A Fibred Tableau Calculus for Modal Logics of Agents*

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Abstract. In [15,19] we showed how to combine propositional multimodal logics using Gabbay's *fibring* methodology. In this paper we extend the above mentioned works by providing a tableau-based proof technique for the combined/fibred logics. To achieve this end we first make a comparison between two types of tableau proof systems, (*graph* & *path*), with the help of a scenario (The Friend's Puzzle). Having done that we show how to uniformly construct a tableau calculus for the combined logic using Governatori's labelled tableau system **KEM**. We conclude with a discussion on **KEM**'s features.

1 Introduction

Modelling and reasoning about cognitive attitudes like knowledge, belief, desire, goals, intention etc. of agents is an active research area within the artificial intelligence community [6,23]. It is often the case that normal[1] multimodal logics are used to formalise these mental notions. Multimodal logics generalise modal logics allowing more than one modal operator to appear in formulae, i.e., a modal operator is named by means of a label, for instance $\Box_i$ which identifies it. Hence a formula like $\Box_i\varphi$ could be interpreted as *φ is known by the agent i or φ is believed by agent i etc.* representing respectively the knowledge and belief of an agent. In addition to the above representation, multimodal logics of agents (**MMA**) impose constraints between the different mental attitudes in the form of *interaction axioms*. For instance, if we consider **MMA**'s like BDI [20] then we can find interaction axioms of the form $\mathrm{INT}(\varphi) \rightarrow \mathrm{DES}(\varphi)$, $\mathrm{DES}(\varphi) \rightarrow \mathrm{BEL}(\varphi)$ denoting respectively intentions being stronger than desires and desires being stronger than beliefs. Moreover, these interaction axioms are *non-homogeneous* in the sense that every modal operator is not restricted to the same system, i.e., the underlying axiom systems for DES is **K** and **D** of modal logic whereas that of BEL is **KD45**. Hence the basic BDI logic $\mathbb{L}$ can be seen as a combination of different component logics plus the two interaction axioms as given below

$$\begin{aligned}\mathbb{L} \equiv\ & (\otimes_{i=1}^{n}\mathbf{KD45}_{\mathrm{BEL}_i}) \otimes (\otimes_{i=1}^{n}\mathbf{KD}_{\mathrm{DES}_i}) \otimes (\otimes_{i=1}^{n}\mathbf{KD}_{\mathrm{INT}_i})\\ & + \{\mathrm{INT}_i\varphi \rightarrow \mathrm{DES}_i\varphi\} + \{\mathrm{DES}_i\varphi \rightarrow \mathrm{BEL}_i\varphi\}\end{aligned} \quad (1)$$

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[1] General modal systems with an arbitrary set of normal modal operators all characterised by the axiom **K**: $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$ and the necessitation rule. i.e., $\vdash \varphi / \vdash \Box\varphi$.

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In a similar manner any **MMA** consists of a combined system of logic of knowledge, beliefs, desires, goals and intentions as mentioned above. They are basically well understood standard modal logics *combined together* to model different facets of the agents. A number of researchers have provided such combined systems for different reasons and different applications. However, investigations into a general methodology for combining the different logics involved has been mainly neglected to a large extent. Recently [15,19] it has been shown that *fibring/dovetailing* [8] can be adopted as a semantic methodology to characterise multimodal logics. But in that work we did not provide any proof techniques for the fibred logics. In this paper we extend our previous work so as to provide a tableau proof technique for the fibred logic which in turn is based on the labelled tableau system **KEM** [11,10,1].

The key feature of our tableau system is that it is neither based on resolution nor on standard sequent/tableau techniques. It combines linear tableau expansion rules with natural deduction rules and an analytic version of the cut rule. The tableau rules are supplemented with a powerful and flexible label algebra that allows the system to deal with a large class of intensional logics admitting possible world semantics (non-normal modal logic [14], multi-modal logics [11] and conditional logics [2]). The label algebra is intended to simulate the possible world semantics and it has a very strong relationship with fibring [10].

As far as the field of *combining logics* is concerned, it has been an active research area since some time now and powerful results about the preservation of important properties of the logics being combined has been obtained [16,4,22]. Also, investigations related to using fibring as a combining technique in various domains has produced a wealth of results as found in works like [8,24,21,5]. The novelty of combining logics is the aim to develop *general techniques* that allow us to produce combinations of *existing* and well understood logics. Such general techniques are needed for formalising complex systems in a systematic way. Such a methodology can help decompose the problem of designing a complex system into developing components (logics) and combining them.

One of the main advantages of using fibring as a semantic methodology for combining multimodal logics as compared to other combining techniques like *fusion* [2] is that the later has the problem of not being able to express interaction axioms, much needed for Multi-Agent-System (MAS) theories. Fibring is more powerful because of the possibility of adding conditions on the fibring function. These conditions could encode interactions between the two classes of models that are being combined and therefore could represent interaction axioms between the two logics. One such result was shown in [15]. Moreover, fibring does not require the logics to be normal. This allows fibring to be used to model combinations of epistemic logic without being forced to suffer from the logical omniscience problem. The drawbacks of other combining techniques like *embedding* and *independent combination* when compared to fibring have been discussed at length in [18]. Another advantage is that fibring makes it possible to combine logics at different levels, obtaining hierarchical modal logics, i.e., a logic with another logic embedded in it, or more precisely a logic with two modal operators such that

[2] Normal bimodal and polymodal logics without any interaction axioms are well studied as *fusions* of normal monomodal logics [16,22].

the first can occur in the scope of the other but not the other way around; see [9] for applications of hierarchical logics. For the second case it is possible to combine logic with different semantics. We can combine, let us say, a normal temporal logic whose semantics is given in terms of Kripke models and an epistemic non-normal modal logic with a neighbourhood semantics. This is not possible with other combining techniques where the semantics for the logics to be combined must be homogeneous. Finally the fibring methodology allows us to study the structure of the combined logic based on the structures of the component logics, and often it gives us conditions under which important meta-theoretical properties of the component logics (soundness, completeness, decidability and so on) are preserved by the combination.

The paper is structured as follows. The next section provides a brief introduction to the technique of fibring. Section 3 outlines the path-based and graph-based tableau procedures. Section 4 describes the **KEM** tableau system. The paper concludes with some final remarks.

