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Quasispin graded-fermion formalism and $gl(m|n)\downarrow osp(m|n)$ branching rules

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The graded-fermion algebra and quasispin formalism are introduced and applied to obtain the $gl(m|n)\downarrow osp(m|n)$ branching rules for the ''two-column'' tensor irreducible representations of $gl(m|n)$, for the case $m\leqslant n(n>2)$. In the case $m<n$, all such irreducible representations of $gl(m|n)$ are shown to be completely reducible as representations of $osp(m|n)$. This is also shown to be true for the case $m=n$, except for the ''spin-singlet'' representations, which contain an indecomposable representation of $osp(m|n)$ with composition length 3. These branching rules are given in fully explicit form. © *1999 American Institute of Physics.*
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I. INTRODUCTION

It is well known that branching rules are of great importance in the study of representation theory. They also play an essential role in the determination of the parities for the components appearing in the twisted tensor product graphs and the construction of corresponding R matrices.[1,2]

There appear to be virtually no results in the literature on the branching rules for Lie superalgebras. The only exception is Ref. 3, in which the branching rules are determined for all typical and atypical irreducible representations of $osp(2|2n)$ with respect to its subalgebra $osp(1|2n)$. It is very interesting (and important) to investigate the branching rules for other Lie superalgebras.

In this paper we investigate the antisymmetric tensor irreducible representations of $gl(m|n)$. This class of representations is of interest since they are also irreducible under the fixed point subalgebra $osp(m|n)$. Moreover, their quantized versions can be shown to be affinizable to provide irreducible representations of the twisted quantum affine superalgebra $U_q[gl(m|n)^{(2)}]$ from which trigonometric R matrices with $U_q[osp(m|n)]$ invariance may be constructed.[4]

These R matrices determine new integrable models that have generated remarkable interest in physics recently,[5–7] particularly in condensed matter physics, where they give rise to new integrable models of strongly correlated electrons.

To explicitly construct such R matrices it is necessary to determine the reduction of the tensor product of two antisymmetric tensor irreducible representations into ''two column'' irreducible representations of $gl(m|n)$ which are then decomposed into irreducible representations of its fixed point subalgebra $osp(m|n)$.

We determine the $gl(m|n)\downarrow osp(m|n)$ branching rules for these two column irreducible tensor representations of $gl(m|n)$, for the case $m\leqslant n$, $n>2$. A natural framework for solving this problem is provided by the graded-fermion algebra and the quasispin formalism, which we introduce and develop in this paper. The Fock space for this graded-fermion algebra affords a convenient realization of the class of irreducible representations of $gl(m|n)$ concerned. The reduction to $osp(m|n)$, and thus the $gl(m|n)\downarrow osp(m|n)$ branching rules, can be achieved using the quasispin formalism.

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II. $osp(m|n=2k)$ AS A SUBALGEBRA OF $gl(m|n)$

Throughout this paper, we assume $n=2k$ is even and set $h=[m/2]$ so that $m=2h$ for even m and $m=2h+1$ for odd m. For homogeneous operators A, B we use the notation $[A,B]=AB-(-1)^{[A][B]}BA$ to denote the usual graded commutator. Let E^a_b be the standard generators of $gl(m|n)$ obeying the graded commutation relations,

$$[E^a_b,E^c_d]=\delta^c_b E^a_d-(-1)^{([a]+[b])([c]+[d])}\delta^a_d E^c_b. \tag{II.1}$$

In order to introduce the subalgebra $osp(m|n)$, we first need a graded symmetric metric tensor $g_{ab}=(-1)^{[a][b]}g_{ba}$, which is assumed to be even. We shall make the convenient choice

$$g_{ab}=\xi_a\delta_{a\bar{b}}, \tag{II.2}$$

where

$$\bar{a}=\begin{cases} m+1-i, & a=i,\\ n+1-\mu, & a=\mu,\end{cases}\quad \xi_a=\begin{cases}1, & a=1\\ (-1)^{\mu}, & a=\mu\end{cases}. \tag{II.3}$$

In the above equations, $i=1,2,...,m$ and $\mu=1,2,...,n$. Note that

$$\xi_a^2=1,\quad \xi_a\xi_{\bar{a}}=(-1)^{[a]},\quad g^{ab}=\xi_b\delta_{a\bar{b}}. \tag{II.4}$$

As generators of the subalgebra $osp(m|n=2k)$, we take

$$\sigma_{ab}=g_{ac}E^c_b-(-1)^{[a][b]}g_{ac}E^c_a=-(-1)^{[a][b]}\sigma_{ba}, \tag{II.5}$$

which satisfy the graded commutation relations,

$$[\sigma_{ab},\sigma_{cd}]=g_{cb}\sigma_{ad}-(-1)^{([a]+[b])([c]+[d])}g_{ad}\sigma_{cb}-(-1)^{[c][d]}(g_{bd}\sigma_{ac}-(-1)^{([a]+[b])([c]+[d])}g_{ac}\sigma_{db}). \tag{II.6}$$

We have an $osp(m|n)$-module decomposition,

$$gl(m|n)=osp(m|n)+T,\quad [T,T]\subset osp(m|n), \tag{II.7}$$

where T is spanned by operators

$$T_{ab}=g_{ac}E^c_b+(-1)^{[a][b]}g_{bc}E^c_a=(-1)^{[a][b]}T_{ba}. \tag{II.8}$$

It is convenient to introduce the Cartan–Weyl generators,

$$\sigma^a_b=g^{ac}\sigma_{cb}=-(-1)^{[a]([a]+[b])}\xi_a\xi_b\sigma^{\bar{b}}_{\bar{a}}. \tag{II.9}$$

As a Cartan subalgebra we take the diagonal operators,

$$\sigma^a_a=E^a_a-E^{\bar{a}}_{\bar{a}}=-\sigma^{\bar{a}}_{\bar{a}}. \tag{II.10}$$

Note that for odd $m=2h+1$ we have $\overline{h+1}=h+1$, and thus $\sigma^{h+1}_{h+1}=E^{h+1}_{h+1}-E^{h+1}_{h+1}=0$.

The positive roots of $osp(m|n)$ are given by the even positive roots [usual positive roots for $o(m)\oplus sp(n)$] together with the odd positive roots $\delta_\mu+\epsilon_i$, $1\leqslant i\leqslant m$, $1\leqslant\mu\leqslant k=n/2$, where we have adopted the useful convention $\epsilon_{\bar{i}}=-\epsilon_i$, $i\leqslant h=[m/2]$ so that $\epsilon_{h+1}=0$ for odd $m=2h+1$. This is consistent with the $\mathbf{Z}$ gradation,

$$osp(m|n)=L_{-2}\oplus L_{-1}\oplus L_0\oplus L_1\oplus L_2. \tag{II.11}$$

Here $L_0=o(m)\oplus gl(k)$; the $gl(k)$ generators are given by

$$\sigma^\mu_\nu=E^\mu_\nu-(-1)^{\mu+\nu}E^{\bar{\nu}}_{\bar{\mu}},\quad 1\leqslant\mu,\nu\leqslant k, \tag{II.12}$$

and $L_{-2}\oplus L_0\oplus L_2=o(m)\oplus sp(n)$, where L_2 gives rise to an irreducible representation of L_0 with highest weight $(\dot{0}|2,\dot{0})$ spanned by the generators

$$\sigma^{\underline{\mu}}_{\nu}=E^{\underline{\mu}}_{\nu}-\xi_\mu\xi_{\bar{\nu}}E^{\nu}_{\underline{\mu}}=E^{\underline{\mu}}_{\nu}+(-1)^{\mu+\nu}E^{\nu}_{\underline{\mu}},\quad 1\leqslant\mu,\nu\leqslant k. \tag{II.13}$$

Finally, L_1 is spanned by odd root space generators,

$$\sigma^\mu_i=E^\mu_i+\xi_\mu E^{\bar{i}}_{\underline{\mu}}=E^\mu_i+(-1)^\mu E^{\bar{i}}_{\underline{\mu}},\quad 1\leqslant\mu\leqslant k,\quad 1\leqslant i\leqslant m, \tag{II.14}$$

and gives rise to an irreducible representation of L_0 with highest weight $(1,\dot{0}|1,\dot{0})$. L_{-1}, L_{-2} give rise to irreducible representations of L_0 dual to L_1, L_2, respectively.

