PHYSICAL REVIEW A VOLUME 58, NUMBER 3 SEPTEMBER 1998

Three-dimensional quantum solitons with parametric coupling

K. V. Kheruntsyan and P. D. Drummond
Department of Physics, University of Queensland, St. Lucia, Queensland 4072, Australia

(Received 5 January 1998)

We consider the quantum field theory of two bosonic fields interacting via both parametric (cubic) and quartic couplings. In the case of photonic fields in a nonlinear optical medium, this corresponds to the process of second-harmonic generation (via $\chi^{(2)}$ nonlinearity) modified by the $\chi^{(3)}$ nonlinearity. The quantum solitons or energy eigenstates (bound-state solutions) are obtained exactly in the simplest case of two-particle binding, in one, two, and three space dimensions. We also investigate three-particle binding in one space dimension. The results indicate that the exact quantum solitons of this field theory have a singular, pointlike structure in two and three dimensions—even though the corresponding classical theory is nonsingular. To estimate the physically accessible radii and binding energies of the bound states, we impose a momentum cutoff on the nonlinear couplings. In the case of nonlinear optical interactions, the resulting radii and binding energies of these photonic particlelike excitations in highly nonlinear parametric media appear to be close to physically observable values. [S1050-2947(98)05109-9]

PACS number(s): 42.50.−p, 03.65.Ge, 11.10.St, 42.65.Tg

I. INTRODUCTION

Quantum solitons were defined in Lee's early work on nonlinear quantum field theory as the bound states of a quantum field [1]. Thus they are generalizations of the nonlinear solitonic solutions of classical wave theory, to include quantum fields. In this sense, there are a wide variety of quantum fields capable of being analyzed. Since quantum field theory is generic to many areas of physics, we can expect these entities to be universally significant, in all areas where there are nonlinear interactions involving quantum fields.

It is possible to treat ordinary nonrelativistic quantum mechanics as a quantum field, so this definition includes the ordinary two-particle bound states of quantum mechanics as an exactly soluble case. Other exactly soluble cases include the many-body bound states of bosons interacting via δ-function interactions in one space dimension. This model (often called the nonlinear Schrödinger model) was solved by Lieb and Liniger, McGuire, and Yang [2]. Recently it was predicted that this soluble model could lead to experimentally observable quantum effects including quantum squeezing in optical fiber solitons [3,4]. This prediction is now verified experimentally [5].

Other examples of exactly soluble models like the Hubbard model [6] are generally restricted to one space dimension—except for Laughlin's highly innovative theory of an idealized model of two-dimensional electron gas in an external magnetic field [7]. This was able to explain the phenomenon of the fractional quantum Hall effect [8].

Each of these soluble cases has led to substantial improvements in our understanding of quantum theory, together with new and interesting physical consequences. However, there are few exact solutions in two or three space dimensions, except for physically inaccessible models like the quantum Davey-Stewartson model [9]. This is especially true if we look for nonlinear quantum field theories which include the most fundamental property that distinguishes quantum mechanics from quantum field theory—that is, the ability to create and destroy particles.

The most elementary interaction of this type is the cubic coupling between two creation operators and an annihilation operator (and vice versa, on grounds of Hermiticity). Cubic couplings are, of course, basic to QED and QCD, where they involve both fermionic and bosonic fields. These theories do not appear to have exactly known solutions in four-dimensional space-time, and are usually treated by various approximations. The most prominent of these is the Feynman diagram method or perturbation theory, which is generally conjectured to be nonconvergent.

It would be useful to have a cubic interaction theory which was exactly soluble, to give some guide as to the possible variety of behavior in this class of widely used quantum field theories. In particular, one would like to investigate whether the resulting quantum solitons have any differences resulting from dimensionality, or from the presence of interactions that change particle number. Surprisingly, the simplest cubic interaction involving two boson fields—the parametric interaction of the form $\Psi^{\dagger}\Phi^{2}$—has not been analyzed for bound states in higher dimensions, even though the corresponding classical parametric theory has stable higher-dimensional soliton solutions [10].

In this paper we consider this problem of bound states in parametric quantum field theory, and find some exactly soluble cases with unusual and previously unexpected properties. The model is a traveling-wave analog of the quantum theory used to describe squeezed states in quantum optics [11,12], and more recently molecular dissociation in atom optics [13]. The problem is all the more interesting because technical advances in nonlinear optics and laser physics are now reaching the point that this type of bound state could become experimentally accessible in the relatively near future. In addition, we mention that the model may be applied to the physics of ultracold atoms and Bose-Einstein condensates, as describing nontrivial excitations in hybrid atomic-molecular systems [14]. This application, however, will be explored in a greater detail elsewhere.

Our results have a number of unexpected features. The most surprising is that while the simplest parametric theory has bound states in one space dimension, it is unstable (like the nonlinear Schrödinger model with an attractive δ-function potential) in higher dimensions. However, unlike the nonlinear Schrödinger model, this instability shows no trace at the classical level [10]. We note that in the case of the nonlinear Schrödinger model, the higher-dimensional instability has a classical analog; the self-focusing singularity. For a stable parametric quantum field theory the Hamiltonian must therefore be modified.

We next investigate the effects of modifying the nonlinear interaction by adding quartic terms to the Hamiltonian. Quartic coupling corresponds to a nonlinear refractive index in the corresponding optical medium, resulting in self- and cross-phase modulation terms. It is also found as a short-range interatomic potential in atom-atom interactions. With a positive quartic interaction, a rigorous lower bound to the energy does exist, and we demonstrate the existence of exact two-particle bound-state solutions in higher dimensions. These new types of quantum solitons have a unique character: the solution has a finite binding energy, but the corresponding two-particle wave function has a zero radius. The point-like structure of these bound states can be termed a "quantum singularity." No analogous behavior exists in the corresponding classical theory, which is known [10,15] to possess stable, finite-size classical soliton solutions.

The reason for this unexpected behavior is that the fundamental structure of the new solution is inherently nonclassical, being a quantum superposition of two states, with one of them having a single (Ψ-type) boson and the other having two (Φ-type) bosons present. This is a bosonic analog of the quark model of mesons, with the Φ-type bosons behaving as "quarks," and the Ψ-type boson taking the role of "gluon." However, unlike the usual meson, the system has a finite probability of having no "quark" present at all.

An alternative way of modifying the Hamiltonian (for higher-dimensional solitons) is to impose a momentum cutoff on the nonlinear couplings. In this case exact two-particle bound states are shown to acquire finite radii in higher dimensions. Moreover, finite-size multidimensional bound states occur even without the stabilizing quartic term, that is, in the simplest version of the theory—pure parametric interaction.

We also investigate the three-particle problem in one space dimension. While no exact solution is found in this case, the existence of a three-particle bound state or a "bosonic hadron" is shown using a variational approach.

In summary, the quantum bound states have a strong dependence on dimensionality, giving rise to the appearance of quantum singularities with zero radius (unless there is a cutoff), and a finite binding energy, in more than one spatial dimensions. With a cutoff included, the corresponding bound states have finite radii and binding energies, even without the stabilizing quartic term. Compared to other models of two-photon bound states in nonlinear optics [16], the parametric system has the advantage of higher binding energy and greater stability.

The paper is organized as follows. In Sec. II, we consider the Hamiltonian and discuss its general symmetry properties and possible eigenstates. In Sec. III, we show that there are exact solutions for the two-particle problem, which have the character of a superposition of either one particle of higher energy, or two particles of lower energy. The three-particle problem is considered in Sec. IV. There is no exact solution here, but the binding energy in the one-dimensional case can be estimated variationally. In Sec. V, we analyze the cutoff-dependent Hamiltonian which corresponds to a restricted range of relative momenta of the interacting fields, and present exact finite-size solutions. In Sec. VI, we present numerical estimates for the binding energies and radii of the solutions in the case of nonlinear optical parametric interaction, and show that effects treated here could result in observable binding energies and radii. Finally, we provide concluding remarks in Sec. VII.

II. HAMILTONIAN

The quantum effective Hamiltonian we consider has the following forms [3,12]:

$$H=H_0+H_{\text{int}}, \tag{1}$$

$$H_0=\hbar\int d^{(D)}\mathbf{x}\left[\frac{\hbar}{2m}|\boldsymbol{\nabla}\Phi|^2+\frac{\hbar}{2M}|\boldsymbol{\nabla}\Psi|^2+\rho\Psi^\dagger\Psi\right], \tag{2}$$

$$\begin{aligned}H_{\text{int}}=\hbar\int d^{(D)}\mathbf{x}\Bigg[&\frac{\chi_D}{2}(\Phi^2\Psi^\dagger+\Phi^{\dagger 2}\Psi)+\frac{\kappa_D}{2}\Phi^{\dagger 2}\Phi^2\\&+\eta_D|\Psi|^2|\Phi|^2+\frac{\sigma_D}{2}\Psi^{\dagger 2}\Psi^2\Bigg].\end{aligned} \tag{3}$$

Here Φ and Ψ are two complex Bose fields which we term subharmonic and second-harmonic fields, respectively, in analogy with the nonlinear optical process of frequency conversion. Their commutation relations are given by

$$\begin{aligned}[\Phi(\mathbf{x}),\Phi^\dagger(\mathbf{x}')]=[\Psi(\mathbf{x}),\Psi^\dagger(\mathbf{x}')]=\delta(\mathbf{x}-\mathbf{x}'),\\ [\Phi(\mathbf{x}),\Psi^\dagger(\mathbf{x}')]=[\Phi(\mathbf{x}),\Psi(\mathbf{x}')]=0.\end{aligned} \tag{4}$$

In addition, m and M are corresponding effective masses, and ρ is the phase mismatch, while χ_D and κ_D, η_D, σ_D are the coupling constants responsible for the parametric interaction (three-wave mixing or frequency conversion) and higher-order (quartic) interactions, respectively, in D ($D=1,2,3$) spatial dimensions.