2 Fibring Multimodal Logics

Consider the basic BDI logic $\mathbb{L}$ given in (1) which is defined from three component logics, viz., $\mathbf{KD45}_n$ for belief, and $\mathbf{KD}_n$ for desires and intentions. For sake of clarity, consider two of the component logics, $\blacktriangledown_1(\mathbf{KD45})$ and $\blacktriangledown_2(\mathbf{KD})$ and their corresponding languages $\mathscr{L}_{\blacktriangledown_1}, \mathscr{L}_{\blacktriangledown_2}$ built from the respective sets $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$ of atoms having classes of models $\mathfrak{M}_{\blacktriangledown_1}, \mathfrak{M}_{\blacktriangledown_2}$ and satisfaction relations $\models_1$ and $\models_2$. Hence we are dealing with two different systems S_1 and S_2 characterised, respectively, by the class of Kripke models $\mathscr{K}_1$ and $\mathscr{K}_2$. For instance, we know how to evaluate $\Box_1\varphi$ (BEL(φ)) in $\mathscr{K}_1$ (**KD45**) and $\Box_2\varphi$ (DES(φ)) in $\mathscr{K}_2$ (**K D**). We need a method for evaluating $\Box_1$ (resp. $\Box_2$) with respect to $\mathscr{K}_2$ (resp. $\mathscr{K}_1$). In order to do so, we are to link (fibre), via a *fibring* function the model for $\blacktriangledown_1$ with a model for $\blacktriangledown_2$ and build a fibred model of the combination. The fibring function can evaluate (give a yes/no) answer with respect to a modality in S_2, being in S_1 and vice versa. The interpretation of a formula φ of the combined language in the fibred model at a state w can be given as

$$w \models \varphi \text{ if and only if } \mathfrak{F}(w) \models^* \varphi$$

where $\mathfrak{F}$ is a fibring function that maps a world to a model *suitable for interpreting* φ and $\models^*$ is the corresponding satisfaction relation ($\models_1$ for $\blacktriangledown_1$ or $\models_2$ for $\blacktriangledown_2$).

Example 1. Let $\blacktriangledown_1, \blacktriangledown_2$ be two modal logics as given above and let $\varphi = \Box_1\Diamond_2\mathfrak{p}_0$ be a formula on a world w_0 of the fibred semantics. φ belongs to the language $\mathscr{L}_{(1,2)}$ as the outer connective ($\Box_1$) belongs to the language $\mathscr{L}_1$ and the inner connective ($\Diamond_2$) belongs to the language $\mathscr{L}_2$.

By the standard definition we start evaluating $\Box_1$ of $\Box_1\Diamond_2$ at w_0. Hence according to the standard definition we have to check whether $\Diamond_2\mathfrak{p}_0$ is true at every w_1 accessible from w_0 since from the point of view of $\mathscr{L}_1$ this formula has the form $\Box_1 p$ (where $p = \Diamond_2\mathfrak{p}_0$ is atomic). But at w_1 we cannot interpret the operator $\Diamond_2$, because we are in a model of $\blacktriangledown_1$, not of $\blacktriangledown_2$. In order to do this evaluation we need the fibring function $\mathfrak{F}$ which at w_1 points to a world v_0, a world in a model suitable to interpret formulae from

$\blacktriangledown_2$. (Fig.1). Now all we have to check is whether $\Diamond_2\mathfrak{p}_0$, is true at v_0 in this last model and this can be done in the usual way. Hence the fibred semantics for the combined language $\mathscr{L}_{(1,2)}$ has models of the form $(\mathscr{F}_1, w_1, v_1, \mathbf{\mathfrak{F}}_1)$, where $\mathscr{F}_1 = (\mathsf{W}_1, \mathsf{R}_1)$ is a frame, and $\mathbf{\mathfrak{F}}_1$ is the fibring function which associates a model $\mathfrak{M}^2_w$ from $\mathscr{L}_2$ with w in $\mathscr{L}_1$ i.e. $\mathbf{\mathfrak{F}}_1(w) = \mathfrak{M}^2_w$.

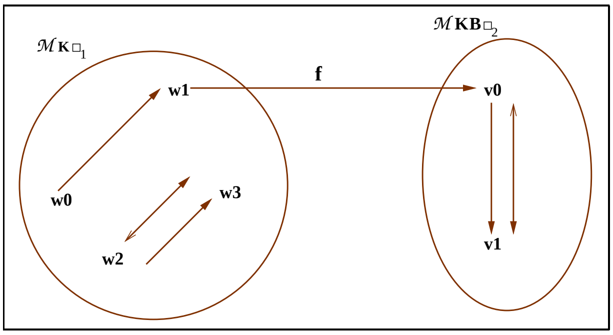

Fig. 1. An Example of Fibring

2.1 Fibring MMA

Let **I** be a set of labels representing the modal operators for the intentional states (belief, goal, intention) for a set of agents, and $\blacktriangledown_i, i \in \mathbf{I}$ be modal logics whose respective modalities are $\Box_i, i \in \mathbf{I}$.

Definition 1 *[8] A fibred model is a structure* $(\mathsf{W}, \mathsf{S}, \mathsf{R}, \mathbf{a}, \nu, \tau, \mathbf{F})$ *where*

- W *is a set of possible worlds;*
- S *is a function giving for each* w *a set of possible worlds,* $\mathsf{S}^w \subseteq \mathsf{W}$*;*
- R *is a function giving for each* w*, a relation* $\mathsf{R}^w \subseteq \mathsf{S}^w \times \mathsf{S}^w$*;*
- $\mathbf{a}$ *is a function giving the actual world* $\mathbf{a}^w$ *of the model labelled by* w*;*
- ν *is an assignment function* $\nu^w(\mathfrak{q}_0) \subseteq \mathsf{S}^w$*, for each atomic* $\mathfrak{q}_0$*;*
- τ *is the semantical identifying function* $\tau : \mathsf{W} \to \mathbf{I}$*.* $\tau(w) = i$ *means that the model* $(\mathsf{S}^w, \mathsf{R}^w, \mathbf{a}^w, \nu^w)$ *is a model in* $\mathscr{K}_i$*, we use* W_i *to denote the set of worlds of type* i*;*
- $\mathbf{F}$*, is the set of fibring functions* $\mathbf{\mathfrak{F}} : \mathbf{I} \times \mathsf{W} \mapsto \mathsf{W}$*. A fibring function* $\mathbf{\mathfrak{F}}$ *is a function giving for each* i *and each* $w \in \mathsf{W}$ *another point (actual world) in* W *as follows:*

$$\mathbf{\mathfrak{F}}_i(w) = \begin{cases} w & \text{if } w \in \mathsf{S}^{\mathfrak{M}} \text{ and } \mathfrak{M} \in \mathscr{K}_i \\ \text{a value in } \mathsf{W}_i, & \text{otherwise} \end{cases}$$

such that if $w \neq w'$ *then* $\mathbf{\mathfrak{F}}_i(w) \neq \mathbf{\mathfrak{F}}_i(w')$*. It should be noted that fibring happens when* $\tau(w) \neq i$*. Satisfaction is defined as follows with the usual truth tables for Boolean connectives:*

$$w \models \mathfrak{q}_0 \text{ iff } \nu(w, \mathfrak{q}_0) = 1, \text{ where } \mathfrak{q}_0 \text{ is an atom}$$
$$w \models \Box_i \varphi \text{ iff } \begin{cases} w \in \mathfrak{M} \text{ and } \mathfrak{M} \in \mathscr{K}_i \text{ and } \forall w'(w \mathsf{R} w' \to w' \models \varphi), \text{or} \\ w \in \mathfrak{M}, \text{ and } \mathfrak{M} \notin \mathscr{K}_i \text{ and } \forall \mathbf{\mathfrak{F}} \in \mathbf{F}, \mathbf{\mathfrak{F}}_i(w) \models \Box_i \varphi. \end{cases}$$

We say the model satisfies φ *iff* $w_0 \models \varphi$.