The simple roots of $osp(m|n=2k)$ are thus given by the usual (even) simple roots of L_0 together with the odd simple root $\alpha_s=\delta_k-\epsilon_1$, which is the lowest weight of L_0-module L_1. Note that the simple roots of $o(m)$ depend on whether m is odd or even, and are given here for convenience: For $m=2h$, $\alpha_i=\epsilon_i-\epsilon_{i+1}$, $1\leqslant i<h$, $\alpha_h=\epsilon_{h-1}+\epsilon_h$. For $m=2h+1$, $\alpha_i=\epsilon_i-\epsilon_{i+1}$, $1\leqslant i<h$, $\alpha_h=\epsilon_h$. The simple roots of $gl(k)$ are given by

$$\alpha_{h+\mu}=\delta_{gm}-\delta_{\mu+1},\quad 1\leqslant\mu<k. \tag{II.15}$$

The graded half-sum of the positive roots of $osp(m|n=2k)$ is given by

$$\rho=\frac{1}{2}\sum_{i=1}^{h}(m-2i)\epsilon_i+\frac{1}{2}\sum_{\mu=1}^{k}(n-m+2-2\mu)\delta_\mu. \tag{II.16}$$

III. GRADED-FERMION REALIZATIONS

We introduce the graded anticommutator:

$$\{A,B\}\equiv AB+(-1)^{[A][B]}BA. \tag{III.1}$$

Note that $\{A,B\}\neq\{B,A\}$. To realize the antisymmetric tensor irreducible representations of $gl(m|n)$, we introduce graded fermions c_a and their adjoints $c_a^\dagger$ obeying the graded anticommutation relations,

$$\{c_a,c_b\}=\{c_a^\dagger,c_b^\dagger\}=0,\quad \{c_a,c_b^\dagger\}=\delta_{ab}. \tag{III.2}$$

Thus, when $a=i$ is even c_i are fermions while for $a=\mu$ odd, c_μ are bosons that anticommute with the fermions.

To get a graded fermion realization of $gl(m|n)$, we set

$$E^a_b=c_a^\dagger c_b, \tag{III.3}$$

and note the graded commutation relations:

$$[E^a_b,c_d^\dagger]=\delta_{bd}c_a^\dagger,\quad [E^a_b,c_d]=(-1)^{([a]+[b])[d]}\delta^a_d c_b. \tag{III.4}$$

Using these relations, it is easy to verify that the operators E^a_b given above indeed satisfy the $gl(m|n)$ graded commutation relations.

Thus, we obtain representations of $gl(m|n)$ on the graded fermion Fock space, which include the antisymmetric tensor representations. The Fock space can be shown to be completely reducible into type I unitary irreducible representations of $gl(m|n)$ according to

$$F=\bigoplus_{a=0}^{m}\hat{V}(\dot{1}_a,\dot{0}|\dot{0})\bigoplus_{b=1}^{\infty}\hat{V}(\dot{1}|b,\dot{0}). \tag{III.5}$$

Thus, for $N\leqslant m$, the space of N-particle states comprises the antisymmetric tensor representation of $gl(m|n)$ with highest weight $\Lambda_N=(\dot{1}_N,\dot{0}|\dot{0})$. For $N>m$ the space of N-particle states comprises the irreducible representations of $gl(m|n)$ with highest weights $\Lambda_N=(\dot{1}|N-m,\dot{0})$.

We introduce an extra ''spin'' index α and consider the family of graded fermions $c_{a\alpha}$ and their adjoints $c_{a\alpha}^{\dagger}$ obeying the graded anticommutation relations,

$$\{c_{a\alpha},c_{b\beta}\}=\{c_{a\alpha}^{\dagger},c_{b\beta}^{\dagger}\}=0,\quad \{c_{a\alpha},c_{b\beta}^{\dagger}\}=\delta_{ab}\delta_{\alpha\beta}. \tag{III.6}$$

Here all spin indices are understood to be even (so that the grading only depends on the orbital labels *a, b, c,* etc.).

We take, for our $gl(m|n)$ generators,

$$E_b^a=\sum_{\alpha} c_{a\alpha}^{\dagger}c_{b\alpha}, \tag{III.7}$$

which can be shown, as before, to satisfy the graded commutation relations

$$[E_b^a,c_{d\alpha}^{\dagger}]=\delta_{bd}c_{a\alpha}^{\dagger},\quad [E_b^a,c_{d\alpha}]=(-1)^{([a]+[b])[d]}\delta_d^a c_{b\alpha}, \tag{III.8}$$

from which we deduce that the E_b^a indeed obey the $gl(m|n)$ graded commutation relations. Thus, we may now construct more general irreducible representations of $gl(m|n)$ in the graded-fermion Fock space. In particular, for ''two-column'' irreducible representations, only two spin labels $\alpha=\pm$ are required.

IV. QUASISPIN (TWO SPIN LABELS)

We employ the above graded-fermion algebra with two spin labels $\alpha=\pm$. We set

$$\begin{aligned} Q_+&=g_{dd'}c_{d,+}^{\dagger}c_{d',-}^{\dagger}=\sum_d \xi_d c_{d,+}^{\dagger}c_{\bar{d},-}^{\dagger},\\ Q_-&=g^{dd'}c_{d,-}^{\dagger}c_{d',+}^{\dagger}=\sum_d \xi_d c_{d,-}c_{\bar{d},+}. \end{aligned} \tag{IV.1}$$

Let $Q_0=\frac{1}{2}(\hat{N}-m+n)$, where $\hat{N}=\Sigma_{a=1}^{m+n}E_a^a$ is the first-order invariant of $gl(m|n)$ (i.e., the number operator). By straightforward computation, the following can be shown.

Proposition 1: *$Q_\pm$, Q_0 generate an $sl(2)$ Lie algebra, called the quasispin Lie algebra,*

$$[Q_+,Q_-]=2Q_0,\quad [Q_0,Q_\pm]=\pm Q_\pm. \tag{IV.2}$$

Moreover, $Q_\pm$, Q_0 commute with the generators of $osp(m|n=2k)$.

To see the significance of the graded fermion algebra for the construction of irreducible representations, we set

$$E_{b\beta}^{a\alpha}=c_{a\alpha}^{\dagger}c_{b\beta}, \tag{IV.3}$$

and note the graded commutation relations,

$$[E_{b\beta}^{a\alpha},c_{c\gamma}^{\dagger}]=\delta_{bc}\delta_{\beta\gamma}c_{a\alpha}^{\dagger},\quad [E_{b\beta}^{a\alpha},c_{c\gamma}]=-(-1)^{[c]([a]+[b])}\delta_c^a\delta_\gamma^\alpha c_{b\beta}, \tag{IV.4}$$

from which we deduce

$$[E^{a\alpha}_{b\beta},E^{c\gamma}_{d\delta}]=\delta^c_b\delta^\gamma_\beta E^{a\alpha}_{d\delta}-(-1)^{([a]+[b])([c]+[d])}\delta^a_d\delta^\alpha_\delta E^{c\gamma}_{b\beta}, \tag{IV.5}$$

which are the defining relations of $gl(2m|2n)$. That is, $E^{a\alpha}_{b\beta}$ are the generators of $gl(2m|2n)$.

