Ladder Operator for the One-Dimensional Hubbard Model

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The one-dimensional Hubbard model is integrable in the sense that it has an infinite family of conserved currents. We explicitly construct a ladder operator which can be used to iteratively generate all of the conserved current operators. This construction is different from that used for Lorentz invariant systems such as the Heisenberg model. The Hubbard model is not Lorentz invariant, due to the separation of spin and charge excitations. The ladder operator is obtained by a very general formalism which is applicable to *any* model that can be derived from a solution of the Yang-Baxter equation.

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The one-dimensional Hubbard model has attracted considerable interest because it is one of the few examples of a model for strongly correlated electrons that is exactly soluble [1]. The fact that it describes a doped Mott insulator and exhibits spin-charge separation (i.e., spin and charge excitations are independent of one another) has been argued to be relevant to understanding the unusual metallic properties of high-temperature superconductors [2]. Furthermore, the model has rich mathematical structure [3]: it is solvable by both the coordinate Bethe ansatz [4] and the algebraic Bethe ansatz [5], it has a hidden so(4) symmetry [6], and is integrable in the sense that it has an infinite family of conserved currents [7]. The latter is a consequence of the fact that the model can be derived from a solution of the Yang-Baxter equation [8]. Integrability is not just of mathematical interest because it may have implications for dissipationless transport [9], the coherence of interchain transport [10], and whether the energy level spacing follows a Poisson distribution or the Gaussian orthogonal ensemble distribution characteristic of quantum chaotic systems [11]. Furthermore, integrability has been essential to recent exact calculations of transport properties in mesoscopic electronic devices [12].

In this Letter we clarify the algebraic structure underlying the integrability of the Hubbard model by using the Yang-Baxter equation to explicitly construct a single operator B (known as the ladder operator) which can be used in a simple recursion relation [Eq. (2) below] to generate the whole family $\{t^{(n)}\}_{n=0}^{\infty}$ of conserved current operators, i.e., operators that commute with the Hamiltonian and one another. This result is surprising in light of the lack of invariance in the model under the lattice version of the Poincaré group.

For continuum field theories in $(1 + 1)$ dimensions the generators of the Poincaré group are B, P, and H, being the generators of Lorentz boosts and translations in space and time, respectively. (P and H are also the total momentum operator and Hamiltonian, respectively.) They obey the closed algebra

$$[B,H] = P, \qquad [B,P] = H, \qquad [H,P] = 0. \qquad (1)$$

It is extraordinary that a wide range of integrable lattice models (including the Heisenberg [13,14], Calogero, Toda [14], and supersymmetric $t - J$ [15] models) are invariant under a generalization of the Poincaré group involving the entire infinite set $\{t^{(n)}\}_{n=0}^{\infty}$ of conserved currents. They satisfy the algebra

$$[B, t^{(n)}] = t^{(n+1)}, \qquad [t^{(n)}, t^{(m)}] = 0, \qquad (2)$$

where $t^{(0)}$ and $t^{(1)}$ are the momentum operator P and Hamiltonian H, respectively. Here the boost operator acts as a ladder operator on the infinite sequence of conserved operators. For the XXZ model the boost operator can be identified with an algebraic element of a lattice Virasoro algebra [16]. This turns out to be of great practical significance because it permits the use of vertex operators for the determination of the energy spectrum and calculation of correlation functions [17].

A crucial property in the manifestation of Lorentz invariance in the above models is the fact that the R matrix which is a solution of the Yang-Baxter equation [Eq. (3) below] has the difference property, $R(u,v) = R(u - v)$, for the spectral parameters u and v. This is because the spectral parameter plays the role of rapidity variable. A uniform shift in both rapidity variables, corresponding to a change in the Lorentz frame, leaves the R-matrix invariant. In this sense those solutions with the difference property are invariant under a lattice version of the Poincaré group [13].

In contrast, the Hubbard model is not Lorentz invariant [18] since it exhibits gapless excitations with different velocities. It is for this reason that spin and charge separate. As a result, the R matrix associated with the Hubbard model does not have the difference property and so its integrability is not as well understood. Although Lieb and Wu [4] gave a coordinate Bethe ansatz solution in 1968, it was not until 1986 that Shastry demonstrated the existence of an infinite family of conserved currents. This involved

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constructing a two-dimensional model in classical statistical mechanics with a transfer matrix that commuted with the Hubbard Hamiltonian [7]. It was achieved by mapping the model onto a pair of coupled spin chains using the Jordan-Wigner transformation. Although Shastry conjectured that the R matrix satisfies the Yang-Baxter equation, it was some time before a convincing proof was available [8]. Furthermore, the use of the algebraic Bethe ansatz method to reproduce the solution of Lieb and Wu has only recently been achieved [5]. A significant consequence of this algebraic development is that it facilitates the use of the quantum transfer matrix method for the analysis of the thermodynamic properties at finite temperature [3,19].