A fibred model for $\blacktriangledown_{\mathbf{I}}^{\mathfrak{F}}$ can be generated from fibring the semantics for the modal logics $\blacktriangledown_i, i \in \mathbf{I}$. The detailed construction is given in [19]. Also, to accommodate the interaction axioms specific constraints need to be given on the fibring function. In [15] we outline the specific conditions required on the fibring function to accommodate axiom schemas of the type $G^{a,b,c,d}$:[3]. We do not want to get into the details here as the main theme of this paper is with regard to tableau based proof techniques for fibred logics.

What we want to point out here, however, is that the fibring construction given in [15,19] works for normal (multi-)modal logics as well as non-normal modal logics.

3 Multimodal Tableaux

In the previous sections we showed that agent logics are usually normal multimodal logics with a set of interaction axioms and introduced general techniques like fibring to explain such combined systems. In this section, before getting into the details related to the constructs needed for a tableau calculus for a fibred/combined logic, we outline with an example two types of tableau systems (*graph* & *path*) that can be used to reason about the knowledge/beliefs of agents in a multi-agent setting. Having done that, in the next section, we describe how to uniformly construct a sound and complete tableau calculus for the combined logic from calculi for the component logics.

Example 2. (The Friends Puzzle) [3] Consider the agents Peter, John and Wendy with modalities $\Box_p, \Box_j$, and $\Box_w$. John and Peter have an `appointment`. Suppose that Peter knows the `time` of appointment. Peter knows that John knows the `place` of their appointment. Wendy knows that if Peter knows the `time` of appointment, then John knows that too (since John and Peter are friends). Peter knows that if John knows the `place` and the `time` of their appointment, then John knows that he has an `appointment`. Peter and John satisfy the axioms T and 4. Also, if Wendy knows something then Peter knows the same thing (suppose Wendy is Peter's wife) and if Peter knows that John knows something then John knows that Peter knows the same thing.

The Knowledge/belief base for Example 2 can be formally given as follows;

1. $\Box_p time$	A_1 $T_p : \Box_p \varphi \to \varphi$
2. $\Box_p \Box_j place$	A_2 $4_p : \Box_p \varphi \to \Box_p \Box_p \varphi$
3. $\Box_w(\Box_p time \to \Box_j time)$	A_3 $T_j : \Box_j \varphi \to \varphi$
4. $\Box_p \Box_j (place \wedge time \to appointment)$	A_4 $4_j : \Box_j \varphi \to \Box_j \Box_j \varphi$
	A_5 $I_{wp} : \Box_w \varphi \to \Box_p \varphi$
	A_6 $S_{pj} : \Box_p \Box_j \varphi \to \Box_j \Box_p \varphi$

Fig. 2. Knowledge base related to the Friend's puzzle.

So we have a modal language consisting of three modalities $\Box_p, \Box_j$ and $\Box_w$ denoting respectively the agents Peter, John and Wendy and characterised by the set $A = \{A_i \mid i = 1, \ldots, 6\}$ of interaction axioms. Suppose now that one wants to show that each of the friends knows that the other one knows that he has an appointment, i.e., one wants to prove

[3] $G^{a,b,c,d} \Diamond_a \Box_b \varphi \to \Box_c \Diamond_d \varphi$.

∧-rules	$\dfrac{\sigma\ \varphi\wedge\psi}{\begin{array}{c}\sigma\ \varphi\\ \sigma\ \psi\end{array}}$	$\dfrac{\sigma\ \neg(\varphi\vee\psi)}{\begin{array}{c}\sigma\ \neg\varphi\\ \sigma\ \neg\psi\end{array}}$	$\dfrac{\sigma\ \neg(\varphi\rightarrow\psi)}{\begin{array}{c}\sigma\ \varphi\\ \sigma\ \psi\end{array}}$		For any prefix σ
∨-rules	$\dfrac{\sigma\ \varphi\vee\psi}{\sigma\ \varphi \mid \sigma\ \psi}$	$\dfrac{\sigma\ \neg(\varphi\wedge\psi)}{\sigma\ \neg\varphi \mid \sigma\ \neg\psi}$	$\dfrac{\sigma\ \varphi\rightarrow\psi}{\sigma\ \neg\varphi \mid \sigma\ \neg\psi}$		For any prefix σ
¬¬-rules	$\dfrac{\sigma\neg\neg\varphi}{\sigma\varphi}$				For any prefix σ
◇-rules	$\dfrac{\sigma\ \Diamond_i\varphi}{\sigma.n_i\ \varphi}$	$\dfrac{\sigma\ \neg\Box_i\varphi}{\sigma.n_i\ \neg\varphi}$			if the prefix $\sigma.n_i$ is new to the branch ($i\in\{1,\ldots,m\}$)
□-rules	$\dfrac{\sigma\ \Box_i\varphi}{\sigma.n_i\ \varphi}$	$\dfrac{\sigma\ \neg\Diamond_i\varphi}{\sigma.n_i\ \neg\varphi}$			If the prefix $\sigma.n_i$ already occurs on the branch ($i\in\{1,\ldots,m\}$)
$T_p$rules:	$\dfrac{\sigma\ \Box_p\varphi}{\sigma\ \varphi}$	$\dfrac{\sigma\ \neg\Diamond_p\varphi}{\sigma\ \neg\varphi}$	$\dfrac{\sigma\ \varphi}{\sigma\ \Diamond_p\varphi}$		
$T_j$rules:	$\dfrac{\sigma\ \Box_j\varphi}{\sigma\ \varphi}$	$\dfrac{\sigma\ \neg\Diamond_j\varphi}{\sigma\ \neg\varphi}$	$\dfrac{\sigma\ \varphi}{\sigma\ \Diamond_j\varphi}$		
4_prules:	$\dfrac{\sigma\ \Box_p\varphi}{\sigma.n_p^*\Box_p\varphi}$	$\dfrac{\sigma\ \neg\Diamond_p\varphi}{\sigma.n_p^*\Box_p\neg\varphi}$	$\dfrac{\sigma.n_p\ \Diamond_p\varphi}{\sigma\ \Diamond_p\varphi}$	$\dfrac{\sigma.n_p\ \neg\Box_p\varphi}{\sigma\ \Diamond_p\neg\varphi}$	
4_jrules:	$\dfrac{\sigma\ \Box_j\varphi}{\sigma.n_j^*\Box_j\varphi}$	$\dfrac{\sigma\ \neg\Diamond_j\varphi}{\sigma.n_j^*\Box_j\neg\varphi}$	$\dfrac{\sigma.n_j\ \Diamond_j\varphi}{\sigma\ \Diamond_j\neg\varphi}$	$\dfrac{\sigma.n_j\ \neg\Box_j\varphi}{\sigma\ \Diamond_j\neg\varphi}$	
I_{wp}rules:	$\dfrac{\sigma\ \Box_w\varphi}{\sigma.n_p^*\varphi}$	$\dfrac{\sigma\ \neg\Diamond_w\varphi}{\sigma.n_p^*\neg\varphi}$	$\dfrac{\sigma.n_p\ \varphi}{\sigma\ \Diamond_w\varphi}$		
S_{pj}rules:	$\dfrac{\sigma\ \Box_p\Box_j\varphi}{\sigma.n_j^*\Box_p\varphi}$	$\dfrac{\sigma\ \neg\Diamond_p\Diamond_j\varphi}{\sigma.n_j^*\Box_p\neg\varphi}$	$\dfrac{\sigma.n_j\ \Diamond_p\varphi}{\sigma\ \Diamond_p\Diamond_j\varphi}$	$\dfrac{\sigma.n_j\ \neg\Box_p\varphi}{\sigma\ \Diamond_p\Diamond_j\neg\varphi}$	