As we have seen, the spin-averaged operators,

$$E^a_b=\sum_{\alpha=\pm}E^{a\alpha}_{b\alpha}, \tag{IV.6}$$

form the generators of $gl(m|n)$. Similarly, the orbital averaged operators,

$$E^\alpha_\beta=\sum_a E^{a\alpha}_{a\beta},\quad \alpha,\beta=\pm, \tag{IV.7}$$

form the generators of the spin Lie algebra $gl(2)$, which commute with the $gl(m|n)$ generators. It is worth noting that the spin $sl(2)$ algebra with generators,

$$S_+=E^+_-,\quad S_-=E^-_+,\quad S_0=\tfrac{1}{2}(E^+_+-E^-_-), \tag{IV.8}$$

also commute with the quasispin Lie algebra. Throughout, we denote the spin Lie algebra (IV.8) by $sl_S(2)$ and the quasispin Lie algebra by $sl_Q(2)$.

Then, the space of N-particle states gives rise to an irreducible representation of $gl(2m|2n)$ [and $osp(2m|2n)$] with highest weight,

$$\begin{cases}(\dot{1}_N,\overline{0}|\dot{0}), & N\leqslant 2m\\ (\dot{1}|N-2m,\dot{0}), & N>2m.\end{cases} \tag{IV.9}$$

This N-particle space decomposes into a multiplicity-free direct sum of irreducible $gl(m|n)\oplus sl_S(2)$ modules,

$$\hat{V}(a,b)\otimes V_s, \tag{IV.10}$$

where V_s denotes the $(2s+1)$-dimensional irreducible representation of $sl_S(2)$, $b=2s$, $N=2a+b$ and $\hat{V}(a,b)$ denotes the irreducible representation of $gl(m|n)$ with highest weight,

$$\Lambda_{a,b}=\begin{cases}(\dot{2}_a,\dot{1}_b,\dot{0}|\dot{0}), & a+b\leqslant m,\\ (\dot{2}_a,\dot{1}|a+b-m,\dot{0}), & a\leqslant m,a+b>m,\\ (\dot{2}|a+b-m,a-m,\dot{0}), & a>m.\end{cases} \tag{IV.11}$$

In this way we may realize all required ''two-column'' irreducible representations of $gl(m|n)$, inside a given antisymmetric tensor irreducible representation of $gl(2m|2n)$ utilizing the graded-fermion calculus.

V. CASIMIR INVARIANTS AND CONNECTION WITH QUASISPIN

From now on we shall use the notation

$$\hat{L}\equiv gl(m|n),\quad L\equiv osp(m|n),\quad \hat{L}_0\equiv gl(m)\oplus gl(n),\quad L_{\bar{o}}\equiv o(m)\oplus sp(n). \tag{V.1}$$

Let $C_{\hat{L}}$, C_L denote the universal Casimir invariants of $\hat{L}$, L, respectively. Then for the two-column irreducible representations of $\hat{L}$ we are considering, a straightforward but tedious calculation shows that

$$C_{\hat{L}}-C_L=(m-n+2-\tfrac{1}{2}\hat{N})\hat{N}-\tfrac{1}{2}(n-m)(n-m-2)+2Q^2, \tag{V.2}$$

where

$$Q^2 = \mathbf{Q}\cdot\mathbf{Q} = Q_0(Q_0+1) + Q_-Q_+ = Q_0(Q_0-1) + Q_+Q_- \tag{V.3}$$

is the square of the quasispin. Equation (V2) shows that Q^2 is expressible in terms of $C_{\hat{L}}$, C_L, and $\hat{N}$. It follows that Q^2, Q_-Q_+, Q_+Q_- must leave invariant (in fact, reduce to a scalar multiple of the identity on) a given irreducible representation of L inside a given (two-column) representation of $\hat{L}$. Given the highest weight of such an L module we may determine its quasispin $\overline{Q}$ [the lowest weight of the relevant $sl_Q(2)$ module] using (V2) and $Q^2 = \overline{Q}(\overline{Q}-1)$.

It is worth noting that we may write, for our quasispin generators,

$$\mathbf{Q} = \mathbf{Q}^{(0)} + \mathbf{Q}^{(1)}, \tag{V.4}$$

where

$$Q_-^{(0)} = \sum_{i=1}^{m} c_{i,-}c_{\bar{i},+}, \quad Q_-^{(1)} = \sum_{\mu=1}^{n} (-1)^{\mu} c_{\mu,-}c_{\bar{\mu},+}, \tag{V.5}$$

and, similarly, for Q_+, while

$$Q_0^{(0)} = \tfrac{1}{2}(\hat{N}_0 - m), \quad Q_0^{(1)} = \tfrac{1}{2}(\hat{N}_1 + n), \tag{V.6}$$

with $\hat{N}_0 = \Sigma_{i=1}^m E_i^i$ and $\hat{N}_1 = \Sigma_{\mu=1}^n E_\mu^\mu$ being the number operators for even fermions and odd bosons, respectively. Then it can be shown that $\mathbf{Q}^{(0)}$, $\mathbf{Q}^{(1)}$ both determine $sl(2)$ algebra that commute, so that the quasispin $\mathbf{Q}$ may be interpreted as the total quasispin obtained by coupling the quasispins of the even and odd components, respectively.

Similar remarks apply to the total spin algebra. The total spin vector is a sum of even and odd components,

$$\mathbf{S} = \mathbf{S}^{(0)} + \mathbf{S}^{(1)}, \tag{V.7}$$

whose corresponding $sl(2)$ algebras [cf. (IV.8)] are generated by

$$E^{(0)}{}^{\alpha}_{\beta} = \sum_{i=1}^{m} E^{i\alpha}_{i\beta}, \quad E^{(1)}{}^{\alpha}_{\beta} = \sum_{\mu=1}^{n} E^{\mu\alpha}_{\mu\beta}, \tag{V.8}$$

respectively. We note that the quasispin and spin algebras $sl_Q^{(0)}(2)$, $sl_Q^{(1)}(2)$, $sl_S^{(0)}(2)$, $sl_S^{(1)}(2)$ all commute with each other.

We remark that the quasispin algebras $sl_Q^{(0)}(2)$, $sl_Q^{(1)}(2)$ play an important role in decomposing irreducible representations of $\hat{L}_0$ into irreducible representations of $L_{\bar{0}}$. They commute with the even subalgebra $L_{\bar{0}}$ of L, but not with L itself.

VI. QUASISPIN EIGENVALUES

Throughout, $\hat{V}(a,b)$ denotes the irreducible representation of $\hat{L}$ with highest weight $\Lambda_{a,b}$ given by (IV.11). Let $\hat{V}_{\bar{0}}(a,b) = \hat{V}_0(\dot{0}|a+b,a,\dot{0})$ be its minimal $\mathbf{Z}$-graded component. Note that $\hat{V}_{\bar{0}}(a,b)$ is an irreducible $gl(n)$ module and thus an irreducible $\hat{L}_0$ module. We have the following.