To construct the general candidate for the eigenstate to our Hamiltonian we note that the parametric interaction here transforms pairs of subharmonic quanta into single second-harmonic quanta, and vice versa. That is, the Hamiltonian does not conserve corresponding particle numbers. However, it does conserve a generalized particle number, or Manley-Rowe invariant, equal to

$$N=N_\Phi+2N_\Psi=\int d^{(D)}\mathbf{x}[|\Phi|^2+2|\Psi|^2]. \tag{5}$$

In addition, since the Hamiltonian is translation invariant, it must have a momentum conservation law for $\mathbf{P}$, where

$$\mathbf{P}=-\frac{i\hbar}{2}\int d^{(D)}\mathbf{x}[\Phi^\dagger(\nabla\Phi)+\Psi^\dagger(\nabla\Psi)]+\text{H.c.} \tag{6}$$

We therefore search for states that are eigenstates of H, $\mathbf{P}$, and N. These must have the form of a superposition state:

$$|\varphi^{(N)}\rangle=\sum_{j=0}^{[N/2]}\int\cdots\int d^{(D)}\mathbf{x}_1\cdots d^{(D)}\mathbf{x}_{N-j}\times g_{N-j}^{(N)}(\mathbf{x}_1,\ldots,\mathbf{x}_{N-j})\,\exp\left(i\sum_{l=1}^{N-j}\frac{\mathbf{K}\cdot\mathbf{x}_l}{N-j}\right)\times\prod_{l=1}^{j}\Psi^{\dagger}(\mathbf{x}_l)\prod_{l=j+1}^{N-j}\Phi^{\dagger}(\mathbf{x}_l)|0\rangle, \tag{7}$$

where $\mathbf{K}=\mathbf{P}/\hbar$ is the total center-of-mass wave-vector, $[N/2]$ is the integer part of $N/2$, j denotes the number of Ψ field operators present in each term, and the function $g_{N-j}^{(N)}$ only depends on the relative coordinates. We note, however, that unless $M=2m$, the Hamiltonian is not Galilean invariant (under velocity boosts the Hamiltonian changes its form), and it is certainly not Lorenz invariant. In the case of nonlinear optical interactions (see Sec. VI), this is related to the fact that the group velocity for the two optical fields in the nonlinear medium will usually only match at one preferred pair of frequencies, called the group-velocity matching frequencies.

For simplicity, we focus on the nontrivial cases of two- and three-particle ($N=2$ and $N=3$) bound-state solutions in this paper, which we term bosonic ''mesons'' and bosonic ''hadrons,'' in analogy to the well-known quark model. It should be pointed out that in these particular cases the quartic self-interaction term ($\sim\sigma_D$) for the Ψ field in Eq. (3) has no effect on the solutions, since the two- and three-particle eigenstates have no more than one second-harmonic quanta. Therefore this term can simply be omitted through the rest of the paper. As to the quartic cross-interaction term ($\sim\eta_D$), it may only affect the three-particle results given in Sec. IV, and therefore will be omitted elsewhere. We also note that, at the classical level, the interplay between cubic and quartic interactions in nonlinear optical solitons was studied in Ref. [15]. To provide closer comparison between quantum and classical results, we would need to proceed with general multiparticle quantum solutions, which, however, are not studied here.

III. TWO-PARTICLE EIGENSTATES: BOSONIC MESONS

We first consider the two-particle problem. In this case we may rewrite the two-particle eigenstate candidate in the following explicit and symmetric form:

$$|\varphi^{(2)}\rangle=\left[\int d^{(D)}\mathbf{x}e^{i\mathbf{K}\cdot\mathbf{x}}\Psi^{\dagger}(\mathbf{x})+\int\int d^{(D)}\mathbf{x}_1 d^{(D)}\mathbf{x}_2\times g(\mathbf{x}_1-\mathbf{x}_2)e^{i\mathbf{K}\cdot(\mathbf{x}_1+\mathbf{x}_2)/2}\Phi^{\dagger}(\mathbf{x}_1)\Phi^{\dagger}(\mathbf{x}_2)\right]|0\rangle, \tag{8}$$

where $g(\mathbf{x}_1-\mathbf{x}_2)\equiv g_2^{(2)}(\mathbf{x}_1,\mathbf{x}_2)$ is the two-particle wave function, and $g_1^{(2)}(\mathbf{x}_1)=1$.

To prove a lower bound on the Hamiltonian energy, we apply Eq. (3) to the ansatz $|\varphi^{(2)}\rangle$ and use the symmetry property of the two-particle wave function: $g(\mathbf{x})=g(-\mathbf{x})$. Then neglecting the positive term $\sim(2\hbar^2/m)\int d^{(D)}\mathbf{x}|\nabla g(\mathbf{x})|^2$, appearing by integration by parts, we reduce the calculations to a chain of algebraic inequalities, and finally obtain that if $\kappa_D>0$ and

$$\hbar(\chi_D)^2>2\Delta\kappa_D, \tag{9}$$

where $\Delta\equiv-\hbar^2K^2/(4m)+\hbar^2K^2/(2M)+\hbar\rho$ and $K=|\mathbf{K}|$, then the lower bound E_l can be defined by

$$E_l=\frac{\hbar^2K^2}{2M}+\hbar\rho-\frac{\hbar(\chi_D)^2}{2\kappa_D}\leqslant\frac{\langle\varphi^{(2)}|H|\varphi^{(2)}\rangle}{\langle\varphi^{(2)}|\varphi^{(2)}\rangle}. \tag{10}$$

A. Variational analysis

To evaluate an upper bound to the lowest-energy eigenvalue of our Hamiltonian, we use a variational approach. In the one-dimensional case ($D=1$) we choose, following the structure of the known exact solution for the pure parametric interaction [17], a trial function $g(r)$ in the form

$$g(r)=g_0\exp(-|r|/R), \tag{11}$$

where $r=x_1-x_2$, and R can be regarded as a characteristic size parameter. Calculating the variational energy $\widetilde{E}=\langle\varphi^{(2)}|H|\varphi^{(2)}\rangle/\langle\varphi^{(2)}|\varphi^{(2)}\rangle$ gives

$$\widetilde{E}=\frac{\hbar^2K^2}{4m}+\frac{1}{1+2g_0^2R}\left[\frac{2\hbar^2}{m}\frac{g_0^2}{R}+2\hbar\kappa_1 g_0^2+2\hbar\chi_1 g_0+\Delta\right]. \tag{12}$$

We then minimize $\widetilde{E}$ with respect to the parameters g_0 and R. As a result we obtain that the variational energy $\widetilde{E}$, subject to localized bound state formation ($R>0$), is minimized at

$$g_0=-\frac{\chi_1}{2}\left[\kappa_1+\frac{2\hbar}{mR}\right]^{-1}, \tag{13}$$

and at the optimum R value determined from

$$\left[\Delta\kappa_1-\frac{\hbar(\chi_1)^2}{2}\right]R^3+\frac{2\hbar\Delta}{m}R^2+\frac{\hbar^2\kappa_1}{m}R+\frac{2\hbar^3}{m^2}=0. \tag{14}$$

Analysis of this cubic equation shows that there always exists one positive solution for R if condition (9) is met. The final result for the minimal value of $\widetilde{E}$, which corresponds to the exact eigenvalue E in this one-dimensional case, is

$$E=\frac{\hbar^2K^2}{4m}-\frac{\hbar^2}{mR^2}. \tag{15}$$

Thus a finite-size two-particle quantum soliton or a bosonic meson is shown to exist in our model in one dimension.

In the cases of two and three dimensions ($D=2$ and 3), we use the trial function

$$g(\mathbf{r})=g_0\exp[-(|\mathbf{r}|/R)^s], \tag{16}$$

where $\mathbf{r}=\mathbf{x}_1-\mathbf{x}_2$ and $s\geqslant0$. Calculating the variational energy $\widetilde{E}$ we obtain

$$\tilde{E}=\left[1+2^{2-D/2}\pi(D-1)\Gamma\left(\frac{D}{s}\right)g_0^2R^D\right]^{-1} \times\left[\frac{\hbar^2K^2}{2M}+\hbar\rho+2\hbar\chi_Dg_0+2\hbar\kappa_Dg_0^2 +\pi(D-1)\Gamma\left(\frac{D}{s}\right)\frac{\hbar^2K^2}{2^{D/s}m}g_0^2R^D+\pi(D-1) \times(D+s-2)\Gamma\left(1+\frac{D-2}{s}\right)\frac{\hbar^2}{2^{(D-2)/2}m}g_0^2R^{D-2}\right], \tag{17}$$

where $D=2$ and 3, and $\Gamma(z)$ is the gamma function. Analysis of this expression shows that $\tilde{E}$, as a function of parameters R, s, and g_0, approaches its minimal value in the limits $R^s\to 0$ and $s\to 0$, and at

$$g_0=-\chi_D/(2\kappa_D). \tag{18}$$

Again, condition (9) is assumed to be fulfilled to provide localized bound states $(R>0)$. The final result for $\tilde{E}_{\min}$ takes the form of the expression for E_l [see Eq. (10)]. This implies that the value of $\tilde{E}_{\min}$ evaluated by variational calculus represents the exact lowest-energy eigenvalue

$$E=\frac{\hbar^2K^2}{2M}+\hbar\rho-\frac{\hbar(\chi_D)^2}{2\kappa_D}. \tag{19}$$

Returning to the form of the trial function $g(\mathbf{r})$ at the optimum values of parameters R,s and g_0, we conclude that

$$g(\mathbf{r})=g_0 \quad \text{if} \quad \mathbf{r}=\mathbf{0},$$

$$g(\mathbf{r})=0 \quad \text{if} \quad \mathbf{r}\neq\mathbf{0}, \tag{20}$$

i.e., the two-particle solutions in two and three dimensions have a singular pointlike structure, with a finite energy.

