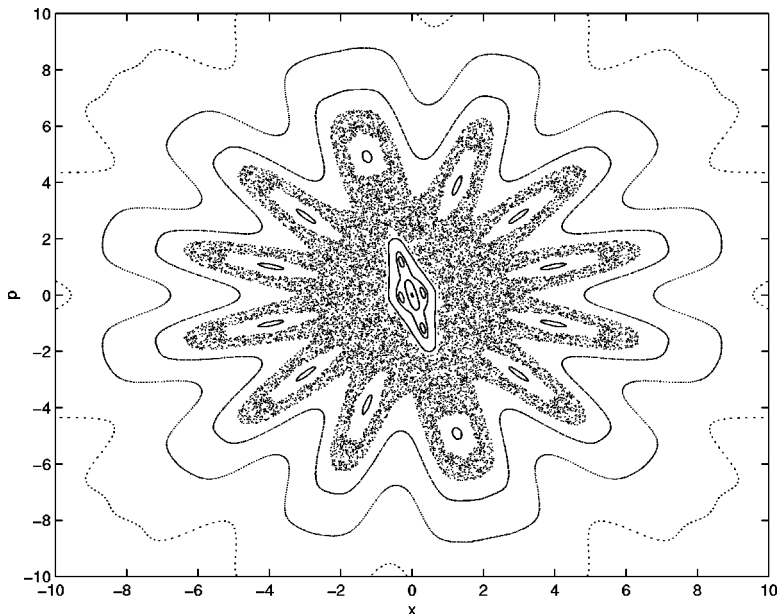
$$|\psi^{n+1}\rangle = \exp\left(\frac{-i}{k}\int_0^1 H(t)dt\right)|\psi^n\rangle \tag{3}$$

$$= \exp\left(\frac{-i\omega}{2k}(p^2+x^2)\right)\exp\left(\frac{-ik}{k}e^{-x^2}\right)|\psi^n\rangle \tag{4}$$

$$\equiv \hat{F}|\psi^n\rangle. \tag{5}$$

Hence the Floquet operator, $\hat{F}$, defines the quantum map. Now, defining the annihilation and creation operators to be

$$a = \frac{1}{\sqrt{2k}}(x+ip) \quad \text{and} \quad a^\dagger = \frac{1}{\sqrt{2k}}(x-ip),$$



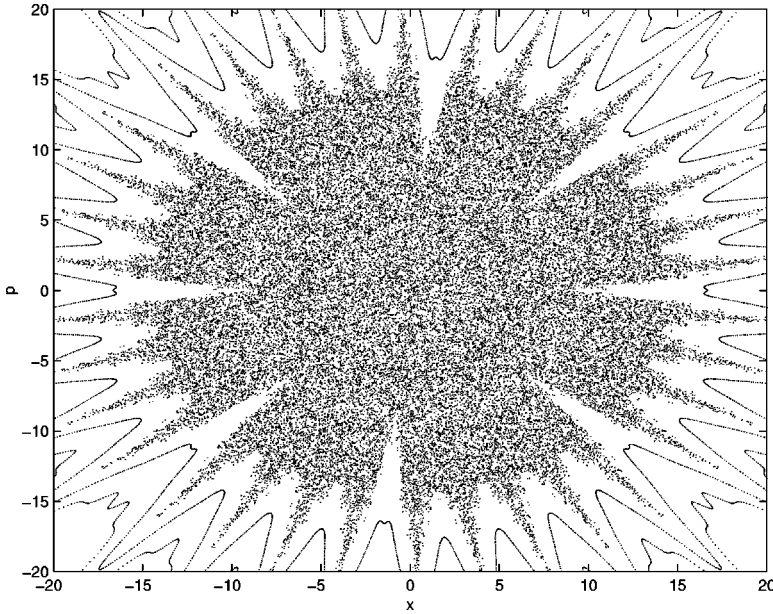FIG. 6. As for Fig. 2 except $\omega/2\pi = 0.42$.