Proposition 2: $\hat{V}_{\bar{0}}(a,b)$ cyclically generates $\hat{V}(a,b)$ as an L module: viz.,

$$\hat{V}(a,b) = U(L)\hat{V}_{\bar{0}}(a,b). \tag{VI.1}$$

Proof: Set

$$W = U(L)\hat{V}_0^-(a,b) \subset \hat{V}(a,b), \tag{VI.2}$$

i.e., W is an L submodule. We show that equality holds. Obviously, $\hat{V}_0^-(a,b)$ is an L_0^- module (since $L_0^- = L_{-2} \oplus L_0 \oplus L_2 \subset \hat{L}_0$). Now, since $\hat{V}_0^-(a,b)$ is the minimal **Z**-graded component of $\hat{V}(a,b)$, we have, by the PBW theorem,

$$\hat{V}(a,b) = U(\hat{L}_+)\hat{V}_0^-(a,b). \tag{VI.3}$$

Using

$$\sigma^i_\mu = E^i_\mu - (-1)^\mu E^{\bar{\mu}}_i \in L_1^- \equiv L_1 \oplus L_{-1}, \tag{VI.4}$$

we have

$$E^i_\mu \hat{V}_0^-(a,b) = \sigma^i_\mu \hat{V}_0^-(a,b) + (-1)^\mu E^{\bar{\mu}}_i \hat{V}_0^-(a,b) = \sigma^i_\mu \hat{V}_0^-(a,b) \subset W, \tag{VI.5}$$

since $E^{\bar{\mu}}_i \hat{V}_0^-(a,b) \subset \hat{L}_- \hat{V}_0^-(a,b) = (0)$. It follows that

$$\hat{L}_+ \hat{V}_0^-(a,b) \subset W. \tag{VI.6}$$

Proceeding recursively, let us assume that

$$(\hat{L}_+)^i \hat{V}_0^-(a,b) \subset W, \quad \forall i \leqslant r. \tag{VI.7}$$

Then

$$\begin{aligned} E^i_\mu \hat{L}^r_+ \hat{V}_0^-(a,b) &= \sigma^i_\mu \hat{L}^r_+ \hat{V}_0^-(a,b) + (-1)^\mu E^{\bar{\mu}}_i \hat{V}_0^-(a,b) \\ &\subset L \hat{L}^r_+ \hat{V}_0^-(a,b) + \hat{L}_- \hat{L}^r_+ \hat{V}_0^-(a,b) \\ &\subset L \hat{L}^r_+ \hat{V}_0^-(a,b) + \hat{L}^{r-1}_+ \hat{V}_0^-(a,b) \subset W, \end{aligned} \tag{VI.8}$$

since $\hat{L}_- \hat{V}_0^-(a,b) = (0)$ and $\hat{L}^r_+ \hat{V}_0^-(a,b) \subset W$, $\hat{L}^{r-1}_+ \hat{V}_0^-(a,b) \subset W$ by the recursion hypothesis. Thus $\hat{L}^{r+1}_+ \hat{V}_0^-(a,b) \subset W$ so that, by induction, $\hat{L}^r_+ \hat{V}_0^-(a,b) \subset W$, $\forall r$. It follows that

$$\hat{V}(a,b) = U(\hat{L}_+)\hat{V}_0(a,b) \subset W. \tag{VI.9}$$

Thus, we must have $W = \hat{V}(a,b)$.

From the traditional quasispin formalism for $gl(n) \supset sp(n)$, we have a decomposition of L_0^- modules,

$$\hat{V}_0^-(a,b) = V_0(a,b) \oplus Q_+^{(1)} \hat{V}_0^-(a-1,b), \tag{VI.10}$$

where $V_0(a,b)$ is an irreducible L_0^- module with highest weight $(\dot{0}|a+b,a,\dot{0})$ and comprises quasispin minimal states with respect to quasispin algebra $\mathbf{Q}^{(1)}$ (and thus also $\mathbf{Q}$), so

$$Q_-^{(1)} V_0(a,b) = Q_- V_0(a,b) = 0. \tag{VI.11}$$

Note that for $n=2$, $\hat{V}_0^-(a,b) = V_0(a,b)$ is an irreducible L_0^- module, but not quasispin minimal. Thus, the case $n=2$ requires a separate treatment. However, for this case, $\hat{V}_0^-(a,b) = V_0(a,b)$ still has well-defined quasispin $\overline{Q}$ (the minimal weight of the quasispin algebra): in fact, $\overline{Q} = \frac{1}{2}(b-m+n)$ for this case.

Proceeding recursively, we arrive at the irreducible $sp(n)$ (and hence L_0^-) module decomposition,

$$\hat{V}_0^-(a,b)=\bigoplus_{c=0}^{a} Q^{(1)a-c}_{+}V_0(c,b), \tag{VI.12}$$

where

$$Q^{(1)a-c}_{+}V_0(c,b)\cong V_0(c,b)\subset \hat{V}_0^-(c,b) \tag{VI.13}$$

is the irreducible L_0^- module with highest weight $(\dot{0}|c+b,c,\dot{0})$. From the above remarks $V_0(c,b)$ in the decomposition (VI.13) is quasispin minimal with respect to $\mathbf{Q}^{(1)}$ (and $\mathbf{Q}$) so $Q_-^{a-c+1}Q^{(1)a-c}_{+}V_0(c,b)=(0)$. It follows that $Q_-^{a+1}\hat{V}_0^-(a,b)=(0)$. Thus, if $q_N=\frac{1}{2}(N-m+n)$ is the eigenvalue of Q_0 on $\hat{V}(a,b)$, $N=2a+b$, then we have the following.

Theorem 1: *The quasispin eigenvalues (i.e., quasispin minimal weights) occurring in $\hat{V}(a,b)$ lie in the range*

$$\overline{Q}=q_N,\ \ q_N-1,...,\ \ q_N-a, \tag{VI.14}$$

or $q_N\geqslant\overline{Q}\geqslant q_N-a$ (in integer steps).

In view of (V.2) and (V.3), the operator Q_-Q_+ must leave invariant an L submodule of $\hat{V}(a,b)$. In view of the above theorem, the (generalized) eigenvalues of Q_-Q_+ on $\hat{V}(a,b)$ must be of the form

$$Q_-Q_+\equiv\overline{Q}(\overline{Q}-1)-q_N(q_N+1)=(\overline{Q}+q_N)(\overline{Q}-q_N-1). \tag{VI.15}$$

This eigenvalue can only vanish if $\overline{Q}+q_N=0$, which would imply, from the above theorem, $q_N-k=-q_N$ for some $0\leqslant k\leqslant a$. Thus, $k=2q_N=N-m+n$ or, equivalently, $a\geqslant N-m+n\Leftrightarrow a\geqslant 2a+b-m+n\Leftrightarrow m-n\geqslant a+b$.

Thus, if $m\leqslant n$, the (generalized) eigenvalues of Q_-Q_+ are all nonzero, except for the trivial module $(a=b=0)$, which we ignore below. Thus, we have proved the following lemma.

Lemma 1: For $m\leqslant n$, Q_-Q_+ determines a nonsingular operator on $\hat{V}(a,b)$, except possibly for the trivial module corresponding to $m=n$, $a=b=0$.

Remarks: The above result is crucial in what follows and will not generally hold for $m>n$. Hence, throughout the remainder we assume $m\leqslant n$, $n>2$. Note that Q_-Q_+ is nonsingular even on the trivial module, except when $m=n$.

VII. INDUCED FORMS AND AN ORTHOGONAL DECOMPOSITION

We recall that the graded fermion calculus admits a grade-$*$ operation, defined by

$$(c^\dagger_{a,\alpha})^*=(-1)^{[a]}c_{a,\alpha},\quad c^*_{a,\alpha}=c^\dagger_{a,\alpha}, \tag{VII.1}$$

which we extend in the usual way with $(AB)^*=(-1)^{[A][B]}B^*A^*$. This induces a grade-$*$ operation on $\hat{L}$ and L. Explicitly,

$$(E^a_b)^*=(-1)^{[a]([a]+[b])}E^b_a,\quad(\sigma^a_b)^*=(-1)^{[a]([a]+[b])}\sigma^b_a. \tag{VII.2}$$

Moreover, the quasispin generators satisfy $Q_+^*=Q_-$, $Q_-^*=Q_+$, and $Q_0^*=Q_0$.

With this convention, the graded fermion Fock space admits a nondegenerate graded sesquilinear form $\langle,\rangle$. In particular, $\hat{V}(a,b)$ is equipped with such a form and is nondegenerate. Note that

$$\langle v,E^a_bw\rangle=(-1)^{[v]([a]+[b])}\langle(E^a_b)^*v,w\rangle, \tag{VII.3}$$

which is the invariance condition of the form. It is the unique (up to scalar multiples) invariant graded form on $\hat{V}(a,b)$.