B. Exact solutions

To understand in more detail how these singular solutions appear in our model, we now analyze our eigenvalue problem $H|\varphi^{(2)}\rangle=E|\varphi^{(2)}\rangle$ directly. Applying our Hamiltonian to the ansatz $|\varphi^{(2)}\rangle$, one can obtain that the eigenvalue problem is equivalent to the following simultaneous equations:

$$\frac{1}{m}\nabla^2g(\mathbf{r})-\mu^2g(\mathbf{r})=q\,\delta(\mathbf{r}), \tag{21}$$

$$E=\frac{\hbar^2K^2}{2M}+\hbar\rho+\hbar\chi_Dg(0)=\frac{\hbar^2K^2}{4m}-\hbar^2\mu^2, \tag{22}$$

where $\mathbf{r}=\mathbf{x}_1-\mathbf{x}_2, K=|\mathbf{K}|$, and we have defined

$$\mu^2=\frac{K^2}{4m}-\frac{E}{\hbar^2} \tag{23}$$

and

$$q=\frac{1}{\hbar}\left[\frac{\chi_D}{2}+\kappa_Dg(0)\right]. \tag{24}$$

Here $\hbar^2\mu^2$ can be interpreted as the binding energy of the solution with momentum $\mathbf{K}$, and $\mu\sqrt{m}$ is an inverse scale length; the solution is bound (against two particle decay) if μ is real and positive.

Equations (21) and (22) can be easily analyzed using the Fourier transform method. In this approach we seek for a solution to Eq. (21) in the form

$$g(r)=\frac{1}{(2\pi)^D}\int d^{(D)}\mathbf{k}\;G(\mathbf{k})\exp(i\mathbf{k}\cdot\mathbf{r}), \tag{25}$$

where $r=|\mathbf{r}|$.

Expanding the δ function into a Fourier integral, we then find

$$g(r)=-\frac{q}{(2\pi)^D}\int d^{(D)}\mathbf{k}\frac{\exp(i\mathbf{k}\cdot\mathbf{r})}{\mu^2+k^2/m}, \tag{26}$$

where $k=|\mathbf{k}|$.

In the one-dimensional case $(D=1)$ the integration gives

$$g(r)=-\frac{q}{2\pi}\int_{-\infty}^{+\infty}dk\,\frac{\exp(ikr)}{\mu^2+k^2/m}=-\frac{q\sqrt{m}}{2\mu}\exp(-\mu\sqrt{m}|r|). \tag{27}$$

Using this result at $r=0$ and the definition of q, we solve for $g(0)$ and find

$$g(0)=-\frac{\chi_1}{2}\left[\kappa_1+\frac{2\hbar\mu}{\sqrt{m}}\right]^{-1}. \tag{28}$$

Correspondingly, the energy eigenvalue then becomes

$$E=\frac{\hbar^2K^2}{2M}+\hbar\rho-\frac{\hbar(\chi_1)^2}{2}\left[\kappa_1+\frac{2\hbar\mu}{\sqrt{m}}\right]^{-1}=\frac{\hbar^2K^2}{4m}-\hbar^2\mu^2. \tag{29}$$

Here μ must be positive for a localized bound state. We note that if we introduce the characteristic size parameter $R=1/(\mu\sqrt{m})$, this equation with respect to R can be rewritten in the form of the cubic equation (14). Hence, if condition (9) is met, there always exists one positive solution for μ. This proves the existence of a one-dimensional bosonic meson of a finite size.

The two- and three-dimensional results are qualitatively different. In these cases we obtain, from Eq. (26),

$$g(0)=-\frac{q}{2\pi^{D-1}}\int_0^\infty dk\,\frac{k^{D-1}}{\mu^2+k^2/m}, \quad (D=2,3), \tag{30}$$

where we have transformed to polar (for $D=2$) and spherical (for $D=3$) coordinates. Using the definition of q we next solve for $g(0)$, and obtain

$$g(0)=-\frac{\chi_D}{2}\left[\kappa_D+\frac{\hbar}{f_D(\mu)}\right]^{-1}, \tag{31}$$

where we have defined

$$f_D(\mu)=\frac{1}{2\pi^{D-1}}\int_0^\infty dk\,\frac{k^{D-1}}{\mu^2+k^2/m}. \tag{32}$$

This integral diverges for $D=2$ and 3. Hence we find that $g(0)=-\chi_D/(2\kappa_D)$, and the energy E is given by Eq. (19). With this result for $g(0)$ it follows also that $q=0$, and hence [see Eq. (26)] $g(r)=0$ if $r\neq 0$. That is, the exact solution coincides with the variational result and confirms that the bound states in two and three dimensions have singular (zero-radius) structure.

Thus the results of this section show that our model Hamiltonian provides quantum solitons or two-particle eigenstates in one and more spatial dimensions. They are superpositions of a second harmonic and two subharmonic quanta. We may regard them as the bosonic analog of two-quark states in the well-known quark model of mesons: the sub-harmonic quanta behave like (bosonic) quarks, while the second-harmonic quanta take a role of gluons.

An important difference between the one dimensional and multidimensional solutions is in their structure and dependence on the additional quartic interaction. In one dimension the bound state has a finite characteristic size, and is available even without a quartic term in the Hamiltonian. In two and three dimensions the bound states have a singular point-like structure. The corresponding binding energy is finite if and only if $\kappa_D>0$. If, however, $\kappa_D=0$ we obtain [see Eq. (19)] an energy collapse: $E\rightarrow-\infty$. Thus, while the additional quartic interaction prevents an energy collapse and makes the multidimensional quantum solitons available in this simple model, it does not prevent singularities in space, unless a momentum cutoff is introduced into the Hamiltonian (see Sec. V).

IV. THREE-PARTICLE PROBLEM: BOSONIC HADRONS

Now let us turn to the three-particle ($N=3$) problem, assuming that $\eta_D\geqslant 0$. In this case a lower bound to our Hamiltonian energy can be proved by considering the reduced Hamiltonian

$$\begin{aligned}H_R&=H-\hbar\int d^{(D)}\mathbf{x}\left[\frac{\hbar}{2m}|\nabla\Phi|^2+\frac{\hbar}{2M}|\boldsymbol{\nabla}\Psi|^2\right]\\&\quad-\hbar\,\eta_D\int d^{(D)}\mathbf{x}|\Psi|^2|\Phi|^2\\&=\hbar\int d^{(D)}\mathbf{x}\left[\rho\Psi^\dagger\Psi+\frac{\chi_D}{2}(\Phi^2\Psi^\dagger+\Phi^{\dagger 2}\Psi)\right.\\&\quad\left.+\frac{\kappa_D}{2}\Phi^{\dagger 2}\Phi^2\right]\leqslant H.\end{aligned} \tag{33}$$

This can be easily applied to the ansatz $|\varphi^{(3)}\rangle$ [Eq. (7)], and if $\kappa_D>0$ and $\hbar\rho-\hbar(\chi_D)^2/(2\kappa_D)<0$ the lower bound can be defined by

$$E_l^{(3)}=\hbar\rho-\frac{\hbar(\chi_D)^2}{2\kappa_D}\leqslant\frac{\langle\varphi^{(3)}|H|\varphi^{(3)}\rangle}{\langle\varphi^{(3)}|\varphi^{(3)}\rangle}, \tag{34}$$

where we label the energies by upper indices to differentiate the two- and three-particle results.

Now we compare $E_l^{(3)}$ with the lowest possible energy eigenvalue of the two-particle problem in two and three dimensions [Eq. (19)]. The latter is realized at $K=0$, and we see that $E_l^{(3)}=E^{(2)}(K=0)$. This implies that the extra subharmonic quantum does not contribute to three-particle binding and remains free. Hence our model does not provide three-particle bound states in two and three dimensions. This conclusion may be modified if there is a cutoff in the relative momenta of interacting fields, but we do not treat this case here.

The situation is different, however, in the one-dimensional case. In this case $E_l^{(3)}<E^{(2)}(K=0)$, and hence one may expect that the true lowest eigenvalue $E^{(3)}$ will satisfy $E_l^{(3)}\leqslant E^{(3)}<E^{(2)}(K=0)$. This will imply the existence of a three-particle eigenstate or a bosonic hadron in one dimension. Alternatively, a one-dimensional hadron can be proved to exist by variational calculus. In this approach we evaluate a variational energy $\tilde{E}^{(3)}$ such that its minimal value $\tilde{E}^{(3)}_{\min}$ — an upper bound to the lowest energy eigenvalue $E^{(3)}$ — will satisfy $E^{(3)}\leqslant\tilde{E}^{(3)}_{\min}<E^{(2)}(K=0)$. To check the inequality $\tilde{E}^{(3)}_{\min}<E^{(2)}(K=0)$, we choose trial functions $g_2^{(3)}(x_1,x_2)$ and $g_3^{(3)}(x_1,x_2,x_3)$ in the eigenstate $|\varphi^{(3)}\rangle$ in the following forms:

$$g_2^{(3)}(x_1,x_2)=\exp(-|x_1-x_2|/l_1), \tag{35}$$

$$g_3^{(3)}(x_1,x_2,x_3)=g_0\exp(-|x_1-x_2|/l_2-|x_2-x_3|/l_3-|x_1-x_3|/l_3). \tag{36}$$

It is important to notice that this choice of the trial functions for the three-particle eigenstate incorporates the two-particle bound-state problem (with $K=0$) as a limiting case of $l_3/l_2\rightarrow\infty$ and $l_3/l_1\rightarrow 2$. In other words, the structure of the three-particle trial functions allows for the situation when the two particles form a bound state while the extra (third) subharmonic quantum remains free, at a large distance. In this case the resulting energy must be equal to the energy for the two-particle bound state. If, however, the extra subharmonic quantum participates in three-particle binding, then the resulting energy must be lower. Correspondingly, the variational energy $\tilde{E}^{(3)}$ should be minimized at values of parameters l_1,l_2, and l_3 different from the above mentioned limits.