Grabowski and Mathieu [20] claimed that there is no "matrix" ladder operator satisfying (2) for the model, motivating them to construct the first seven conserved currents by "brute force methods." We now show how for *any* model derived from a solution of the Yang-Baxter equation there is a one parameter family of ladder operators $B(v)$ such that the conserved currents satisfy (2). In cases where a solution to the Yang-Baxter equation has the difference property, the conserved currents have no dependence on the spectral parameter v. In this instance the construction for the ladder operator occurs as a particular case of the more general method we describe below.

As an application of our general result we then consider a one parameter family of Hamiltonians which includes the Hubbard model as a special case ($v = 0$). Our approach has the further appeal that we work directly with the fermion operators of the model (rather than a two-dimensional statistical mechanics model) and that the so(4) invariance of the model is manifest throughout.

The Yang-Baxter equation (or star-triangle relation) is central to exactly soluble models because it is a sufficient condition for the validity of the Bethe ansatz [21,22]. The corresponding equations for $(1 + 1)$-dimensional quantum field theories are also known as the factorization equations because they imply that all possible decompositions of the N-particle scattering (S) matrix give the same result as a product of two-particle S matrices [23]. Consider a lattice model defined on L sites, each of which has a Hilbert space V. The matrix $R(u,v)$ acts on the tensor product space $V \otimes V$ and satisfies the Yang-Baxter equation [21,22]

$$R_{12}(u,w)R_{13}(u,v)R_{23}(w,v) = R_{23}(w,v)R_{13}(u,v) \times R_{12}(u,w), \tag{3}$$

where the subscripts refer to the embedding of $R(u,v)$ on the threefold space $V \otimes V \otimes V$. From a solution to this equation we define a transfer matrix

$$t(u,v) = \mathrm{tr}_0[R_{0L}(u,v)\ldots R_{02}(u,v)R_{01}(u,v)],$$

where tr denotes the trace over V (when V is a superspace we use the supertrace; i.e., the trace over the bosonic states minus the trace over the fermionic states). It follows from the Yang-Baxter equation (3) that

$$[t(u,v), t(w,v)] = 0 \tag{4}$$

for all values of the parameters u and w.

An assumed feature of the R matrix is the regularity property, i.e., $R(u,u) = P$, with P being the permutation operator. (A phase of -1 is gained whenever two fermionic states are interchanged.) Using this property, the Hamiltonian is defined to be

$$H(v) = -T^{-1} \cdot \frac{\partial t(u,v)}{\partial u}\bigg|_{u=v}, \tag{5}$$

where $T \equiv t(u,u) = P_{1L}\ldots P_{13}P_{12}$ is the translation operator. This yields

$$H(v) = \sum_{j=1}^{L} h_{j(j+1)}(v) \tag{6}$$

with the local Hamiltonian given by

$$h(v) = -P\,\frac{\partial R(u,v)}{\partial u}\bigg|_{u=v}.$$

Above and throughout periodic boundary conditions are imposed. For later use, it is convenient to consider the series expansion for the R matrix

$$R(u,v) = P[I + (v-u)h(v) + 1/2(v-u)^2 f(v) + 1/6(v-u)^3 g(v) + \ldots]. \tag{7}$$

Expressing the logarithm of the transfer matrix in a power series expansion

$$\ln t(u,v) = \sum_{n=0}^{\infty} \frac{(u-v)^n}{n!}\, t^{(n)}(v), \tag{8}$$

it is apparent, in view of (4), that

$$[t^{(n)}(v), t^{(m)}(v)] = 0, \qquad \forall\, m,n$$

and, in particular,

$$[H(v), t^{(n)}(v)] = 0.$$

Consequently, the Hamiltonian $H(v)$ is integrable since the set of operators $\{t^{(n)}(v)\}$ provides a set of conservation laws for the system. Note that in the case where the R matrix does not have the difference property, we have the generic feature that the Hamiltonian and higher conserved charges will always have nontrivial dependence on the variable v, as can be seen from (8).