We now note that $Q_+\hat{V}(a-1,b)$ is an L submodule of $\hat{V}(a,b)$. In view of Lemma 1 and Eqs. (V.2) and (V.3), we have the following.

Lemma 2: The form $\langle\, ,\rangle$ *restricted to* $Q_+\hat{V}(a-1,b)\subset\hat{V}(a,b)$ *is nondegenerate, except for the case* $a=1$, $b=m-n=0$.

Proof: Under the above conditions, Q_-Q_+ is nonsingular on $\hat{V}(a-1,b)$, so $Q_-Q_+\hat{V}(a-1,b)=\hat{V}(a-1,b)$. Hence, for $v\in\hat{V}(a-1,b)$, we have $0=\langle Q_+\hat{V}(a-1,b),Q_+v\rangle\Rightarrow 0=\langle Q_-Q_+\hat{V}(a-1,b),v\rangle=\langle\hat{V}(a-1,b),v\rangle\Rightarrow v=0$ since $\langle\, ,\rangle$ on $\hat{V}(a-1,b)$ is nondegenerate. This shows that the form $\langle\, ,\rangle$ restricted to $Q_+\hat{V}(a-1,b)$ is nondegenerate, as required.

In view of Proposition 2, we have the following.

Proposition 3: $Q_-\hat{V}(a,b)=\hat{V}(a-1,b)$.

Proof: From Proposition 2, we have

$$Q_-\hat{V}(a,b)=Q_-U(L)\hat{V}_0^-(a,b)=U(L)Q_-\hat{V}_0^-(a,b)=U(L)Q_-^{(1)}\hat{V}_0^-(a,b)=U(L)\hat{V}_0^-(a-1,b), \tag{VII.4}$$

where the last step follows from a classical Lie algebra result. Again, utilizing Proposition 2, we have $U(L)\hat{V}_0^-(a-1,b)=\hat{V}(a-1,b)$, from which the result follows.

We are now in a position to prove the following.

Proposition 4: We have an L-module orthogonal decomposition;

$$\hat{V}(a,b)=\mathcal{K}\oplus Q_+\hat{V}(a-1,b), \tag{VII.5}$$

where $\mathcal{K}=\mathrm{Ker}\,Q_-\cap\hat{V}(a,b)$, *except for the case* $a=1$, $b=m-n=0$.

Proof: For $v\in\hat{V}(a,b)$, $\langle v,Q_+\hat{V}(a-1,b)\rangle=0\Leftrightarrow\langle Q_-v,\hat{V}(a-1,b)\rangle=0\Leftrightarrow Q_-v=0$ (by Proposition 3) $\Leftrightarrow v\in\mathcal{K}$. Since $\langle\, ,\rangle$ restricted to $Q_+\hat{V}(a-1,b)$ is nondegenerate, the result follows.

Finally, in view of Theorem 1 we have Proposition 5.

Proposition 5: $\hat{V}(a=0,b)$ *is an irreducible L module.*

Proof: In such a case, $\hat{V}_0^-(0,b)=V_0(0,b)$ is an irreducible L_0^- module cyclically generated by an L maximal state. Thus, $\hat{V}(0,b)=U(L)V_0(0,b)$ must be an indecomposable L module. Since the form $\langle\, ,\rangle$ on $\hat{V}(0,b)$ is nondegenerate, this forces $\hat{V}(0,b)$ to be an irreducible L module.

The result above shows that the minimal $\hat{L}$ irreducible representations are indeed irreducible under L.

VIII. PRELIMINARIES TO BRANCHING RULES

It is our aim below to prove, barring the exceptional case of Lemma 2, that $\mathcal{K}$ is an irreducible L module. Note that the maximal state of the L_0^- module $V_0(a,b)$ occurring in the decomposition (VI.10), in fact, coincides with the $\hat{L}_0$ maximal vector v_+^Λ of $\hat{V}_0^-(a,b)$: For $n>2$ it can be seen directly that

$$Q_-v_+^\Lambda=Q_-^{(1)}v_+^\Lambda=0, \tag{VIII.1}$$

for this maximal vector. Moreover, for $n>2$ we have

$$E_{\bar{\mu}}^{i}v_+^\Lambda=0,\quad 1\leqslant i\leqslant m,\quad 1\leqslant\mu\leqslant k; \tag{VIII.2}$$

otherwise, this vector would have weight $(\dot{0}|a+b,a,\dot{0})+\epsilon_i-\delta_{\bar{\mu}}$ $(\bar{\mu}>k=n/2)$, which is impossible since all $\hat{L}$ weight components are positive. Also, since v_+^Λ belongs to the $\hat{L}$ minimal **Z**-graded component, we must have

$$E_i^{\mu} v_+^{\Lambda} = 0, \quad \forall i, \mu. \tag{VIII.3}$$

Thus, for $\sigma_i^{\mu} \in L_1$, we have

$$\sigma_i^{\mu} v_+^{\Lambda} = (E_i^{\mu} + (-1)^{\mu} E_{\mu}^{\bar{i}}) v_+^{\Lambda} = 0, \quad \forall i, 1 \leq \mu \leq k \Rightarrow L_1 v_+^{\Lambda} = (0). \tag{VIII.4}$$

It follows that the L_0^- module $V_0(a,b)$ must cyclically generate an indecomposable module over L:

$$V(a,b) = U(L) V_0(a,b), \tag{VIII.5}$$

with highest weight

$$\lambda_{a,b} \equiv (\dot{0} | a+b, a, \dot{0}). \tag{VIII.6}$$

Since

$$Q_- V_0(a,b) = Q_-^{(1)} V_0(a,b) = (0), \tag{VIII.7}$$

we have

$$Q_- V(a,b) = Q_- U(L) V_0(a,b) = U(L) Q_- V_0(a,b) = (0). \tag{VIII.8}$$

It follows that $V(a,b) \subset \mathcal{K}$.

We now show that $V(a,b) = \mathcal{K}$ is irreducible. First, in view of Proposition 3, we have the following lemma.

Lemma 3: $v \in \mathcal{K} \Leftrightarrow Q_+ Q_- v = 0$.

Proof: Obviously $v \in \mathcal{K} \Rightarrow Q_- v = 0 \Rightarrow Q_+ Q_- v = 0$. Conversely, $Q_+ Q_- v = 0 \Rightarrow$

$$0 = \langle Q_+ Q_- v, \hat{V}(a,b) \rangle = \langle Q_- v, Q_- \hat{V}(a,b) \rangle = \langle Q_- v, \hat{V}(a-1,b) \rangle \tag{VIII.9}$$

$\Rightarrow Q_- v = 0 \Rightarrow v \in \mathcal{K}$.

It follows that $\mathcal{K}$ consists of eigenstates of $Q_+ Q_-$ with a zero eigenvalue. Also, since $Q_- \mathcal{K} = (0)$ and $\mathcal{K} \subset \hat{V}(a,b)$, it follows that all states in $\mathcal{K}$ are eigenvectors of Q_0 with eigenvalue $q_N = \frac{1}{2}(N-m+n)$ and are, moreover, quasispin minimal states, and so have quasispin $\overline{Q} = q_N$. Thus, Q^2 reduces to a scalar multiple $\overline{Q}(\overline{Q}-1) = q_N(q_N-1)$ on $\mathcal{K}$. It then follows from (V.2) that the universal Casimir element C_L of L must reduce to a scalar multiple of the identity on $\mathcal{K}$. Since $V(a,b) \subset \mathcal{K}$ has highest weight $\lambda_{a,b}$, this eigenvalue can be shown to be given by

$$\chi_{\lambda_{a,b}}(C_L) = (\lambda_{a,b}, \lambda_{a,b} + 2\rho) = -(a+b)(a+b+n-m) - a(a+n-m-2). \tag{VIII.10}$$

Hence we have proved the following.