Calculating the variational energy $\tilde{E}^{(3)} = \langle \varphi^{(3)} | H | \varphi^{(3)} \rangle / \langle \varphi^{(3)} | \varphi^{(3)} \rangle$, with use of the trial functions (35) and (36), we obtain

$$\tilde{E}^{(3)} = \frac{1}{\mathcal{N}} (\mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3 + \mathcal{E}_4), \tag{37}$$

$$\mathcal{E}_1 = \frac{\hbar^2 (m+M)}{2mMl_1} + \hbar \rho l_1, \tag{38}$$

$$\mathcal{E}_2 = \frac{\hbar^2 g_0^2}{m} \left[\frac{l_3}{l_2} + \frac{4l_2(23l_2^2 + 30l_2l_3 + 11l_3^2)}{9l_2^3 + 15l_2^2l_3 + 7l_2l_3^2 + l_3^3} \right], \tag{39}$$

$$\mathcal{E}_3 = 4\hbar \chi_1 g_0 \left[\frac{l_1 l_3}{2l_1 + l_3} + \frac{2l_1l_2l_3}{l_1l_2 + l_1l_3 + l_2l_3} \right], \tag{40}$$

$$\mathcal{E}_4 = 2\hbar \kappa_1 g_0^2 l_3 \left[\frac{9l_2 + l_3}{2(l_2 + l_3)} + \frac{8l_2}{3l_2 + l_3} \right] + \hbar \eta_1, \tag{41}$$

$$\mathcal{N} = l_1 + 2g_0^2 \frac{l_2 l_3^2 (2l_2 + l_3)}{(l_2 + l_3)} \left[\frac{1}{2(l_2 + l_3)} + \frac{8l_2}{(3l_2 + l_3)^2} \right]. \tag{42}$$

It is easy to check that if we take the limits $l_3/l_2 \rightarrow \infty$ and $l_3/l_1 \rightarrow 2$ and then optimize $\tilde{E}^{(3)}$ with respect to the remaining two free parameters l_2 and g_0, we will reproduce (with l_2 being replaced by *R)* the variational results obtained for the two-particle problem, [Eqs. (12) and (15)] for $K=0$.

In the general case of treating l_1, l_2, l_3, and g_0 as free variational parameters we minimize $\tilde{E}^{(3)}$ numerically. In the case of zero quartic couplings ($\kappa_1 = 0$, $\eta_1 = 0$) we find that $\tilde{E}^{(3)}$ is minimized at some finite values of $l_{1,2,3}$ and g_0, and that the above-mentioned limit of two bound particles plus a third free particle ($l_3/l_2 \rightarrow \infty$ and $l_3/l_1 \rightarrow 2$) is not the optimum. Correspondingly, the $\tilde{E}^{(3)}_{\rm min}$ value turns out to be less than the two-particle energy eigenvalue $E^{(2)}(K=0)$, implying three-particle binding.

Inclusion of the quartic couplings, which are assumed to be positive here, will obviously increase the $\tilde{E}^{(3)}_{\rm min}$ value (decrease its absolute value or the binding energy). The results of our numerical analysis for the cases $\eta_1 = 2\kappa_1$ and $\rho=0$ are represented in Fig. 1, where we plot $\tilde{E}^{(3)}_{\rm min}$ versus κ_1 For comparison, the curve for $E^{(2)}(K=0)$ is also plotted. The results are given for the choice of the relevant parameters as applied to the case of optical interactions in a nonlinear material (see Sec. VI), where χ_1 and κ_1 are proportional to the second- and third-order nonlinear susceptibilities ($\chi^{(2)}$ and $\chi^{(3)}$), respectively. With a characteristic value of $\chi_1 = 7.39 \times 10^7 \sqrt{\rm m}$/s, $M/m=2$, and $\hbar/m = 0.1$ m^2/s, our analysis shows that $\tilde{E}^{(3)}_{\rm min} < E^{(2)}$ over a wide range of κ_1 values, and correspondingly the optimum values of l_1, l_2, l_3, and g_0 are different from the limit of forming a two-particle soliton plus a free third particle. Approaching this limit of $\tilde{E}^{(3)}_{\rm min} \rightarrow E^{(2)}$ occurs through developing a (second) local minimum (at $l_2/l_3 \rightarrow 0$ and $l_3/l_1 \rightarrow 2$), which becomes the absolute minimum at large values of κ_1 ($\kappa_1 \sim 4 \times 10^4$ m/s in the case of Fig. 1). We note, however, that realistic values of cubic non-

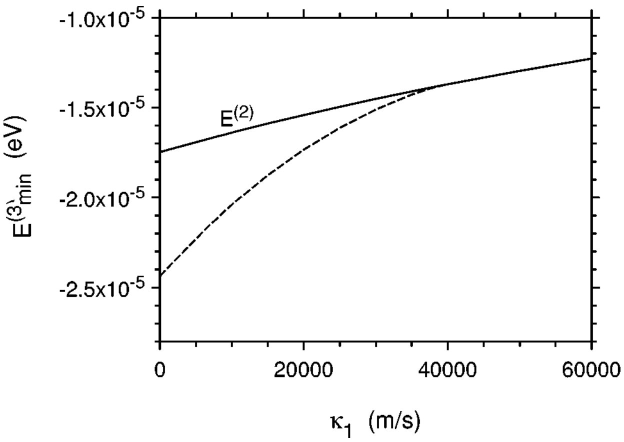

FIG. 1. Three-particle variational energy $\tilde{E}^{(3)}_{\rm min}$ (broken line) vs κ_1 ($\eta_1 = 2\kappa_1$), for ρ=0, $M/m=2$, $\chi_1 = 7.39 \times 10^7 \sqrt{\rm m}$/s, and $\hbar/m = 0.1$ m^2/s. The full line represents the corresponding result for the two-particle energy eigenvalue $E^{(2)}$.

linearities, such as for silica optical fibers, can only give $\kappa_1 \sim 10^{-6} - 10^{-5}$m/s. With these realistic values of κ_1, the $\tilde{E}^{(3)}_{\rm min}$ value and optimum characteristic size parameters are: $\tilde{E}^{(3)}_{\rm min} \simeq -2.4 \times 10^{-5}$ eV, $l_1 \simeq 4.2$ μm, $l_2 \simeq 1.8$ μm, and $l_3 \simeq 9.4$ μm. For comparison, the exact energy eigenvalue $E^{(2)}$ and the effective radius R in the two-particle problem in one dimension, with the same values of parameters, are $E^{(2)} \simeq -1.75 \times 10^{-5}$ eV and $R \simeq 1.94$ μm.

Thus the results of this section prove the existence of a three-particle bound state or a bosonic hadron solution in our model in one dimension, provided that the quartic couplings are not extremely strong. An obvious difference of this three-particle soliton, compared to other known solutions [such as in the ordinary nonlinear Schrödinger model (NLS) [2] or a perturbed NLS model [18]], is in its structure which represents a superposition or entangled state, involving two different interacting fields.

As applied to the nonlinear optical case, there is also an advantage of higher binding energies and accessible radii of the parametric quantum solitons. This is due to the stronger $\chi^{(2)}$ nonlinear effects in parametric media, compared to $\chi^{(3)}$ effects in nonlinear optical fibers. For example, in the ordinary nonlinear Schrödinger model the three-particle bound-state solution [2], available with an attractive δ-function interaction (corresponding to a self-focusing nonlinear optical material, with $\kappa_1 < 0$ in our notations), has the following energy eigenvalue (in the case of zero momentum) and characteristic radius:

$$E^{(N=3)}_{\rm NLS} = -m(\kappa_1)^2 \frac{N(N^2-1)}{24} \bigg|_{N=3} = -m(\kappa_1)^2, \tag{43}$$

$$R_{\rm NLS} \sim \frac{2\hbar}{m|\kappa_1|}. \tag{44}$$

With the choice of $\hbar/m = 0.1$ m^2/s and a silica fiber characteristic value of $\kappa_1 \sim 5 \times 10^{-6}$ m/s, this results in $E^{(3)}_{\rm NLS} \simeq -1.6 \times 10^{-25}$ eV and $R_{\rm NLS} \sim 40$ km. Comparison of these values with the earlier estimates for the optical parametric interaction clearly illustrates the advantages of the parametric quantum solitons to be accessible.