We now construct the parameter-dependent ladder operator $B(v)$. Differentiating the Yang-Baxter equation (3) with respect to w, then setting $w = v$ and premultiplying by the permutation operator $P_{j(j+1)}$, yields an analog of the Sutherland equation [24]

$$[h_{j(j+1)}(v), R_{0(j+1)}(u,v)R_{0j}(u,v)] = R_{0(j+1)}(u,v)\,\frac{\partial R_{0j}(u,v)}{\partial v} - \frac{\partial R_{0(j+1)}(u,v)}{\partial v}\,R_{0j}(u,v). \tag{9}$$

An immediate consequence of (9) is that

$$[H(v), t(u, v)] = 0,$$

which also follows from (4).

The ladder operator $B(v)$ is defined in terms of the local Hamiltonians $h_j \equiv h_{j(j+1)}(v)$ through the relation

$$B(v) = -\sum_{j=1}^{L} \mathbf{j} h_j + \frac{\partial}{\partial v}, \tag{10}$$

where $\mathbf{j}$ are the elements of the integers modulo L. A consequence of the generalized Sutherland relation (9) is now

$$[B(v), t(u, v)] = 0. \tag{11}$$

The relation (11) permits us to deduce the recurrence relation (2) from (8).

The definition of the ladder operator here is different from the cases considered in [13–16] by the inclusion of the differential operator. This term is not required for those cases with the difference property since it is apparent from (8) that the conserved currents $t^{(n)}$ have no dependence on v and hence (2) still holds. However, in this instance (11) becomes

$$[B, t(u - v)] = \frac{\partial t(u - v)}{\partial u},$$

which can be integrated to

$$t(u + \lambda) = \exp(\lambda B) t(u) \exp(-\lambda B).$$

The parameter u characterizes the Lorentz frame for the transfer matrix $t(u)$, showing that B is the generator of Lorentz boosts in this context [13]. This is clearly not the case for the Hubbard model where Lorentz invariance is not present.

From (2) the operators $t^{(n)}(v)$ may be calculated iteratively. We find the following expressions for the leading terms

$$t^{(0)} = \ln T, \qquad t^{(1)} = -H,$$

$$\begin{aligned} t^{(2)} &= \sum_{ij} \mathbf{j}[h_j, h_i] - H' \\ &= \sum_{j} \mathbf{j}[h_j, h_{j+1} + h_{j-1}] - H' \\ &= \sum_{j} \mathbf{j}[h_j, h_{j+1}] + \sum_{j} (\mathbf{j} + \mathbf{1})[h_{j+1}, h_j] - H' \\ &= -\sum_{j} [h_j, h_{j+1}] - H', \end{aligned}$$

where the prime denotes a derivative with respect to v. The computation of $t^{(3)}$ can be simplified by invoking the generalized Reshetikhin condition (cf. [22])

$$[h_{12} + h_{23}, [h_{12}, h_{23}]] + [h_{12}, h'_{12}] + [h_{12}, h'_{23}] + [h_{23}, h'_{12}] = x_{23} - x_{12} \tag{12}$$

with the two site operator given by

$$x = 2h^3 + g - 3hf + 2[h', h] - h''.$$

The Reshetikhin relation is obtained by applying $\partial^3/\partial u^2 \partial w$ to (3) and using (7). Omitting the details, this yields the result

$$\begin{aligned} t^{(3)} = cH &+ 2\sum_j [h_j, [h_{j-2}, h_{j-1}] + h'_{j-1}] \\ &+ \sum_j [h_j, [h_{j-1}, h_j] - h'_{j+1}] \\ &+ \sum_j (h_j^3 - g_j - h_j \cdot h'_j - 2h'_j \cdot h_j). \end{aligned}$$

Above, c is a constant determined by the normalization of $R(u, v)$. It can always be chosen to be zero. Before turning to the particular case of the Hubbard model we stress that the above construction of the ladder operator is valid for any solution of the Yang-Baxter equation (3).