Lemma 4: C_L reduces to a scalar multiple of the identity on $\mathcal{K}$ with an eigenvalue given by (VIII.10).

Now $\mathcal{K}$ is a completely reducible L_0^- module. Hence we have the following.

Lemma 5: Suppose for any irreducible L_0^- module $V_0(\lambda)$ contained in an irreducible $\hat{L}_0$ module $\hat{V}_0(\Lambda) \subset \hat{V}(a,b)$ that $\chi_\lambda(C_L) = \chi_{\lambda_{a,b}}(C_L) \Leftrightarrow \Lambda = \Lambda_{a,b}$ and $\lambda = \lambda_{a,b}$. Then $\mathcal{K} = V(a,b)$ is irreducible.

Proof: Indeed, in such a case it follows from Lemma 4 that the highest weight vector of $V(a,b)$ must be the unique primitive vector in $\mathcal{K}$. This is enough to prove that $\mathcal{K}$ is irreducible.

Finally, we recall that $\hat{V}(a,b)$ comprises states with total spin $s = b/2$ and with particle number $N = 2a+b$. Then the possible irreducible representations of $\hat{L}_0$ occurring in $\hat{V}(a,b)$ must have highest weights of the form

$$\Lambda=(\dot{2}_{a'},\dot{1}_{b'},\dot{0}|c',d',\dot{0}). \tag{VIII.11}$$

Then we must have

$$2a'+b'+c'+d'=N=2a+b. \tag{VIII.12}$$

Moreover, the total spins for the even and odd components of this irreducible representation are $s_0=b'/2$ and $s_1=(c'-d')/2$, respectively. So, using the triangular rule for angular momenta, we have

$$s\leqslant s_0+s_1,\quad s_0\leqslant s+s_1,\quad s_1\leqslant s+s_0, \tag{VIII.13}$$

or

$$b\leqslant b'+c'-d',\quad b'\leqslant b+c'-d',\quad c'-d'\leqslant b+b'. \tag{VIII.14}$$

These inequalities turn out to be important below.

IX. $\hat{L}\downarrow L$ BRANCHING RULES

We start this section with some facts concerning $\hat{L}_0\downarrow L_0^-$. The possible $\hat{L}_0$ highest weights Λ occurring in $\hat{V}(a,b)$ are of the form of (VIII.11). The possible L_0^- highest weights λ in $\hat{V}(a,b)$ are obtained from such Λ by a classical contraction procedure and have the form

$$\lambda=(\dot{2}_c,\dot{1}_d,\dot{0}|e,f,\dot{0}),\quad c+d\leqslant h, \tag{IX.1}$$

where $d=b'\wedge(m-2c-b')$, $e-f=c'-d'$ [here and below $x\wedge y\equiv\min(x,y)$] and

$$c\leqslant a',\quad e+f\leqslant c'+d'=2a+b-2a'-2b'. \tag{IX.2}$$

Note that for $n>4$, there are additional restrictions on the allowed L_0^- dominant weights in order that they give rise to highest weights of L.[8] In the interests of a unified treatment of all cases, including $n=4$, we do not impose these supplementary conditions here.

Since $e-f=c'-d'$, the inequalities (VIII.14) lead to

$$b'\leqslant b+e-f,\quad b\leqslant b'+e-f,\quad e-f\leqslant b+b'. \tag{IX.3}$$

Hence, we have the following inequalities.

Lemma 6: $e\leqslant a+b-c$, $f\leqslant a-c$.

Proof: We have

$$e+f\leqslant 2a+b-2a'-b',\quad e-f\leqslant b+b'.$$

Adding these two inequalities gives $e\leqslant a+b-a'$. Thus, $e\leqslant a-c$ since $c\leqslant a'$. Similarly, adding

$$e+f\leqslant 2a+b-2a'-b',\quad f-e\leqslant b'-b$$

leads to $f\leqslant a-a'\leqslant a-c$.

We are now in a position to compute the eigenvalue $\chi_\lambda(C_L)$ compared with that of (VIII.10). By direct computation we have

$$\chi_\lambda(C_L)=(\lambda,\lambda+2\rho)=m(2c+d)-c(c+1)-(c+d)(c+d+1)$$
$$-(n-m)(e+f)+4c+d+2f-e^2-f^2, \tag{IX.4}$$

where we have used

$$\lambda=\sum_{i=1}^{c} 2\epsilon_i+\sum_{i=c+1}^{d+c}\epsilon_i+e\delta_1+f\delta_2, \tag{IX.5}$$

together with the expression for ρ of L. By a straightforward but tedious calculation, using (VIII.10) and (IX.4), we obtain

$$\chi_\lambda(C_L)-\chi_{\lambda_{a,b}}(C_L)=2cn+d(m-d)+2c(2a+b-2c-d)+(a+b-c-e)\times(a+b-c+e+n-m)+(a-c-f)(a-c+f+n-m-2) \tag{IX.6}$$

$$=[2c(n+1)+2f-2a]+d(m-d)+2c(2a+b-2c-d)+(a+b-c-e)(a+b-c+e+n-m)+(a-c-f)(a-c+f+n-m). \tag{IX.7}$$

All terms on the rhs of (IX.6) are positive, in view of the inequalities given above, except possibly the last due to the term $(a-c+f+n-m-2)$. Similarly, in (IX.7) all terms on the rhs are positive, except possibly the first.

We proceed stepwise.

(i) $c\geqslant 1$: Then the first term on the rhs of (IX.7) gives

$$2c(n+1)+2f-2a\geqslant 2(n+1+f-a).$$

This leads to two subclasses.

(i.1) $a\leqslant n+1$: The rhs terms are all non-negative, so (IX.7) can only vanish if $a=n+1$, $f=0=d$, $2a+b=2c+d$. But then, since $d=0$ this would imply $2c=2a+b\Rightarrow c>a=n+1$, which is impossible since $c\leqslant h\leqslant m\leqslant n$. Thus we conclude that the rhs must be strictly positive in this case.

(i.2) $a\geqslant n+2$: In this case all terms on the rhs of (IX.6) are non-negative, including the last term, since, for the case at hand,

$$a-c+f+n-m-2\geqslant n+2-c+f+n-m-2\geqslant n-c+f+n-m\geqslant 0,$$

since $n\geqslant m\geqslant h\geqslant c$. Since $c\geqslant 1$, the rhs of (IX.6) must be strictly positive in this case.

We thus conclude, for $c\geqslant 1$, that $\chi_\lambda(C_L)-\chi_{\lambda_{a,b}}(C_L)>0$. It remains then to consider the case $c=0$, in which case we have

$$\chi_\lambda(C_L)-\chi_{\lambda_{a,b}}(C_L)=d(m-d)+(a+b-e)(a+b+e+n-m)+(a-f)(a+f+n-m-2). \tag{IX.8}$$

Note that for the case $c=0$, the inequalities of Lemma 6 reduce to $e\leqslant a+b$, $f\leqslant a$ and for the case at hand we have

$$e-f=c'-d',\quad d=b'\wedge(m-b').$$

It is convenient to treat the cases $m=n$ and $m<n$ separately.

(ii) $c=0$, $n>m$: Here we assume $a\geqslant 1$, since when $a=0$, $\hat{V}(a=0,b)$ is already known to be an irreducible L module, so the branching rule is trivial.