V. CUTOFF DEPENDENT RESULTS

Now let us turn to the analysis of the singular behavior of the two-particle quantum solitons in two and three spatial dimensions. We note that these singularities represent a rather unusual situation, since the classical counterpart of the theory has well-behaved and widely used multidimensional nonlinear-optical soliton solutions. Since we expect the quantum theory to be correct, this leads to the ''paradox'' of how a singular quantum field theory can describe real physical processes. We note that a related paradox is known in the theory of a Bose gas with a repulsive δ-function interaction [19], which is commonly used to model Bose-Einstein condensation. In the atomic interaction case, either a momentum cutoff or other regularization procedures [20] are needed to provide a physical interpretation for the three-dimensional δ-function interaction potential.

Provided the cutoff is chosen correctly, it should not be necessary to renormalize the values of the observed nonlinear parameters, if optically measured nonlinearities are involved. This is because nonlinear optical parameters are operationally measurable under different conditions to those encountered in measuring atomic cross sections. It is optically possible to measure the nonlinearities in strong coherent fields, and also to operate under different types of velocity and phase-matching conditions to those assumed here.

To resolve the paradox in the optical parametric interaction case, we note that the origins of the theory involve the rotating-wave and paraxial approximations, and neglect the higher-order dispersion. Therefore, in higher dimensions we should include, for example, nonparaxial diffraction if the characteristic size of the solutions becomes less than the field carrier wavelengths. Alternatively, we may modify our interaction Hamiltonian in a way that will not result in singular structures, consistent with the paraxial approximation.

A possible way is to incorporate the fact that parametric couplings of the type found in Eq. (3) are usually restricted to a finite range of relative momenta or wave numbers. To represent this we can introduce a cutoff at $|\mathbf{k}|=k_{\max}$ in the relative momenta of the interacting fields. We choose $k_{\max} \sim 2\pi/\lambda_1$, where λ_1 is the carrier wavelength of the subharmonic field Φ. The interaction part of the Hamiltonian (3) can then be expressed in terms of $a(\mathbf{k})$ and $b(\mathbf{k})$, the Fourier component of Φ and Ψ, so that its cutoff dependence is implemented through the limits of the corresponding integrals:

$$H_{\text{int}}=\frac{(2\pi)^D\hbar}{2}\int d^{(D)}\mathbf{K}\left\{\chi_D\int_{|\mathbf{k}|=0}^{k_{\max}} d^{(D)}\mathbf{k}[a^\dagger(\mathbf{k}_+)a^\dagger(\mathbf{k}_-)b(\mathbf{K})+\text{H.c.}]\right.$$
$$\left.+\kappa_D\int\int_{|\mathbf{k}_{1,2}|=0}^{k_{\max}} d^{(D)}\mathbf{k}_1 d^{(D)}\mathbf{k}_2 a^\dagger(\mathbf{k}_{1+})a^\dagger(\mathbf{k}_{1-})a(\mathbf{k}_{2+})a(\mathbf{k}_{2-})\right\}. \tag{45}$$

Here $\mathbf{k}_{(i)\pm}\equiv\mathbf{K}/2\pm\mathbf{k}_{(i)}$, the commutation relations for $a(\mathbf{k})$ and $b(\mathbf{k})$ are given by $[a(\mathbf{k}),a^\dagger(\mathbf{k}')]=[b(\mathbf{k}),b^\dagger(\mathbf{k}')]=\delta(\mathbf{k}-\mathbf{k}')/(2\pi)^D$, and other quartic terms have been omitted since they have no effect on the two-particle solutions.

We can now analyze the eigenvalue problem $H|\varphi^{(2)}\rangle=E|\varphi^{(2)}\rangle$ directly, in Fourier space, by introducing a cutoff-dependent Fourier transform of $g(\mathbf{r})$, so that

$$g(\mathbf{r})=\frac{1}{(2\pi)^D}\int_{|\mathbf{k}|=0}^{k_{\max}} d^{(D)}\mathbf{k}G(\mathbf{k})\exp(i\mathbf{k}\cdot\mathbf{r}), \tag{46}$$

This implies that, due to the cutoff in the nonlinearities, we need only investigate eigenstates for which $G(\mathbf{k})$ vanishes if $|\mathbf{k}|>k_{\max}$. This leads the following simultaneous equations for an eigenstate:

$$\left(\frac{k^2}{m}+\mu^2\right)G(\mathbf{k})=-q, \tag{47}$$

$$E=\frac{\hbar^2K^2}{2M}+\hbar\rho+\hbar\chi_D g(0)=\frac{\hbar^2K^2}{4m}-\hbar^2\mu^2, \tag{48}$$

where $k=|\mathbf{k}|$, and we have used the same notations for μ and q as in Eqs. (23) and (24). The above equations are Fourier transform equivalents of Eqs. (21) and (22), except that now they are valid for $|\mathbf{k}|<k_{\max}$.

In order to evaluate the binding energy and the effective radius, we must next solve for $g(0)$. After a little algebra, we find

$$g(0)=-\frac{\chi_D}{2}\left[\kappa_D+\frac{\hbar}{f_D(\mu,k_{\max})}\right]^{-1}, \tag{49}$$

where the cutoff structure functions

$$f_D(\mu,k_{\max})=\frac{1}{2^{[D/2]}\pi^{[(D+1)/2]}}\int_0^{k_{\max}} dk\,\frac{k^{D-1}}{\mu^2+k^2/m} \qquad (D=1,2,3)$$

(with $[x/2]$ being the integer part of $x/2$) are given by

$$f_1(\mu,k_{\max})=\frac{\sqrt{m}}{\pi\mu}\tan^{-1}\left(\frac{k_{\max}}{\mu\sqrt{m}}\right),$$

$$f_2(\mu,k_{\max})=\frac{m}{4\pi}\ln\left(1+\frac{k_{\max}^2}{\mu^2 m}\right), \tag{50}$$

$$f_3(\mu,k_{\max})=\frac{m}{2\pi^2}\left[k_{\max}-\mu\sqrt{m}\tan^{-1}\left(\frac{k_{\max}}{\mu\sqrt{m}}\right)\right].$$

This result clearly shows the difference caused by the dimensionality of the space. In one dimension, $f_1(\mu,k_{\max})$ approaches a constant value if $k_{\max}\to\infty$, while in two and three dimensions $f_D(\mu,k_{\max})$ has a logarithmic or linear divergence, respectively.

The effect of this divergence depends on whether or not the additional quartic interaction is present. If it is present (with $\kappa_D>0$), there are exact solutions without cutoff, and $g(0)=-\chi_D/(2\kappa_D)$, so that the energy eigenvalue E takes the form of Eq. (19), and $g(\mathbf{r})=0$ if $|\mathbf{r}|>0$. In other words, the solutions in two and three dimensions have a finite energy (unlike the energy divergence in the nonlinear Schrödinger model with an attractive δ-function potential) but zero radius in the limit of $k_{\max}\to\infty$. If, however, $\kappa_D=0$ (or is negative, as in the case of attractive nonlinear Schrödinger model), we must impose a finite cutoff on the couplings to prevent an energy divergence. Simultaneously, a finite cutoff prevents singularities in space.

With a finite cutoff, the general result for the energy eigenvalue E is given by

$$\begin{aligned}E&=\frac{\hbar^2K^2}{2M}+\hbar\rho-\frac{\hbar(\chi_D)^2}{2}\left[\kappa_D+\frac{\hbar}{f_D(\mu,k_{\max})}\right]^{-1}\\&=\frac{\hbar^2K^2}{4m}-\hbar^2\mu^2,\end{aligned} \tag{51}$$

where μ must be positive for a localized bound state. Analysis of this equation with respect to μ shows that under a certain condition a positive solution for μ is always available. This condition, in the cases of one and two dimensions, can be written in the form of Eq. (9), while in the three-dimensional case it is modified to

$$\hbar(\chi_3)^2>2\Delta\left(\kappa_3+\frac{2\pi^2\hbar}{mk_{\max}}\right). \tag{52}$$

In the simplest case of $\kappa_D=0$ and $\Delta=0$, and in the limit $k_{\max}\gg\mu\sqrt{m}$ one can write simple approximate results for the binding energies $E_D^b=\hbar^2\mu^2$ in one, two, and three dimensions ($D=1,2,3$):

$$E_1^b=\hbar^2\mu^2\simeq\frac{(\chi_1)^2\sqrt{m}}{4\mu}, \tag{53}$$

$$E_2^b=\hbar^2\mu^2\simeq\frac{(\chi_2)^2 m}{4\pi}\ln\left(\frac{k_{\max}}{\mu\sqrt{m}}\right), \tag{54}$$

$$E_3^b=\hbar^2\mu^2\simeq\frac{(\chi_3)^2 m k_{\max}}{4\pi^2}. \tag{55}$$

The effective radii of the solitons are defined as $R_D=1/(\mu\sqrt{m})$, since this define the characteristic distance over which the two-particle wave function can decay.