$(*)$ prefix already occurs on the branch

Fig. 3. Tableau rules corresponding to the Friend's Puzzle.

$$\Box_j\Box_p appointment \wedge \Box_p\Box_j appointment \tag{2}$$

is a theorem of the knowledge-base. The tableaux rules for a logic corresponding to the Friends puzzle are given in Fig.3 [17], and the tableaux proof for (2) is given in Fig.4 [17]. The tableaux in Fig.4. is a prefixed tableau [7] where the accessibility relations are encoded in the structure of the name of the worlds. Such a representation is often termed as a *path* representation. We show the proof of the first conjunct and the proof runs as follows. Item 1 is the negation of the formula to be proved; 2, 3, 4 and 5 are from Example 2; 6 is from 1 by a ◇-rule; 7 is from 6 by an S_{pj}-rule; 8 is from 7 by a ◇-rule; 9 is from 8 by a ◇-rule; 10 is from 5 by a □-rule; 11 is from 10 by a □-rule. 12 and 24 are from 11 by a ∨-rule; 13 and 16 are from 12 by a ∨-rule; 14 is from 3 by a □-rule; 15 is from 14 by a □-rule; the branch closes by 13 and 15; 17 is from 4 by an I_{wp}-rule; 18 and 22 are from 17 by a ∨-rule; 19 is from 18 by a ◇-rule; 20 is from 2 by a 4_p-rule; 21 is from 20 by a □-rule; the branch closes by 19 and 21; 23 is from 22 by a □-rule; the branch closes by 16 and 23; by 9 and 24 the remaining branch too closes.

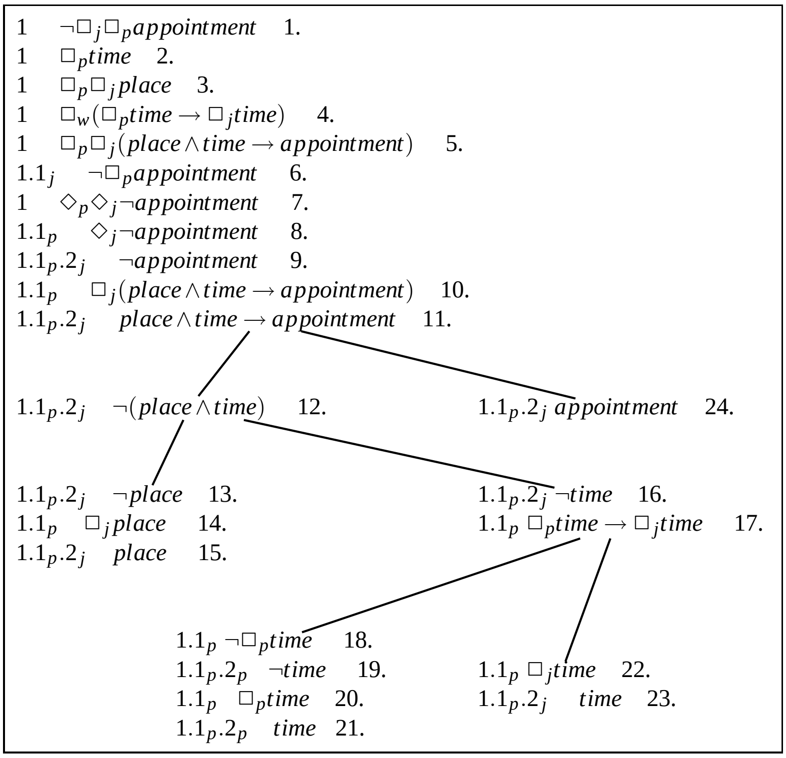

Fig. 4. Proof of $\Box_j\Box_p$ appointment using *path* representation

In a similar manner the tableaux proof for (2) using a *graph* representation where the accessibility relations are represented by means of an explicit and separate graph of named nodes is given in Fig.6. Each node is associated with a set of prefixed formulae and choice allows any inclusion axiom to be interpreted as a *rewriting rule* into the path structure of the graph. The proof uses the rules given in Fig.5. which is often referred to as the Smullyan-Fitting uniform notation. We will be using this notation in the next section for our **KEM** tableaux system. The proof for (2) as given in [3] runs as follows. Steps 1-4 are from Fig.2 and 5 is the first conjunct of (2). Using π-rule we get items 6 and 7 (from 5) and 8 and 9 (from 6). We get 10 from 7 using axiom A_6 in Fig.2 and ρ-rule in Fig.5. Similarly 11 is from 9 via A_6 and ρ-rule. By making use of the ν-rule in Fig.5 we get 12 (from 4 and 10) and 13 (from 12 and 11). 14a and 14b are from 13 using β-rule (“a” and “b” denote the two branches created by the application of β-rule). Branch “a” (14a) closes with 8. Applying β-rule again we get 15ba and 15bb from 14b (“ba” and “bb” denote the two branches created by the application of β-rule). Applying ν-rule we get 16ba (from 3 and 10) and 17ba (from 16ba and 11). Branch “ba” closes because of 15ba and 17ba. We get 16bb from 10 via axiom A_5 in Fig.2 and π-rule in Fig.5. Similarly from 2 and 16bb by using ν-rule we get 17bb. We get 18bba and 18bbb from 17bb by applying the β-rule (“bba” and “bbb” denote the branches created by the β-rule). By using ν-rule we get 19bba (from 18bba and 11). Branch “bba” (19bba) closes with 15bb. From 18bbb using π-rule we get 19bbb and 20bbb. From 10 and

20bbb via axiom A_2 (in Fig.2) and ρ-rule (in Fig.5) we get 21bbb. By applying ν-rule to 1 and 21bbb we get 22bbb as a result of which the branch “bbb” closes (22bbb and 19bbb).