Under these assumptions all terms on the rhs of (IX.8) are non-negative, including the last, since

$$a+f+n-m-2\geqslant f+n-m-1\geqslant 0.$$

Note that this factor can only vanish when $a=1$, $f=0$, $n=m+1$. There are thus two possibilities to consider for vanishing of the rhs of (IX.8):

(ii.1) $d=0$, $e=a+b$, $f=a$: Since $c'+d'=2a+b-2a'-b'\geqslant e+f=2a+b$ and $c'-d'=e-f=b$, this implies that $a'=b'=0$, $c'=a+b$, $d'=a$, and $\lambda=\lambda_{a,b}$. So in this case $\Lambda=(\dot{0}|a+b,a,\dot{0})=\Lambda_{a,b}$ and $\lambda=\lambda_{a,b}$.

(ii.2) $d=0$, $e=a+b$, $f=0$, $a=1$, $n=m+1$: Then $c'+d'\geqslant e+f=a+b$. Since $a=1$, we thus have

$$2+b=N=2a'+b'+c'+d'\geqslant 2a'+b'+a+b=2a'+b'+1+b$$

$\Rightarrow 1\geqslant 2a'+b'\Rightarrow a'=0$ and $b'\leqslant 1$. In such a case we must have $d=b'\wedge(m-b')$ and since $d=0\Rightarrow b'=0$, or $m=b'=1\Rightarrow n=2$, which we ignore. Then $\Lambda=(\dot{0}|c',b',\dot{0})$ with $c'-b'=e-f=a+b=1+b$, which corresponds to states with spin $(1+b)/2$, which is impossible since all states in $\hat{V}(a,b)$ have spin $b/2$. Thus, this latter case cannot occur.

Thus we have shown, for all cases, that when $n>m$, $\mathcal{K}=V(a,b)$ must be an irreducible module with highest weight $\lambda_{a,b}$, using Lemma 4.

In view of Proposition 3 we thus have the L module decomposition,

$$\hat{V}(a,b)=V(a,b)\oplus Q_+\hat{V}(a-1,b). \tag{IX.9}$$

Since Q_-Q_+ is nonsingular, $Q_+\hat{V}(a-1,b)\cong\hat{V}(a-1,b)$. By repeated application of (IX.9), we arrive at the irreducible L module decomposition,

$$\hat{V}(a,b)=\bigoplus_{c=0}^{a} Q_+^{a-c}V(c,b). \tag{IX.10}$$

Hence we have proved the following theorem.

Theorem 2: $(n>m,n>2)$**:** *We have the irreducible L-module decomposition,*

$$\hat{V}(a,b)=\bigoplus_{c=0}^{a} V(c,b). \tag{IX.11}$$

We emphasize that throughout $V(a,b)$ denotes the L module with highest weight $\lambda_{a,b}=(\dot{0}|a+b,a,\dot{0})$. It remains now to consider the case $m=n$, which is somewhat more interesting.

(iii) $c=0$, $m=n>2$: Again, we assume $a\geqslant 1$ since $\hat{V}(a=0,b)$ is an irreducible L module, as we have seen. We recall for the case at hand $e\leqslant a+b$, $f\leqslant a$, $a\geqslant 1$, $m=n>2$, $e-f=c'-d'$, $d=b'\wedge(m-b')$ and

$$\chi_\lambda(C_L)-\chi_{\lambda_{a,b}}(C_L)=d(m-d)+(a+b-e)(a+b+e)+(a-f)(a+f-2). \tag{IX.12}$$

There are now several cases to consider for the vanishing of (IX.12).

(iii.1) $a=f$: Then (IX.12) vanishes when $d=0$, $e=a+b$. Thus

$$c'+d'\geqslant e+f=2a+b=2a'+b'+c'+d'$$

$\Rightarrow a'=b'=0$, $c'+d'=2a+b$, and $c'-d'=e-f=b$. This corresponds to $\Lambda=\Lambda_{a,b}$ and $\lambda=\lambda_{a,b}$.

(iii.2) $f=2-a$: Then (IX.12) vanishes when $d=0$, $e=a+b$. Since $a\geqslant 1$ there are two cases.

(iii.2.1) $f=0$, $a=2$: This is only possible when $c'+d'\geqslant e+f=a+b\Rightarrow$

$$2a+b\geqslant 2a'+b'+c'+d'\geqslant 2a'+b'+a+b$$

$\Rightarrow a\geqslant 2a'+b'$ or $2\geqslant 2a'+b'$. This leads to two further cases.

(iii.2.1a) $f=0$, $a=2$, $a'=0$, $b'\leqslant 2$: In view of the contraction procedure, this is only consistent with $d=0$ if $b'=0$ (so $c'=a+b$, $d'=a$) or if $b=2$ and $m=n=2$. The latter case is being ignored and the former case cannot occur since then $c'-d'=e-f=a+b>b$ in contradiction to the fact that all states in $\hat{V}(a,b)$ have spin $b/2$.

(iii.2.1b) $f=0$, $a=2$, $a'=1$, $b'=d=0$: Then $c'-d'=e-f=a+b>b$, which again is impossible since all states have spin $b/2$.

(iii.2.2) $f=a=1$: Then $c'-d'=a+b-a=b$, $c'+b'\geqslant e+f=2a+b\Rightarrow a'=b'=0$, $c'=a+b$, $d'=a\Rightarrow\Lambda=\Lambda_{a,b}$, $\lambda=\lambda_{a,b}$.

(iii.3) $a+f-2<0$, $a>f$: This can only occur when $a=1$, $f=0$, in which case the rhs of (IX.12) becomes

$$d(m-d)+(a+b+e)(a+b-e)-1.$$

There are two cases for the vanishing of this.

(iii.3.1) $e=a+b$, $d=1$, $m=2$, which can occur, but we are ignoring since $n=m>2$.

(iii.3.2) $d=f=e=b=0$: Then $c'-d'=e-f=0$ and

$$N=2=2a+b=2a'+b'+c'+d'=2(a'+c')+b',$$

which can occur in the following cases:

$$a'=b'=0,\quad c'=d'=1\Rightarrow\lambda=(\dot{0}|\dot{0}),\quad \Lambda=(\dot{0}|1,1,\dot{0});$$

$$b'=c'=d'=0,\quad a'=1\Rightarrow\lambda=(\dot{0}|\dot{0}),\quad \Lambda=(2,\dot{0}|\dot{0}).$$

This exhausts all possibilities. It follows from the above that for $n=m>2$ the rhs of (IX.12) is always strictly positive and can only vanish in the last case, corresponding to $a=1$ and $b=0$. This is the irreducible representation $\hat{V}(2,\dot{0}|\dot{0})$ of $gl(n|n)$, which is known to give rise to an indecomposable $osp(n|n)$ module with a composition series of length 3 whose factors are isomorphic to the $osp(n|n)$ modules $V(1,0)$ and $V(0,0)$ (see Appendix).

Thus we have proved the decomposition

$$\hat{V}(a,b)=V(a,b)\oplus Q_+\hat{V}(a-1,b) \tag{IX.13}$$

with $V(a,b)$ an irreducible L-module of highest weight $\lambda_{a,b}$, provided $(a,b)\neq(1,0)$. Proceeding recursively we have the following theorem.

Theorem 3 (n=m>2): For $b>0$ we have the irreducible L-module decomposition,

$$\hat{V}(a,b)=\bigoplus_{c=0}^{a} V(c,b). \tag{IX.14}$$

For $b=0$ we have the L-module decomposition,

$$\hat{V}(a,0)=\bigoplus_{c=1}^{a} V(c,0), \tag{IX.15}$$

where $V(c,0)$ is irreducible for $c>1$ but $V(1,0)$ is indecomposable with a composition series of length 3 with composition factors isomorphic to irreducible L modules $V(1,0)$ and $V(0,0)$, the latter occurring twice.