VI. APPLICATION TO NONLINEAR OPTICAL INTERACTION

As an application of our results (in the two-particle case) to a realistic physical system, we consider the nonlinear optical process of frequency conversion (second-harmonic generation). In this case the actual Hamiltonian H_0 is asymmetric with respect to longitudinal and transverse coordinates, and is given by [10,11]

$$\begin{aligned}H_0=\int d^{(D)}\mathbf{x}\Bigg[&\left(\frac{\hbar^2}{2m_\parallel}|\nabla_\parallel\Phi|^2+\frac{\hbar^2}{2m_\perp}|\nabla_\perp\Phi|^2\right)\\&+\left(\frac{\hbar^2}{2M_\parallel}|\nabla_\parallel\Psi|^2+\frac{\hbar^2}{2M_\perp}|\nabla_\perp\Psi|^2\right)+\hbar\rho\Psi^\dagger\Psi\Bigg].\end{aligned} \tag{56}$$

Here Φ and Ψ represent two optical fields (subharmonic and second harmonic) with carrier wave numbers k_1 and $k_2=2k_1$. The corresponding frequencies are $\omega_i=\omega(k_i)$, $i=1,2$. The quantity ρ is now identified as a phase mismatch term, given by $\rho=\omega_2-2\omega_1$. We choose, for definiteness, the x axis in the direction of propagation, so that the coordinate x is defined here in a moving frame with $x=x_L-vt$, where x_L is the laboratory frame coordinate and $v=\partial\omega_i/\partial k$ is the group velocity which is assumed equal at both frequencies. The transverse coordinates are y and (y,z) in two and three dimensions, respectively, so that $\nabla_\parallel$ is defined as $\nabla_\parallel=\partial/\partial x$, while $\nabla_\perp$ is given by $\nabla_\perp=\partial/\partial y$ in two dimensions, or has the vector components $(\partial/\partial y,\partial/\partial z)$ in three dimensions. The effective longitudinal masses $m_\parallel=\hbar/\omega_1''$ and $M_\parallel=\hbar/\omega_2''$ are caused by the group-velocity dispersion, where $\omega_i''=\partial^2\omega_i/\partial k^2$ is the dispersion coefficient in the ith frequency band. The lower-frequency dispersion coefficient ω_1'' is assumed to have a positive value. The transverse masses $m_\perp=\hbar\omega_1/v^2$ and $M_\perp=\hbar\omega_2/v^2$ are caused by diffraction, and the corresponding term in H_0 is relevant only in the case of two and three ($D=2,3$) spatial dimensions. The coupling constants χ_D and κ_D are proportional to the second- and third-order nonlinear susceptibilities ($\chi^{(2)}$ and $\chi^{(3)}$) of the nonlinear medium, respectively.

As we can see this modification in the Hamiltonian does not affect the one-dimensional results of the previous sections, with the effective masses m and M being interpreted as the dispersive ones, $m\equiv m_\parallel$ and $M\equiv M_\parallel$. However, it does affect the two- and three-dimensional results, so that they need to be slightly modified.

The modifications are not of qualitative character, and the final form of solutions obtained as in Sec. III B for the two-particle bound states can be reproduced by some formal replacements. In particular, the relations K^2/m and k^2/m must be replaced by

$$\frac{K^2}{m}\to\frac{K_\parallel^2}{m_\parallel}+\frac{K_\perp^2}{m_\perp},\quad \frac{k^2}{m}\to\frac{k_\parallel^2}{m_\parallel}+\frac{k_\perp^2}{m_\perp}, \tag{57}$$

and similarly for the case of mass M instead of m. Equations (21) and (22) are now rewritten in the following forms:

$$\left(\frac{1}{m_\parallel}\nabla_\parallel^2+\frac{1}{m_\perp}\nabla_\perp^2\right)g(\mathbf{r})-\mu^2 g(\mathbf{r})=q\,\delta(\mathbf{r}), \tag{58}$$

$$E=\frac{\hbar^2}{2}\left(\frac{K_\parallel^2}{M_\parallel}+\frac{K_\perp^2}{M_\perp}\right)+\hbar\rho+\hbar\chi_D g(0)$$
$$=\frac{\hbar^2}{4}\left(\frac{K_\parallel^2}{m_\parallel}+\frac{K_\perp^2}{m_\perp}\right)-\hbar^2\mu^2, \tag{59}$$

where q is given by Eq. (24), and μ^2 is now defined as

$$\mu^2\equiv\frac{1}{4}\left(\frac{K_\parallel^2}{m_\parallel}+\frac{K_\perp^2}{m_\perp}\right)-\frac{E}{\hbar^2}, \tag{60}$$

so that, as in Sec. III B, we can again arrive at the same conclusions on the pointlike structure of the multidimensional bound-state solutions without cutoffs.

The cutoff dependent results of Sec. V are modified in the following way. The form of Eq. (51) (with the abovementioned replacements), representing our main result, remains unchanged. That is, the energy eigenvalue E is given by

$$E=\frac{\hbar^2}{2}\left(\frac{K_\parallel^2}{M_\parallel}+\frac{K_\perp^2}{M_\perp}\right)+\hbar\rho-\frac{\hbar(\chi_D)^2}{2}\left[\kappa_D+\frac{\hbar}{f_D(\mu,k_{\max})}\right]^{-1}$$
$$=\frac{\hbar^2}{4}\left(\frac{K_\parallel^2}{m_\parallel}+\frac{K_\perp^2}{m_\perp}\right)-\hbar^2\mu^2. \tag{61}$$

The only relevant quantitative effect of adopting the asymmetric form of the Hamiltonian H_0 is related to the cutoff structure function $f_D(\mu,k_{\max})$, $D=2$ and 3. Now it becomes dependent on the two masses $m_\parallel$ and $m_\perp$, and is defined as

$$f_D(\mu,k_{\max})=\frac{1}{(2\pi)^D}\int_{|\mathbf{k}|=0}^{k_{\max}}d^{(D)}\mathbf{k}\frac{1}{\mu^2+k_\parallel^2/m_\parallel+k_\perp^2/m_\perp}. \tag{62}$$

We see that the integrations here cannot be carried out as easily as in the symmetric case of Sec. V in polar ($D=2$) or spherical ($D=3$) coordinates. Instead, in the case of arbitrary values of $m_\parallel$ and $m_\perp$, the integrals and resulting binding energies can be evaluated numerically. If, however, $\sqrt{m_\perp/m_\parallel}\ll1$ and $k_{\max}/(\mu\sqrt{m_\parallel})\gg1$ we can obtain approximate expressions for the cutoff structure functions:

$$f_2(\mu,k_{\max})\simeq\frac{\sqrt{m_\parallel m_\perp}}{2\pi}\ln\left(\frac{2k_{\max}}{\mu\sqrt{m_\parallel}}\right), \tag{63}$$

$$f_3(\mu,k_{\max})\simeq\frac{m_\perp k_{\max}}{2\pi^2}\left(1-\ln\sqrt{\frac{m_\perp}{m_\parallel}}\right). \tag{64}$$

With these functions the condition of having a positive solution for μ [such that $k_{\max}/(\mu\sqrt{m_\parallel})\gg1$] in Eq. (61) remains unchanged [i.e., in the form of Eq. (9)] in two dimensions, while in three dimensions it becomes

$$\hbar(\chi_3)^2>2\Delta\left(\kappa_3+\frac{2\pi^2\hbar}{m_\perp k_{\max}(1-\ln\sqrt{m_\perp/m_\parallel})}\right), \tag{65}$$

where the above replacements [Eq. (57)], are included in the definition of Δ.

We notice that the phase mismatch ρ has a strong effect on the solutions, changing both the characteristic radius and the binding energy. In the three-dimensional case, if the quartic interaction term κ_3 is absent, then Eq. (65) implies that ρ cannot have a large positive value. On the other hand, if ρ is large and negative then the effective radius is very small. Thus it is optimal to choose $\rho\leqslant0$, although in one and two dimensions this does not appear essential if $\kappa_D=0$ [see Eq. (9)].

In the cases $\kappa_D=0$ and $\Delta=0$, Eqs. (61) and (63) and (64) lead to the following simple results for the soliton binding energies in two and three dimensions:

$$E_2^b=\hbar^2\mu^2\simeq\frac{(\chi_2)^2\sqrt{m_\parallel m_\perp}}{4\pi}\ln\left(\frac{2k_{\max}}{\mu\sqrt{m_\parallel}}\right), \tag{66}$$

$$E_3^b=\hbar^2\mu^2\simeq\frac{(\chi_3)^2 m_\perp k_{\max}}{4\pi^2}\left(1-\ln\sqrt{\frac{m_\perp}{m_\parallel}}\right). \tag{67}$$

The binding energy $E_1^b=\hbar^2\mu^2$ and the corresponding μ value in the one-dimensional case is determined from Eq. (29), with $m\equiv m_\parallel$. The effective radius in one dimension is defined as $R_1=1/(\mu\sqrt{m_\parallel})$, while in two and three dimensions we should introduce two characteristic size parameters [scaled as $1/(\mu\sqrt{m_\parallel})$ and $1/(\mu\sqrt{m_\perp})$] corresponding to the longitudinal and transverse directions. It is clear that once the transverse (longitudinal) size is evaluated and the ratio of the effective masses $m_\perp/m_\parallel$ is specified, then the longitudinal (transverse) size can be obtained as well. In the case $\sqrt{m_\perp/m_\parallel}\ll1$ considered here, the transverse size is larger than the longitudinal one, and our numerical estimates in two and three dimensions will be given for the transverse characteristic radii defined as $R_{2,3}=1/(\mu\sqrt{m_\perp})$.

To give numerical estimates for the binding energies $E_D^b=\hbar^2\mu^2$ and radii $R_{1(2,3)}=1/(\mu\sqrt{m_{\parallel(\perp)}})$, we note that the nonlinear couplings χ_D and κ_D ($D=1,2,3$) are defined here as [11,17]

$$\chi_D\simeq\frac{\chi_B^{(2)}\omega_1}{n^3}\left(\frac{\hbar\omega_2}{2\varepsilon_0}\right)^{1/2}\frac{1}{d^{(3-D)/2}}, \tag{68}$$

$$\kappa_D\simeq\frac{3\hbar\chi_B^{(3)}\omega_1^2 v^2}{4\varepsilon c^2 d^{3-D}}=\frac{\hbar n_2\omega_1^2 v^2}{c d^{3-D}}, \tag{69}$$

where $\chi_B^{(2)}$ and $\chi_B^{(3)}$ are the Bloembergen [21] second- and third-order nonlinear susceptibilities (in S.I. units), n is the refractive index, n_2 is the nonlinear refractive index, and d is the effective modal (waveguide) diameter.