(1) $$\frac{w:\alpha}{\begin{array}{c}w:\alpha_1\\ w:\alpha_2\end{array}}\ \alpha\text{-rule}$$

(2) $$\frac{w:\beta}{w:\beta_1 \mid w:\beta_2}\ \beta\text{-rule}$$

(3) $$\frac{w:\nu_i \quad w\rho_i w'}{w':\nu_i^0}\ \nu\text{-rule}$$ where $w\rho_i w'$ is *available* on the branch

(4) $$\frac{w:\pi_i}{\begin{array}{c}w':\pi_i^0\\ \hline w:\rho_i w'\end{array}}\ \pi\text{-rule}$$ where w' is *new* on the branch

(5) $$\frac{w\rho_{s_1}w_1\ldots w_{m-1}\rho_{s_m}w'}{\begin{array}{c}w\rho_{i_1}w'_1\\ \vdots\\ w'_{n-1}\rho_{i_n}w'\end{array}}\ \rho\text{-rule}$$ where $w'_1,\ldots,w'_{n-1}$ are *new* on the branch and $\Box_{i_1}\ldots\Box_{i_n}\varphi \rightarrow \Box_{i'_1}\ldots\Box_{i'_m}\varphi \in$ A

α	α_1	α_2
T $(\varphi\wedge\psi)$	**T** φ	**T** ψ
F $(\varphi\vee\psi)$	**F** φ	**F** ψ
F $(\varphi\rightarrow\psi)$	**T** φ	**F** ψ
F $(\neg\varphi)$	**T** φ	**T** φ

(a) $\wedge$-formulae

β	β_1	β_2
F $(\varphi\wedge\psi)$	**F** φ	**F** ψ
T $(\varphi\vee\psi)$	**T** φ	**T** ψ
T $(\varphi\rightarrow\psi)$	**F** φ	**T** ψ
T $(\neg\varphi)$	**F** φ	**F** φ

(b) $\vee$-formulae

ν_i	ν_0
T$\Box_i\varphi$	**T**φ
F$\Diamond_i\varphi$	**F** φ

(c) $\Box$-formulae

π_i	π_0
F$\Box_i\varphi$	**F**φ
T$\Diamond_i\varphi$	**T** φ

(d) $\Diamond$-formulae

Fig. 5. Tableaux rules based on uniform notation for propositional inclusion modal logics. [3].

It should be noted that axiom schemas like $A_1,\ldots,A_6$ of Example 2 given in Fig. 2 belong to the class of axioms called *inclusion axioms*. In particular they belong to axiom sets of the form, $\Box_{i_1}\ldots\Box_{i_n} \rightarrow \Box_{i'_1}\ldots\Box_{i'_m}$ $(i_n > 0, i'_m \geq 0)$, which in turn characterise the class of *normal modal logics* called *inclusion modal logics*. As shown in [3], for each axiom schema of the above type the corresponding *inclusion* property on the *accessibility relation* can be given as

$$\mathsf{R}_{i_1}\circ\mathsf{R}_{i_2}\circ\ldots\mathsf{R}_{i_n} \supseteq \mathsf{R}_{i'_1}\circ\mathsf{R}_{i'_2}\ldots\circ\mathsf{R}_{i'_m} \qquad (3)$$

where “$\circ$” denotes the relation composition $\mathsf{R}_{i_1}\circ\mathsf{R}_{i_2} = \{(w,w'')\in W\times W \mid \exists w'\in W$ such that $(w,w')\in\mathsf{R}_{i_1}$ and $(w',w'')\in\mathsf{R}_{i_2}\}$. This inclusion property is used to rewrite items 7. $(w_0\mathsf{R}_{john}w_1)$ and 9. $(w_1\mathsf{R}_{peter}w_2)$ of the proof given in Fig.6 so as to derive a new path $(w_0\mathsf{R}_{peter}w_3)$ and $(w_3\mathsf{R}_{john}w_2)$ as in items 10. and 11. The corresponding tableaux rule for this property is given as ρ-rule (5) in Fig.5. Also, the type of interaction axiom schemas of Example 2 involves the interaction between the *same mental attitude* of *different agents*. There is also another type where there is interaction between *different mental attitudes* of the *same agent*. The interaction axioms given in (1) is of the later type. In the coming sections we will show that the **KEM** tableau can deal with both types of interaction axioms.

1.	$w_0 : \mathbf{T}\Box_p time$	14b.	$w_2 : \mathbf{F}(\text{place} \wedge \text{time})$
2.	$w_0 : \mathbf{T}\Box_w(\Box_p time \rightarrow \Box_j time)$	15ba.	$w_2 : \mathbf{F}\ \text{place}$
3.	$w_0 : \mathbf{T}\Box_p\Box_j place$	16ba.	$w_3 : \mathbf{T}\Box_j place$
4.	$w_0 : \mathbf{T}\Box_p\Box_j(place \wedge time \rightarrow appointment)$	17ba.	$w_2 : \mathbf{T}\text{place}$
5.	$w_0 : \mathbf{F}\Box_j\Box_p appointment$		$\times$
6.	$w_1 : \mathbf{F}\Box_p appointment$	15bb.	$w_2 : \mathbf{F}time$
7.	$w_0 \mathsf{R}_{john} w_1$	16bb.	$w \mathsf{R}_{wife} w_3$
8.	$w_2 : \mathbf{F}\ appointment$	17bb.	$w_3 : \mathbf{T}(\Box_p time \rightarrow \Box_j time)$
9.	$w_1 \mathsf{R}_{peter} w_2$	18bba.	$w_3 : \mathbf{T}\ \Box_j time$
10.	$w_0 \mathsf{R}_{peter} w_3$	19bba.	$w_2 : \mathbf{T}time$
11.	$w_3 \mathsf{R}_{john} w_2$		$\times$
12.	$w_3 : \mathbf{T}\Box_j(place \wedge time \rightarrow appointment)$	18bbb.	$w_3 : \mathbf{F}\Box_p time$
13.	$w_2 : \mathbf{T}\ (place \wedge time \rightarrow appointment)$	19bbb.	$w_4 : \mathbf{F}time$
14a.	$w_2 : \mathbf{T}\ appointment$	20bbb	$w_3 \mathsf{R}_{peter} w_4$
	$\times$	21bbb.	$w_0 \mathsf{R}_{peter} w_4$
		22bbb.	$w_4 : \mathbf{T}\ time$
			$\times$

Fig. 6. Proof of $\Box_j\Box_p$ using *graph* representation.

As pointed out in [3], the main difference between the two types of tableaux, (graph and path), is in the use of ν-rule. In the case of *path* representation one needs to use a specific ν-rule for each logic as can be seen from Fig.3. These rules code the properties of the accessibility relations so as to express complex relations between prefixes depending on the logic. Whereas in the case of *graph* representation the accessibility relations are given explicitly. Also, it has been pointed out in [3] that the approach based on path representation can be used only for some subclasses of inclusion axioms and therefore difficult to extend the approach to the whole class of multi-modal systems.

4 Labelled Tableau for Fibred MMA Logic

In this section we show how to adapt **KEM**, a labelled modal tableaux system, to deal with the fibred combination of multimodal agent logics. In labelled tableaux systems, the object language is supplemented by labels meant to represent semantic structures (possible worlds in the case of modal logics). Thus the formulas of a labelled tableaux system are expressions of the form $A : i$, where A is a formula of the logic and i is a label. The interpretation of $A : i$ is that A is true at (the possible world(s) denoted by) i.