Theorems 2 and 3 are our main results in this section concerning the $\hat{L}\downarrow L$ branching rules for the two-column tensor representations of $\hat{L}$. We remark that for the special case $n-m=0=b$, $a=1$, $\hat{V}(a-1,b)=\hat{V}(0,0)$ coincides with the identity module, which is the exceptional case of Lemma 2. For this case the form $\langle\,,\rangle$ on $\hat{V}(a,b)=\hat{V}(1,0)$ is degenerate on $Q_+\hat{V}(a-1,b)$

$=Q_+\hat{V}(0,0)$. Thus, Proposition 4 fails in this case (and only this case). This, of course, agrees with the result that $\hat{V}(a,b)=\hat{V}(1,0)\equiv\hat{V}(2,\dot{0}|\dot{0})$ is indecomposable for $m=n$.

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APPENDIX: STRUCTURE OF $\hat{V}(z,\dot{0}|\dot{0})$ AS A $osp(n|n)$-MODULE

Here for completeness we determine the structure of the irreducible $\hat{L}=gl(n|n=2k)$ module $\hat{V}(2,\dot{0}|\dot{0})$ as a module over $L=osp(n|n)$, in fully explicit form.

First $\hat{V}(2,\dot{0}|\dot{0})$ admits the following **Z**-graded decomposition into irreducible $\hat{L}_0$ modules with highest weights shown:

$$\hat{V}(2,\dot{0}|\dot{0})=\hat{V}_0(2,\dot{0}|\hat{0})\oplus\hat{V}_1(1,\dot{0}|1,\dot{0})\oplus\hat{V}_2(\dot{0}|1,1,\dot{0}).$$

In the notation of the paper, the last space corresponds to the irreducible $\hat{L}_0$ module $\hat{V}_{\overline{0}}(a=1,b=0)$. In terms of the graded fermion formalism, we have the following basis states:

$$\begin{aligned}
&\hat{V}_0(2,\dot{0}|\dot{0}):\ (c^\dagger_{i,+}c^\dagger_{j,-}+c^\dagger_{j,+}c^\dagger_{i,-})|0\rangle,\quad 1\leqslant i,j\leqslant n,\\
&\hat{V}_1(1,\dot{0}|1,\dot{0}):\ (c^\dagger_{i,+}c^\dagger_{\mu,-}+c^\dagger_{\mu,+}c^\dagger_{i,-})|0\rangle,\quad 1\leqslant i,\mu\leqslant n,\\
&\hat{V}_2(\dot{0}|1,1\dot{0}):\ (c^\dagger_{\mu,+}c^\dagger_{\nu,-}-c^\dagger_{\nu,+}c^\dagger_{\mu,-})|0\rangle,\quad 1\leqslant \mu,\nu\leqslant n,
\end{aligned}\tag{A1}$$

where $|0\rangle$ is the vacuum state. The latter space decomposes into $L_{\overline{0}}$ modules according to

$$\hat{V}_2(\dot{0}|1,1,\dot{0})=V_0(\dot{0}|1,1\dot{0})\oplus V_0(\dot{0}|\dot{0}),$$

where $V_0(\dot{0}|\dot{0})$ is spanned by $Q_+^{(1)}|0\rangle$ (the trivial $L_{\overline{0}}$ module) and $V_0(\dot{0}|1,1,\dot{0})$ is an irreducible $L_{\overline{0}}$ module with the highest weight indicated and the following basis vectors:

$$(c^\dagger_{\mu,+}c^\dagger_{\nu,-}-c^\dagger_{\nu,+}c^\dagger_{\mu,-})|0\rangle,\quad 1\leqslant\nu\neq\overline{\mu}\leqslant n,\tag{A2}$$

$$(\Omega^\dagger_\mu-\Omega^\dagger_{\mu+1})|0\rangle,\quad 1\leqslant\mu<k,\tag{A3}$$

where

$$\Omega^\dagger_\mu\equiv c^\dagger_{\mu,+}c^\dagger_{\overline{\mu},-}-c^\dagger_{\overline{\mu},+}c^\dagger_{\mu,-}.$$

Note that this irreducible $L_{\overline{0}}$ module cyclically generates an indecomposable L module $\tilde{V}(\delta_1+\delta_2)$ with highest weight $\delta_1+\delta_2$ and highest weight vector given by (A2) with $\mu=1$, $\nu=2$.

Now $\hat{V}_1(1,\dot{0}|1,\dot{0})$ is also irreducible as an $L_{\overline{0}}$ module that is contained in $\tilde{V}(\delta_1+\delta_2)$. Then by applying the odd lowering generators $\sigma^i_\mu=E^i_\mu-(-1)^\mu E^{\overline{\mu}}_i(1\leqslant\mu\leqslant k,1\leqslant i\leqslant n)$ of L to the states (A1), the following states in $\hat{V}_0(2,\dot{0}|\dot{0})$ are easily seen to be in $\tilde{V}(\delta_1+\delta_2)$:

$$(c^\dagger_{i,+}c^\dagger_{j,-}+c^\dagger_{j,+}c^\dagger_{i,-})|0\rangle,\quad 1\leqslant j\neq\overline{i}\leqslant n,\tag{A4}$$

$$(\Omega^\dagger_i-\Omega^\dagger_{i+1})|0\rangle,\quad 1\leqslant i<k,\tag{A5}$$

where

$$\Omega_i^\dagger \equiv c_{i,+}^\dagger c_{,-}^\dagger + c_{\overline{i},+}^\dagger c_{i,-}^\dagger .$$

Further, the following states are also seen to be in $\tilde{V}(\delta_1+\delta_2)$:

$$(\Omega_i^\dagger + (-1)^\mu \Omega_\mu^\dagger)|0\rangle, \quad 1 \leqslant i,\mu < k, \tag{A6}$$

which follows by applying $\sigma_i^{\overline{\mu}}$ to the states (A1) with $1 \leqslant \mu \leqslant k$. Summing (A6) on $\mu = i$ from 1 to k, we thus obtain

$$\left(\sum_{i=1}^{k} \Omega_i^\dagger + \sum_{\mu=1}^{k} (-1)^\mu \Omega_\mu^\dagger\right)|0\rangle = Q_+|0\rangle \in \tilde{V}(\delta_1+\delta_2). \tag{A7}$$

It is worth noting that the states (A6) are expressible in terms of the states (A3), (A5), and (A7).

The states (A1)–(A7) form a basis for the standard cyclic L module $\tilde{V}(\delta_1+\delta_2)$. We note that $\dim \tilde{V}(\delta_1+\delta_2) = \dim \hat{V}(2,\dot{0}|\dot{0}) - 1$ and $\tilde{V}(\delta_1+\delta_2)$ is the unique maximal L submodule of $\hat{V}(2,\dot{0}|\dot{0})$. In view of (A7), this module is not irreducible since it contains the trivial one-dimensional L module $V(\dot{0}|\dot{0})$ as a unique submodule.

The remaining state in $\hat{V}(2,\dot{0}|\dot{0})$, not in $\tilde{V}(\delta_1+\delta_2)$, is $Q_+^{(1)}|0\rangle$ (or $Q_+^{(0)}|0\rangle$), which thus generates the basis vector for the L factor module $\hat{V}(2,\dot{0}|\dot{0})/\tilde{V}(\delta_1+\delta_2)$, which is obviously isomorphic to the trivial L module $V(\dot{0}|\dot{0})$. We thus arrive at the L-module composition series $\hat{V}(2,\dot{0}|\dot{0}) \supset \tilde{V}(\delta_1+\delta_2) \supset V(\dot{0}|\dot{0}) \supset (0)$ with corresponding factors isomorphic to the irreducible L modules with highest weights $(\dot{0}|\dot{0})$, $\delta_1+\delta_2$, and $(\dot{0}|\dot{0})$, respectively.

This result is of importance to the explicit construction of new R matrices.[4] In particular, it gives rise to an L-invariant nilpotent contribution to the R matrices, a new effect not seen in the untwisted or nonsuper cases.

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