With the above definition for χ_D we may rewrite the binding energies in one- and three-dimensional cases in an explicit form, as expressed in terms of relevant material constants. In the one-dimensional case, the binding energy E_1^b is given by Eq. (53), with $m\equiv m_\parallel$ and μ

$=[(\chi_1)^2\sqrt{m_\parallel}/(4\hbar^2)]^{1/3}$ being the corresponding solutions. In addition, we substitute explicit expressions for the effective masses $m_\parallel$, $m_\perp$ and $v\simeq c/n$ for the group velocity, and transform to the wavelengths $\lambda_{1,2}=2\pi c/\omega_{1,2}$. In the cutoff-dependent three-dimensional case [Eq. (67)], we choose the cutoff at $k_{\max}=2\pi/\lambda_1$, thus arriving at the following expressions:

$$E_1^b\simeq\frac{4\pi^2c^2\hbar^{5/3}}{(4\varepsilon_0 d^2)^{2/3}(\omega_1'')^{1/3}}\frac{[\chi_B^{(2)}]^{4/3}}{n^4\lambda_1^2}, \tag{70}$$

$$E_3^b(k_{\max}=2\pi/\lambda_1)\simeq\frac{(2\pi)^3c^2\hbar^2}{\varepsilon_0}\frac{[\chi_B^{(2)}]^2}{n^4\lambda_1^5}\left[1-\ln\sqrt{\frac{2\pi n^2\omega_1''}{c\lambda_1}}\right]. \tag{71}$$

These results explicitly demonstrate the dependence of the binding energies on the nonlinearity, dispersion, refractive index, and subharmonic field wavelength λ_1. We also recall that the most important requirements for quantum soliton formation were a positive dispersion coefficient ω_1'' at the subharmonic wavelength, and group-velocity matching. Clearly, the lower the dispersion the larger the effective mass $m_\parallel$, and hence higher the binding energy the smaller the soliton size. Large nonlinearities also enhance the binding energy. In addition, we stress the strong dependence of the binding energy (especially in the higher-dimensional case) on the subharmonic wavelength. Other factors that may have practical significance, but were omitted here for simplicity, include higher-order linear dispersion, nonlinear dispersion, tensorial (direction-dependent) properties of the medium, absorption, and thermal phonon effects due to Raman scattering (see, e.g., Ref. [22]).

Ideally, the soliton binding energy should be greater than any thermal phonon energies, and clearly the soliton radius should be within available geometrical sizes of the nonlinear material. Another important parameter is the characteristic interaction (formation) length, which scales as $\sim c/[(R_1)^2\omega_1'']$ in one dimension, and which should be less than an absorption length.

With a value of $\chi_B^{(2)}\sim 10^{-11}$ m/V characteristic for conventional $\chi^{(2)}$ nonlinear crystals (such as $LiNbO_3$) the binding energies, with $\lambda_1\sim 1$ μm, are low compared to thermal phonon energies. In addition, it may be difficult in practice to satisfy the other above-mentioned requirements, such as positive dispersion at shorter wavelengths, together with group-velocity matching. However, recent experiments on second harmonic generation (with $\lambda_1\sim 9$ μm) demonstrate that three to four orders of magnitude greater nonlinearities can be obtained in semiconductor devices, such as GaAs asymmetric quantum wells and related systems [23]. This also has an advantage that the actual optical properties of such devices can be fabricated over a rather wide range of parameter values. Other promising devices, with fabricable material properties, include Bragg-grating structures [24].

In order to give numerical estimates we choose the following values of parameters: $\chi_B^{(2)}=10^{-7}$ m/V, $n=3$, $\omega_1''=0.1$ m^2/s, and $d=5$ μm, and take the subharmonic wavelength $\lambda_1=2$ μm. These give the ratio $m_\perp/m_\parallel\simeq 9.4\times 10^{-3}$, and the coupling constant $\chi_1\simeq 7.39\times 10^7\sqrt{\text{m}}$/s, while in higher dimensions the coupling constants $\chi_{2,3}$ can be evaluated using relation (68), so that $\chi_D=d^{(D-1)/2}\chi_1(D=2,3)$. As a result, the magnitudes of the binding energy E_1^b and the soliton radius R_1 in one space dimension become $E_1^b\simeq 1.75\times 10^{-5}$ eV and $R_1\simeq 1.94$ μm. In Figs. 2 and 3 we plot the binding energies $E_{2,3}^b$ and radii $R_{2,3}$ in two and three dimensions as a function of $k_{\max}$, for different values of cubic nonlinearities $\kappa_D\sim\chi_B^{(3)}$. These are chosen arbitrarily (much greater than realistic values) in order to demonstrate explicitly the stabilizing effect of the quartic interaction term. As we see, with a choice of the cutoff at $k_{\max}=2\pi/\lambda_1$ (with $\lambda_1=2$ μm), the resulting solutions have binding energies ($E_2^b\simeq 4.43\times 10^{-6}$ eV and $E_3^b\simeq 2.25\times 10^{-6}$ eV for $\kappa_{2,3}=0$) and radii ($R_2\simeq 39.7$ μm, and $R_3\simeq 55.6$ μm) comparable to the results for a one-dimensional parametric waveguide. In fact, we find that $R_1<R_2<R_3$ and $E_1^b>E_2^b>E_3^b$ for the above values of parameters. This indicates that we expect the higher dimensional solitons to be less strongly bound and of larger radius than their one-dimensional counterparts.

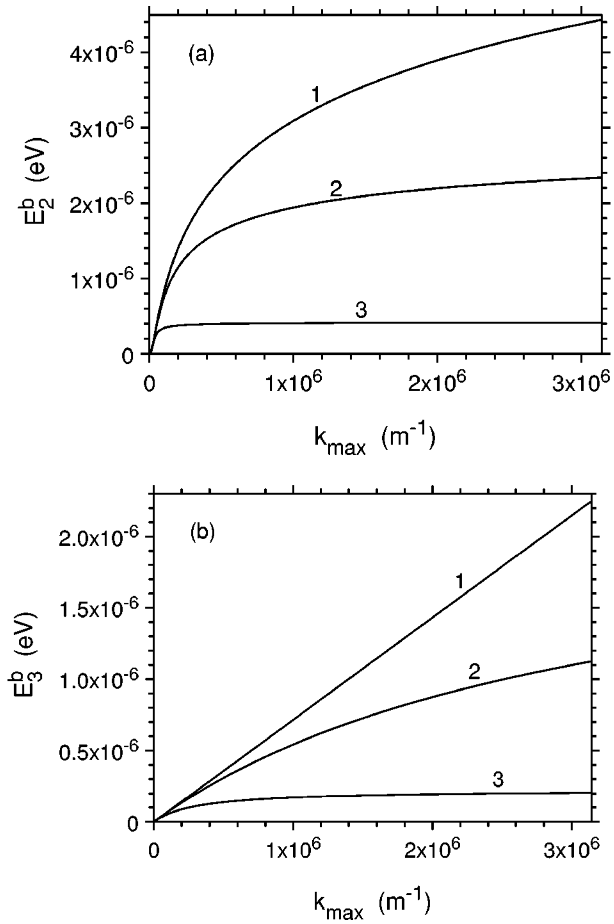

FIG. 2. Soliton binding energies $E_{2,3}^b$ in two (a) and three (b) dimensions as a function of the cutoff momentum $k_{\max}$ for $\Delta=0$ and $m_\perp/m_\parallel=9.42\times 10^{-3}$. (a) $\chi_2=1.65\times 10^5$ m/s and $\kappa_2=0$ (1), $\kappa_2=2$ m^2/s (2), and $\kappa_2=20$ m^2/s (3). (b) $\chi_3=369.5$ m$^{3/2}$/s and $\kappa_3=0$ (1), $\kappa_3=2\times 10^{-5}$ m^3/s (2) and $\kappa_3=2\times 10^{-4}$m^3/s (3).

VII. SUMMARY

In summary, we have presented quantum soliton or bound state solutions to the parametric quantum field theory. Exact

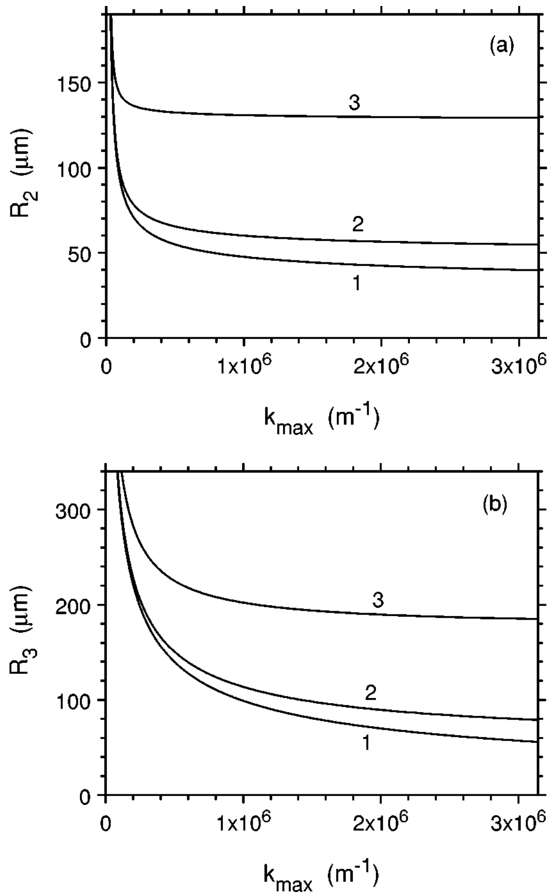

FIG. 3. Soliton radii $R_{2,3}$ in two (a) and three (b) dimensions as a function of the cutoff momentum $k_{\max}$ for the same values of parameters as in Fig. 2.

two-particle solutions are obtained in one, two, and three space dimensions, while three-particle binding in one space dimension is analyzed variationally. The results have the remarkable character that the theory, while having a well-behaved (and widely used) classical counterpart, has quantum singularities in the eigenstates in more than one space dimensions, corresponding to zero-radius structures. The reason for this behavior is the inherently nonclassical structure of the bound state, which is a quantum superposition state. To resolve this paradox, we impose an appropriate momentum cutoff on the nonlinear couplings, which results in a finite radius of the two-particle bound state, even in the simplest case of pure parametric interaction (i.e., even without the stabilizing quartic interaction). While the cutoff-dependent results require a knowledge of the precise mechanism that reduces the coupling at large relative momentum, we can estimate (in the case of nonlinear optical parametric interaction) that the nonlinear couplings should extend no higher than $2\pi/\lambda_1$.