FIG. 7. Phase portrait for $k=8$ and $\omega=\pi(3-\sqrt{5})$.

respectively, and using the simple harmonic oscillator eigenstates

$$|n\rangle = \frac{1}{\sqrt{n!}} a^{\dagger n}|0\rangle, \quad n=0,1,2,\ldots$$

as an orthogonal basis we can rewrite Eq. (5) as

$$c_m^{n+1} = F_{mk}c_k^n,$$

where $c_m^n = \langle m|\psi^n\rangle$ and

$$F_{nm} = \langle n|\hat{F}|m\rangle = e^{-i\omega(n+1/2)}\langle n|\exp\left(\frac{-ik}{k}e^{-x^2}\right)|m\rangle.$$

The last term is found by taking the exponential of the matrix with components

$$\langle n|\frac{-ik}{k}e^{-x^2}|m\rangle$$

$$= \frac{-ik}{k^{3/2}\sqrt{\pi n!m!2^{n+m}}}\int_{-\infty}^{\infty} H_n\left(\frac{x}{\sqrt{k}}\right)H_m\left(\frac{x}{\sqrt{k}}\right)$$

$$\times e^{-(1+1/k)x^2}dx$$

$$= \frac{-k(n+m-1)(n+m-3)\cdots 1}{k^{3/2}\sqrt{n!m!}}\left(\frac{-k}{1+k}\right)^{(n+m+1)/2}$$

$$\times {}_2F_1\left[-n,-m;\frac{1-n-m}{2};\frac{1}{2}\left(1+\frac{1}{k}\right)\right]$$

for $n+m$ even and vanishing otherwise. Here $H_n$ are Hermite polynomials and ${}_2F_1$ is the hypergeometric function. Note that $F_{nm}=0$ whenever $n+m$ is odd. This means that even and odd parity states do not couple under $\hat{F}$ and thus evolve independently.

## IV. LOCALIZATION

We now show numerically the presence of dynamical localisation [5] in the system. Or, more precisely, we show that classical diffusion is suppressed when the system is evolved quantum mechanically. For this, we have chosen $\omega=\pi(3-\sqrt{5})$ and $k=8$. In this parameter regime a large chaotic sea centred at the origin consumes the phase space (see Fig. 7). The initial state was chosen to be $|0\rangle$, which has a Husimi probability density of

$$|\langle z|0\rangle|^2 = e^{-|z|^2} \tag{6}$$

in phase space. Here $z=(1/\sqrt{2k})(x+ip)$ and $|z\rangle$ are the coherent states defined as

$$|z\rangle = e^{-(1/2)|z|^2}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}|n\rangle.$$

Thus our initial state is a highly localized Gaussian hump centered at the origin. This was then evolved forward using 1800 of the even basis states. Figure 8 shows the average dimensionless energy, $\langle x^2+p^2\rangle$, after each kick. The light gray is for $k=0.5$ and the dark gray is for $k=0.2$. The energy under classical evolution is shown in black. Here an initial density (6) with $k=0.2$ was chosen. One can clearly see in this figure that diffusion is suppressed after about 100 kicks. The procedure was repeated using only 1000 even basis states to confirm accuracy and it was found that the difference in the energies did not exceed $10^{-8}$ until after 1000 kicks. Thus the localization is truly a property of quantum mechanics.