KEM's inferential engine is based on a combination of standard tableaux linear expansion rules and natural deduction rules supplemented by an analytic version of the cut rule. In addition it utilises a sophisticated but powerful label formalism that enables the logic to deal with a large class of modal and non-classical logics. Furthermore the label mechanism corresponds to fibring and thus it is possible to define tableaux systems for multi-modal logic by a seamless combination of the (sub)tableaux systems for the component logics of the combination.

It is not possible in this paper to give a full presentation of **KEM** for fully fledged multimodal agent logics supplemented with the interaction axioms given in Example 2. (for a comprehensive presentation see [10]). Accordingly we will limit ourselves to a single modal operator for each agent and we will show how to characterise the axioms and the interaction of example 2.

4.1 Label Formalism

KEM uses *Labelled Formulas* (L-formulas for short), where an L-formula is an expression of the form $A : i$, where A is a wff of the logic, and i is a label. For fibred **MMA** (from now on **FMMA**) we need to have labels for various modalities (belief, desire, intention) for each agent. However, as we have just explained we will consider only one modality and thus will have only labels for the agents.

The set of atomic labels, $\Im_1$, is then given as

$$\Im_1 = \bigcup_{i \in Agt} \Phi^i,$$

where Agt is the set of agents. Every Φ^i is partitioned into two (non-empty) sets of atomic labels: $\Phi^i_C = \{w^i_1, w^i_2, \dots\}$ the set of constants of type i, and $\Phi^i_V = \{W^i_1, W^i_2, \dots\}$ the set of variables of type i. We also add a set of auxiliary un indexed atomic labels Φ^A, again partitioned into variables $\Phi^A_V = \{W_1, W_2, \dots\}$ and constants $\Phi^A_C = \{w_1, w_2, \dots\}$, that will be used in unifications and proofs.

Definition 1 (labels) *A label $u \in \Im$ is either (i) an atomic label, i.e., $u \in \Im_1$ or (ii) a path term (u', u) where (iia) $u' \in \Phi_C \cup \Phi_V$ and (iib) $u \in \Phi_C$ or $u = (v', v)$ where (v', v) is a label.*

As an intuitive explanation, we may think of a label $u \in \Phi_C$ as denoting a world (a *given* one), and a label $u \in \Phi_V$ as denoting a set of worlds (*any* world) in some Kripke model. A label $u = (v', v)$ may be viewed as representing a path from v to a (set of) world(s) v' accessible from v (the world(s) denoted by v).

For any label $u = (v', v)$ we shall call v' the *head* of u, v the *body* of u, and denote them by $h(u)$ and $b(u)$ respectively. Notice that these notions are recursive (they correspond to projection functions): if $b(u)$ denotes the body of u, then $b(b(u))$ will denote the body of $b(u)$, and so on. We call each of $b(u)$, $b(b(u))$, etc., a *segment* of u. The length of a label u, $\ell(u)$, is the number of atomic labels in it. $s^n(u)$ will denote the segment of u of length n and we shall use $h^n(u)$ as an abbreviation for $h(s^n(u))$. Notice that $h(u) = h^{\ell(u)}(u)$. Let u be a label and u' an atomic label. We use $(u'; u)$ as a notation for the label (u', u) if $u' \neq h(u)$, or for u otherwise. For any label $u, \ell(u) > n$, we define the *counter-segment-n* of u, as follows (for $n < k < \ell(u)$):

$$c^n(u) = h(u) \times (\cdots \times (h^k(u) \times (\cdots \times (h^{n+1}(u), w_0))))$$

where w_0 is a dummy label, i.e., a label not appearing in u (the context in which such a notion occurs will tell us what w_0 stands for). The counter-segment-n defines what remains of a given label after having identified the segment of length n with a 'dummy' label w_0. The appropriate dummy label will be specified in the applications where such a notion is used. However, it can be viewed also as an independent atomic label. In the context of fibring w_0 can be thought of as denoting the actual world obtained via the fibring function from the world denoted by $s^n(u)$.

Example 3. Given the label $u = (w_4^i, (W_3^k, (w_3^j, (W_2^j, w_1^j))))$, according to the above definitions its length $\ell(u)$ is 5, the head $h(u)$ is w_4^i, the body $b(t)$ is $(W_3^k, (w_3^j, (W_2^j, w_1^j)))$, the segment of length 3 is $s^3(u) = (w_3^j, (W_2^j, w_1^j))$, and the relative counter-segment-3 is $c^3(u) = (w_4^i, (W_3^k, w_0))$, where $w_0 = s^3(u) = (w_3^j, (W_2^j, w_1^j))$.

To clarify the notion of counter-segment, which will be used frequently in the course of the present work, we present, in the following table the list of the segments of u in the left-hand column and the relative counter-segments in the right-hand column.

$s^1(u) = w_1$	$c^1(u) = (w_4^i, (W_3^k, (w_3^j, (W_2^j, w_0))))$
$s^2(u) = (W_2^j, w_1^j)$	$c^2(u) = (w_4^i, (W_3^k, (w_3^j, w_0)))$
$s^3(u) = (w_3^j, (W_2^j, w_1^j))$	$c^3(u) = (w_4^i, (W_3^k, w_0))$
$s^4(u) = (W_3^k, (w_3^j, (W_2^j, w_1^j)))$	$c^4(u) = (w_4^i, w_0)$
$s^5(u) = u$	$c^5(u) = w_0$

So far we have provided definitions about the structure of the labels without regard to the elements they are made of. The following definitions will be concerned with the type of world symbols occurring in a label.

We say that a label u is *i-preferred* iff $h(u) \in \Phi^i$; a label u is *i-pure* iff each segment of u of length $n > 1$ is i-preferred. Thus when we consider the label u of Example 3 then u is i-preferred, $b(u)$ is k-preferred and $s^3(u)$ is j-pure and consequently k-preferred. We will use $\Im^i$, $i \in Agt$, for the set of i-pure labels.

4.2 Label Unifications

The basic mechanism of **KEM** is its logic dependent label unification. In the same way as each modal logic is characterised by a combination of modal axioms (or semantic conditions on the model), **KEM** defines a unification for each modality and axiom/semantic condition and then combines them in a recursive and modular way. In particular we use what we call unification to determine whether the denotation of two labels have a non empty intersection, or in other terms whether two labels can be mapped to the same possible world in the possible worlds semantics.

The second key issue is the ability to split labels and to work with parts of labels. The mechanism permits the encapsulation of operations on sub-labels. This is an important feature that, in the present context, allows us to correlate unifications and fibring functions. Given the modularity of the approach the first step of the construction is to define unifications (pattern matching for labels) corresponding to the single modality in the logic we want to study.

Every unification is built from a basic unification defined in terms of a substitution $\rho : \Im_1 \mapsto \Im$ such that:

$$\begin{aligned} \rho :\ & \mathbf{1}_{\Phi_C} \\ & \Phi_V^i \mapsto \Im^i \text{ for every } i \in Agt \\ & \Phi_V^A \mapsto \Im \end{aligned}$$

The substitution ρ is such that every constant is mapped to itself, while the mapping of variables depends on their types. For a variable of type i, $i \in Agt$, the variable is mapped to an arbitrary i-pure label, but this restriction is dropped for auxiliary variables, thus any label can be associated to an auxiliary variable.