A similar procedure was employed by Bethe, in using an estimated cutoff of $k_{\max}=m_e c/\hbar$ in the first Lamb shift calculation [25]. Just as in the Lamb shift, this can be improved by more careful treatment of the theory at large relative momenta. Such an improved treatment would especially be appropriate in the three-dimensional case, where we obtain a linear divergence with $k_{\max}\rightarrow\infty$, in contrast to more acceptable logarithmic divergences usually encountered in quantum field theory and statistical physics.

With a finite cutoff of $k_{\max}=2\pi/\lambda_1$, we have estimated the binding energies and radii of the solutions in the case of the nonlinear optical process of second-harmonic generation. The estimated energies appear to be achievable—either by using cryogenic means, to reduce the energy of competing thermal processes, or else by means of transient experiments on time and length scales shorter than those of thermal Raman processes and absorption.

The physical interpretation of these bound states is that they are a superposition of a second-harmonic photon and two subharmonic photons, which can propagate without either down-conversion of the higher-frequency photon, or dispersive spreading of the subharmonic photons. In practical terms, of course, most photon pairs created by down-conversion are in unbound (continuum) states, which are not treated in detail here. The possibility of creating bound states that are immune to further down-conversion does not seem to have been treated in earlier theories of this process, although earlier nondispersive theories predicted nonclassical spatial oscillations [26]. Most significantly, the solitons form in physically testable regimes, with the required experimental environment being nearly accessible with currently available technology.

It is not impossible that this parametric quantum theory, as well as being theoretically interesting, could result in the first experimental test of multidimensional quantum soliton theory for Bose fields. Thus, complementary to high-energy physics and particle accelerators, investigation of particlelike quantum structures may become available in a larger variety of physical systems, including nonlinear optics and laser physics.

[1] N. H. Christ and T. D. Lee, Phys. Rev. D **12**, 1606 (1975); T. D. Lee, *Particle Physics and Introduction to Field Theory* (Harwood, Chur, Switzerland, 1988).

[2] E. H. Lieb and W. Liniger, Phys. Rev. **130**, 1605 (1963); I. B. McGuire, J. Math. Phys. **5**, 622 (1964); C. N. Yang, Phys. Rev. **168**, 1920 (1967); H. B. Thacker, Rev. Mod. Phys. **53**, 253 (1981).

[3] S. J. Carter, P. D. Drummond, M. D. Reid, and R. M. Shelby, Phys. Rev. Lett. **58**, 1841 (1987); P. D. Drummond and S. J. Carter, J. Opt. Soc. Am. B **4**, 1565 (1987).

[4] Y. Lai and H. A. Haus, Phys. Rev. A **40**, 844 (1989); H. A. Haus and Y. Lai, J. Opt. Soc. Am. B **7**, 386 (1990).

[5] M. Rosenbluth and R. M. Shelby, Phys. Rev. Lett. **66**, 153 (1991); P. D. Drummond, R. M. Shelby, S. R. Friberg, and Y. Yamamoto, Nature (London) **365**, 307 (1993).

[6] E. H. Lieb and F. Y. Wu, Phys. Rev. Lett. **20**, 1445 (1968).

[7] R. B. Laughlin, Phys. Rev. Lett. **50**, 1395 (1983).

[8] D. C. Tsui, H. L. Störmer, and H. C. Gossard, Phys. Rev. Lett. **48**, 1559 (1982).

[9] C. L. Schultz, M. J. Ablowitz, and O. BarYaacov, Phys. Rev.

Lett. **59**, 2825 (1987); G. D. Pang, F. C. Pu, and B.-H. Zhao, *ibid.* **65**, 3227 (1990); G. D. Pang, Phys. Lett. A **184**, 163 (1994).

[10] A. A. Kanashov and A. M. Rubenchik, Physica D **4**, 122 (1981); K. Hayata and M. Koshiba, Phys. Rev. Lett. **71**, 3275 (1993); L. Torner, C. R. Menyuk, W. E. Torruellas, and G. I. Stegeman, Opt. Lett. **20**, 13 (1995); L. Bergé, V. K. Mezentsev, J. J. Rasmussen, and J. Wyller, Phys. Rev. A **52**, R28 (1995); H. He, M. J. Werner, and P. D. Drummond, Phys. Rev. E **54**, 896 (1996).

[11] C. M. Caves and D. D. Crouch, J. Opt. Soc. Am. B **4**, 1535 (1987); M. G. Raymer, P. D. Drummond, and S. J. Carter, Opt. Lett. **16**, 1189 (1991); M. Hillery and L. D. Mlodinov, Phys. Rev. A **30**, 1860 (1984); P. D. Drummond. *ibid.* **42**, 6845 (1990).

[12] D. F. Walls, Nature (London) **306**, 141 (1983); D. F. Walls and G. J. Milburn, *Quantum Optics* (Springer, Berlin, 1994).

[13] Ch. J. Bordé, Phys. Lett. A **204**, 217 (1995).

[14] A. Fioretti, D. Comparat, A. Crubellier, O. Dulieu, F. Masnou-Seeuws, and P. Pillet, Phys. Rev. Lett. (to be published); P. Tommasini, E. Timmermans, M. Hussein, and A. Kerman, cond-mat/9804015; Y. B. Band and P. S. Julienne, Phys. Rev. A **51**, R4317 (1995); A. Vardi, D. Abrashkevich, E. Frishman, and M. Shapiro, J. Chem. Phys. **107**, 6166 (1997); R. Côté and A. Dalgarno, Chem. Phys. Lett. **279**, 50 (1997).

[15] A. V. Buryak, Yu. S. Kivshar, and S. Trillo, Opt. Lett. **20**, 1961 (1995); M. A. Karpierz, *ibid.* **20**, 1677 (1995); S. Trillo, A. V. Buryak, and Yu. S. Kivshar, Opt. Commun. **122**, 200 (1996); O. Bang, J. Opt. Soc. Am. B **14**, 51 (1997).

[16] I. H. Deutsch and R. Y. Chiao, Phys. Rev. Lett. **69**, 3627 (1992); I. H. Deutsch, R. Y. Chiao, and J. C. Garrison, Phys. Rev. A **47**, 3330 (1993); Z. Cheng and G. Kurizki, Phys. Rev. Lett. **75**, 3430 (1995).

[17] P. D. Drummond and H. He, Phys. Rev. A **56**, R1107 (1997).

[18] A. G. Shnirman, B. A. Malomed, and E. Ben-Jacob, Phys. Rev. A **50**, 3453 (1994).

[19] A. A. Abrikosov, L. P. Gorkov, and I. E. Dzyaloshinski, *Methods of Quantum Field Theory in Statistical Physics* (Dover, New York, 1963); N. N. Bogolyubov, J. Phys. (USSR) **11**, 23 (1947); K. A. Brueckner and K. Sawada, Phys. Rev. **106**, 1117 (1957); K. Huang, C. N. Yang, and J. M. Luttinger, *ibid.* **105**, 776 (1957).

[20] E. Fermi, Ric. Sci. **7**, 12 (1936); S. Albeverio, F. Gesztesy, and R. Hoegh-Krohn, *Solvable Models in Quantum Mechanics* (Springer, Heidelberg, 1988); K. Wódkiewicz, Phys. Rev. A **43**, 68 (1991).

[21] N. Bloembergen, *Nonlinear Optics* (Benjamin, New York, 1965); P. N. Butcher and D. Cotter, *The Elements of Nonlinear Optics* (Cambridge University Press, Cambridge, 1990).

[22] S. J. Carter and P. D. Drummond, Phys. Rev. Lett. **67**, 3757 (1991).

[23] H. Xie, W. I. Wang, J. R. Meyer, and L. R. Ram-Mohan, Appl. Phys. Lett. **65**, 2048 (1994); M. Seto *et al.*, *ibid.* **65**, 2969 (1994); M. J. Shaw *et al.*, Phys. Rev. B **48**, 2001 (1993).

[24] H. He and P. D. Drummond, Phys. Rev. Lett. **78**, 4311 (1997).

[25] H. A. Bethe, Phys. Rev. **72**, 339 (1947); W. E. Lamb, Jr. and R. C. Retherford, *ibid.* **72**, 241 (1947).

[26] D. F. Walls and C. T. Tindle, J. Phys. A **5**, 534 (1972).