## V. DISCUSSION AND CONCLUSION

What are the physical requirements to realize this system in current ion trap experiments? Consider a ${}^9\text{Be}^+$ ion such as used in the NIST experiments [3], with a harmonic fre-
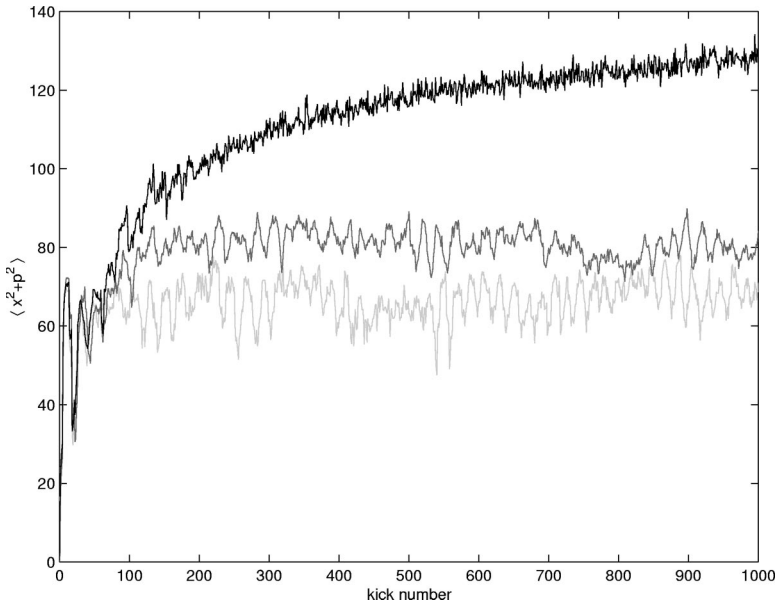
FIG. 8. The average dimensionless energy after each kick for the initial state $|0\rangle$ with $k = 0.5$ (light gray), $k = 0.2$ (dark gray), and classical evolution (black).

quency in the relevant direction of $\Omega = 1$ MHz. The key parameter which determines the effective Planck constant is the parameter $\alpha$. If one uses a focused laser beam, a typical value for this parameter is $10^{10}$ m$^2$ and a resulting effective Planck's constant of $k = 6.7 \times 10^{-5}$, which is hopelessly too small. On the other hand, if we use a near-field probe, as used in near-field optical scanning microscopy (NOSM), to inject the field we can get a value as high as $\alpha = 10^{14}$ m$^2$ with a typical probe tip of diameter 10 nm. This corresponds to an effective Planck's constant for $^9$Be$^+$ of about $k = 0.7$, which is more promising. To achieve a value for the kick parameter of the order used above we would need to focus a few nanowatts into the NOSM probe which is quite typical. This would correspond to an intensity of about 1000 mW cm$^{-2}$ at the ion. If we choose the kick period to be of the order of 10 ms, the kick parameter $k$ has a value of the order of unity. We conclude that this experiment is possible for a current single trapped-ion experiment with the addition of near-field optical fibre probes.

The next question we need to ask of such a system is how are we to observe the motion of the ion? Fortunately the current single trapped-ion experiments are designed precisely to enable careful monitoring of the motion states. The details are described in Ref. [3]. The basic idea is to map the motion states onto particular internal states of the ion which are then probed by a fluorescent shelving technique. In particular, it is possible to measure the center-of-mass energy of the ion in the trap. Each measurement however destroys the quantum state of the ion at that time, so repreparation of the ion initial state is required. One then needs to perform repeated experiments for differing number of kicks before reading out the center-of-mass energy. In this way it is pos-

sible to monitor the energy of the motion as a function of kick number. Dynamical localization of the motion energy of the ion could thus be observed.

Finally we need to ask if it is feasible to prepare the initial states we have used in this paper. Again Ref. [3] shows that it is possible to prepare the ion in the ground state of the harmonic trap, so this part is relatively easy. Laser pulses may then be used to displace this minimum uncertainty state anywhere in the phase plane. This ability to place a localized state anywhere in the phase plane would enable a detailed study of mixed chaotic and regular phase space structures. Unfortunately in the current experiment unwanted stray linear potentials cause a heating of the ion and thus it does not stay in the ground state for long, but rather undergoes a diffusive motion in the phase plane [6,7]. A very considerable amount of effort is currently being devoted to removing this unwanted heating so that trapped ions can be used in a quantum logic gate. We thus expect this problem to be solved or at least significantly mitigated.

Needless to say this is not an easy experiment. Introducing the near field probe close to the ion will cause additional unwanted van der Walls forces to be exerted on the ion. However, these forces, while making a detailed comparison to experiment more difficult, will not effect the generic transition to chaos described above so long as they remain weak. The heating of the ion due to stray linear potentials will remain a problem to some extent. Such fluctuating forces are a source of decoherence and thus will tend to destroy localization. Taking a longer view, however, the ease with which decoherence can be induced via this mechanism should enable a detailed study of the effect of noise on dynamical localization to be made, thus turning a bug into a feature.

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