Consider the four-dimensional local Hilbert space V spanned by the states

$$|0\rangle, \quad |\uparrow\rangle, \quad |\downarrow\rangle, \quad |\uparrow\downarrow\rangle.$$

Introduce the spaces $W(\sigma)$ with basis $\{|0\rangle, |\sigma\rangle\}$, $\sigma = \uparrow\downarrow$, so that $V \equiv W(\uparrow) \times W(\downarrow)$. For each tensor space $W(\sigma) \otimes W(\sigma)$ there is a solution of the Yang-Baxter equation (with difference property) given by

$$\begin{aligned} \mathcal{R}^{\sigma}_{ij}(u - v) = {} & \cos(u - v)(1 - n_{i\sigma} - n_{j\sigma}) \\ & + \sin(u - v)(n_{i\sigma} + n_{j\sigma} - 2n_{i\sigma}n_{j\sigma}) \\ & + c^{\dagger}_{i\sigma}c_{j\sigma} + c^{\dagger}_{j\sigma}c_{i\sigma}. \end{aligned}$$

Here $c^{\dagger}_{j\sigma}$ and $c_{j\sigma}$ are the creation and annihilation operators with spin $\sigma(= \uparrow, \downarrow)$ at site j and $n_{j\sigma} = c^{\dagger}_{j\sigma}c_{j\sigma}$ is the density operator. The associated Hamiltonian obtained through (5) is that for free fermions.

It has recently been shown [25] that the following R matrix is also a solution of the Yang-Baxter equation acting on $V \otimes V$,

$$\begin{aligned} R_{ij}(u, v) = {} & \mathcal{R}^{\uparrow}_{ij}(u - v)\mathcal{R}^{\downarrow}_{ij}(u - v) \\ & - \frac{\cos(u - v)}{\cos(u + v)} \tanh[\theta(u) - \theta(v)] \\ & \times \mathcal{R}^{\uparrow}_{ij}(u + v)\mathcal{R}^{\downarrow}_{ij}(u + v)(1 - 2n_{i\uparrow})(1 - 2n_{i\downarrow}) \end{aligned} \tag{13}$$

with $\theta(u)$ defined through the relation

$$\sinh 2\theta(u) = \frac{U \sin 2u}{4}.$$

An important consequence of (3) is that it allows for the construction of a generalized Hubbard model (with spectral parameter dependence) as noted in [25]. The identification of this generalized model is paramount in the construction of the ladder operator. Explicitly, the local Hamiltonians read

$$h_{ij}(v) = -\sum_{\sigma=\uparrow\downarrow}[c^\dagger_{i\sigma}c_{j\sigma} + c^\dagger_{j\sigma}c_{i\sigma}] + \frac{U}{4\cosh 2\theta(v)}\Gamma_{ij\uparrow}(v)\Gamma_{ij\downarrow}(v), \quad (14)$$

where

$$\Gamma_{ij\sigma}(v) = \cos^2 v(1 - 2n_{i\sigma}) - \sin^2 v(1 - 2n_{j\sigma}) + \sin 2v(c^\dagger_{i\sigma}c_{j\sigma} - c^\dagger_{j\sigma}c_{i\sigma}). \quad (15)$$

It is clear that (6) with (14) reduces to the usual Hubbard model when $v = 0$. In particular, we find

$$h'_{ij}(0) = U/2[(1 - 2n_{i\uparrow})(c^\dagger_{i\downarrow}c_{j\downarrow} - c^\dagger_{j\downarrow}c_{i\downarrow}) + (1 - 2n_{i\downarrow})(c^\dagger_{i\uparrow}c_{j\uparrow} - c^\dagger_{j\uparrow}c_{i\uparrow})], \quad (16)$$

which plays an important role in the explicit construction of the higher conserved operators discussed previously. It is important to note that there is no way of determining $h'(0)$ directly from the usual Hubbard model.

Substituting g [which is obtained through (7) and (13)] and Eqs. (15) and (16), all evaluated at $v = 0$, into the above expressions for $t^{(2)}$ and $t^{(3)}$, we recover the expressions found previously for the first [7] and second [20,26] nontrivial conserved currents (modulo a constant term and multiple of H). A well known feature of the Hubbard model is that all the integrals of motion, except for the translation operator, are invariant with respect to the so(4) Lie algebra when the lattice length is even [25]. Significantly, the ladder operator (10) in the case of the Hubbard model is also so(4) invariant for an even length lattice.

To conclude, a systematic method for obtaining the conserved currents in the Hubbard model has been described which employs the use of a ladder operator. We emphasize, however, that the construction presented here is entirely general and may be applied to any Yang-Baxter integrable system.

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