Accordingly, we have that two atomic ("world") labels u and v σ-unify iff there is a substitution ρ such that $\rho(u) = \rho(v)$. We shall use $[u;v]\sigma$ both to indicate that there is a substitution ρ for u and v, and the result of the substitution. The σ-unification is extended to the case of composite labels (path labels) as follows:

$$[i;j]\sigma = k \text{ iff } \exists \rho : h(k) = \rho(h(i)) = \rho(h(j)) \text{ and } b(k) = [b(i);b(j)]\sigma$$

Clearly σ is symmetric, i.e., $[u;v]\sigma$ iff $[v;u]\sigma$. Moreover this definition offers a flexible and powerful mechanism: it allows for an independent computation of the elements of the result of the unification, and variables can be freely renamed without affecting the result of a unification, and the σ-unification of any two labels can be computed in linear time [13].

We are now ready to introduce the unifications corresponding to the modal operators at hand, i.e., $\Box_w$, $\Box_j$ and $\Box_p$ characterised by the axioms in Figure 2. We can capture the relationship between $\Box_w$ and $\Box_p$ by extending the substitution ρ by allowing a variable of type w to be mapped to labels of the same type and of type p.

$$\rho^w(W^w) \in \Im^w \cup \Im^p$$

Then the unification σ^w is obtained from the basic unification σ by replacing ρ with the extended substitution ρ^w. This procedure must be applied to all pairs of modalities $\Box_1, \Box_2$ related by the interaction axiom $\Box_1\varphi \to \Box_2\varphi$.

For the unifications for $\Box_p$ and $\Box_j$ (σ^p and σ^j) we assume that the labels involved are i-pure. First we notice that these two modal operators are **S4** modalities thus we have to use the unification for this logic.

$$[u;v]\sigma^{S4} = \begin{cases} [u;v]\sigma^D \text{ if } \ell(u) = \ell(v) \\ [u;v]\sigma^T \text{ if } \ell(u) < \ell(v), h(u) \in \Phi_C \\ [u;v]\sigma^4 \text{ if } \ell(u) < \ell(v), h(u) \in \Phi_V \end{cases} \tag{4}$$

It is worth noting that the conditions on axiom unifications are needed in order to provide a deterministic unification procedure. The σ^T and σ^4 are defined as follows:

$$[u;v]\sigma^T = \begin{cases} [s^{\ell(v)}(u);v]\sigma \text{ if } \ell(u) > \ell(v), \text{ and} \\ \qquad \forall n \geq \ell(v), [h^n(u);h(v))]\sigma = [h(u);h(v)]\sigma \\ [u;s^{\ell(u)}(v)]\sigma \text{ if } \ell(u) > \ell(v), \text{ and} \\ \qquad \forall n \geq \ell(u), [h(u);h^n(v)]\sigma = [h(u);h(v)]\sigma \end{cases}$$

The above unification allows us to unify to labels such that the segment of the longest with the length of the other label and the other label unify, provided that all remaining elements of the longest have a common unification with the head of the shortest. This means that after a given point the head of the shortest is always included in its extension, and thus it is accessible from itself, and consequently we have reflexivity.

Example 4. For the notion of σ^T-unification, take for example the labels

$$u = (w_3^p, (W_1^p, w_1^p)) \qquad\qquad v = (w_3^p, (W_2^p, (w_2^p, w_1^p)))$$

Here $[W_2^p; w_3^p]\sigma = [w_3^p; w_3^p]\sigma$. Then the two labels σ^T-unify to $(w_3^p, (w_2^p, w_1^p))$. This intuitively means that the world w_3^p, accessible from a sub-path $s(v) = (W_2^p, (w_2^p, w_1^p))$, after the deletion of W_2^p from v, is accessible from any path u which turns out to denote the same world(s) as $s(u)$; in fact the step from w_2^p to W_2^p is irrelevant because of the reflexivity relation of the model.

$$[u;v]\sigma^4 = \begin{cases} c^{\ell(u)}(v) \text{ if } \ell(v) > \ell(u), h(u) \in \Phi_V \text{ and} \\ \qquad w_0 = [u; s^{\ell(u)}(v)]\sigma \\ c^{\ell(v)}(u) \text{ if } \ell(u) > \ell(v), h(v) \in \Phi_V \text{ and} \\ \qquad w_0 = [s^{\ell(v)}(u); v]\sigma \end{cases}$$

In this case we have that the shortest label unifies with the segment with the same length of the longest and that the head of the shortest is variable. A variable stands for all worlds accessible from the predecessor of it. Thus, given transitivity every element extending the segment with length of the shortest is accessible from this point.

Example 5. For the notion of σ^4-unification, take for example the labels

$$u = (W_3^j, (w_2^j, w_1^j)) \qquad\qquad v = (w_5^j, (w_4^j, (w_3^j, (W_2^j, w_1^j))))$$

Here $s^{\ell(u)}(v) = (w_3^j, (W_2^j, w_1^j))$. Then u and v σ^4-unify to $(w_5^j, (w_4^j, (w_3^j, (w_2^j, w_1^j))))$ since $[u; s^{\ell(u)}(v)]\sigma = [(W_3^j, (w_2^j, w_1^j)); (w_3^j, (W_2^j, w_1^j))]\sigma$. This intuitively means that all the worlds accessible from a sub-path $s^{\ell(u)}(v)$ of v are accessible from any path u which leads to the same world(s) denoted by $s^{\ell(u)}(v)$. Here W_3^j stands for the set of worlds accessible from w_2^j; Then w_3^j, after the unification of (w_2^j, w_1^j) and (W_2^j, w_1^j), is one of such worlds. w_4^j is accessible from w_3^j and, via transitivity, from w_2^j. The same for w_5^j.

Then a unification corresponding to axiom A6 from Example 2 is

$$[u;v]\sigma^{S_{p,j}} = \begin{cases} c^m(v) \text{ if } h(u) \in \Phi_V^j \text{ and } c^n(v) \text{ is } p\text{-pure, and} \\ \qquad h^{\ell(u)-1}(u) \in \Phi_V^p \text{ and } c^m(s^n(v)) \text{ is } j\text{-pure, and} \\ \qquad w_0 = [s^{\ell(u)-2}(u); s^m(v)]\sigma \\ c^m(u) \text{ if } h(v) \in \Phi_V^j \text{ and } c^n(u) \text{ is } p\text{-pure, and} \\ \qquad h^{\ell(v)-1}(v) \in \Phi_V^p \text{ and } c^m(s^n(u)) \text{ is } j\text{-pure and} \\ \qquad w_0 = [s^m(u); s^{\ell(v)-2}(v)]\sigma \end{cases}$$

This unification allows us to unify two labels such that in one we have a sequence of a variable of type p followed by a variable of type j and a label where we have a sequence of labels of type j followed by a sequence of labels of type p.

Example 6. As an example of $\sigma^{S_{p,j}}$-unification consider the labels

$$u = (W_2^j, (W_2^p, (w_2^p, w_1^w))) \qquad\qquad v = (w_3^p, (W_4^j, (w_3^j, (W_1^p, w_1^w))))$$

Given the two labels u and v we have that the last two elements of u are, in this order, a variable of type j, $h(u) \in \Phi_V^j$, and a variable of type p, $h^3(u) \in \Phi_V^p$. Thus we have to check that there are two sequences of p-pure and j-pure labels in v. Clearly $c^4(v) = (w_3^p, w_0)$ is p-pure and $c^2(s^4(u)) = (W_4^j, (w_3^j, w_0))$ is j-pure. Thus the last thing to do is to verify whether $s^2(v)$ and $s^{\ell(u)-2}(u) = s^2(u)$ σ-unify; it is immediate to verify that $[s^2(u); s^2(v)]\sigma$. Thus $[u;v]\sigma^{S_{p,j}} = (w_3^p, (W_4^j, (w_3^j, (w_2^p, w_1^w))))$.

The unification for $\Box_p$ and $\Box_j$ are just the combination of the three unifications given above. Finally the unification for the logic **L** defined by the axioms A1–A6 is obtained from the following recursive unification

$$[u;v]\sigma_L = \begin{cases} [u;v]\sigma^{w,p,j} \\ [c^m(u); c^n(v)]\sigma^{w,p,j} \text{ where } w_0 = [s^m(u); s^n(v)]\sigma_L \end{cases}$$

$\sigma^{w,p,j}$ is the simple combination of the unifications for the three modal operators. Having accounted for the unification we now give the inference rules used in **KEM** proofs.

Example 7. To illustrate the σ_L-unification consider the labels

$$u = (w_3^j, (w_2^j, (W_1^j, (W_1^p, w_1^w)))) \qquad v = (W_2^j, (w_1^p, (w_1^j, w_1^w)))$$

A simple inspection of the label shows that none of the other unifications can be used here to unify the two labels. The only way is to split the labels in appropriate segments and counter-segments and then use the σ_L-unification. We split the labels as follows $c^3(u) = (w_3^j, (w_2^j, w_0))$ and $c^2(v) = (W_2^j, w_0)$. Now it is easy to verify that $[c^3(u); c^{2(v)}]\sigma^4$. On the other hand we have that $s^3(u) = (W_1^j, (W_1^p, w_1^w))$ and $s^2(v) = (w_1^j, w_1^w)$, and $[s^3(u); s^2(u)]\sigma^{S_{p,j}}$. Thus we can identify w_0 with $[s^3(u); s^2(u)]\sigma^{S_{p,j}}$, and then $[u;v]\sigma_L$.

Notice that the unification mechanism, in particular the splitting of the labels into segments and counter-segments and the use of subunifications for them follows the same idea as fibring. As the fibring function takes us to a new model specific to the modal operator we evaluate, the decomposition of the unification allows us to reduce the unification of complex labels with atomic labels of multiple types to unifications of pure labels, where we can use the unifications for the component logics.

4.3 Inference Rules

For the inference rules we use the Smullyan-Fitting unifying notation [7].

$$\frac{\alpha : u}{\begin{array}{c}\alpha_1 : u \\ \alpha_2 : u\end{array}}(\alpha) \qquad \frac{\begin{array}{c}\beta : u \\ \beta_i^c : v\end{array} (i=1,2)}{\beta_{3-i} : [u;v]\sigma_L}(\beta)$$

The α-rules are just the familiar linear branch-expansion rules of the tableau method. The β-rules are nothing but natural inference patterns such as Modus Ponens, Modus

Tollens and Disjunctive syllogism generalised to the modal case. In order to apply such rules it is required that the labels of the premises unify and the label of the conclusion is the result of their unification.

$$\frac{\nu^i : u}{\nu^i_0 : (W^i_n, u)}(\nu) \qquad\qquad \frac{\pi^i : u}{\pi^i_0 : (w^i_n, u)}(\pi)$$

where W^i_n is a new label. The ν and π rules are the normal expansion rule for modal operators of labelled tableaux with free variable. The intuition for the ν rule is that if $\Box_i A$ is true at u, then A is true at all worlds accessible via R_i from u, and this is the interpretation of the label (W^i_n, u); similarly if $\Box_i A$ is false at u (i.e., $\neg \mathbf{B}A$ is true), then there must be a world, let us say w^i_n accessible from u, where $\neg A$ is true. A similar intuition holds when u is not i-preferred, but the only difference is that we have to make use of the fibring function instead of the accessibility relation

$$\frac{}{A : u \quad | \quad \neg A : u}(PB) \qquad\qquad \frac{\begin{array}{c}A : u\\ \neg A : v\end{array}}{\times}[\text{ if } [u;v]\sigma_L](PNC)$$

The *Principle of Bivalence* (PB) represents the semantic counterpart of the cut rule of the sequent calculus (intuitive meaning: a formula A is either true or false in any given world). PB is a zero-premise inference rule, so in its unrestricted version can be applied whenever we like. However, we impose a restriction on its application. PB can be only applied w.r.t. immediate sub-formulas of unanalysed β-formulas, that is β formulas for which we have no immediate sub-formulas with the appropriate labels in the tree. The *Principle of Non-Contradiction* (PNC) states that two labelled formulas are $\sigma_{\mathbf{L}}$-complementary when the two formulas are complementary and their labels $\sigma_{\mathbf{L}}$-unify.

It is possible to show that the resulting calculus is sound and complete for the class of (fibred) models corresponding to the (fibred) logic determined by the axiom in Fig. 2; see [10] for the techniques needed to prove the results. Notice that the Knowledge base of Fig 2 does not specify whether the modal operators are normal or not. While this could be a problem for other combination techniques and tableaux systems, this does not affect fibring, and **KEM**. It is possible to differentiate normal and non-normal modal logic in **KEM** based on additional conditions on the substitution function ρ, see [14].

4.4 Proof Search

Let $\Gamma = \{X_1, \ldots, X_m\}$ be a set of formulas. Then $\mathscr{T}$ is a **KEM**-*tree for* Γ if there exists a finite sequence $(\mathscr{T}_1, \mathscr{T}_2, \ldots, \mathscr{T}_n)$ such that (i) $\mathscr{T}_1$ is a 1-branch tree consisting of $\{X_1 : t_1, \ldots, X_m : t_m\}$; (ii) $\mathscr{T}_n = \mathscr{T}$, and (iii) for each $i < n$, $\mathscr{T}_{i+1}$ results from $\mathscr{T}_i$ by an application of a rule of **KEM**. A branch θ of a **KEM**-tree $\mathscr{T}$ of L-formulas is said to be $\sigma_{\mathbf{L}}$-*closed* if it ends with an application of *PNC*, open otherwise. As usual with tableau methods, a set Γ of formulas is checked for consistency by constructing a **KEM**-tree for Γ. Moreover we say that a formula A is a **KEM**-*consequence of a set of formulas* $\Gamma = \{X_1, \ldots, X_n\}$ ($\Gamma \vdash_{\mathbf{KEM}(L)} A$) if a **KEM**-tree for $\{X_1 : u_1, \ldots, X_n : u_n, \neg A : v\}$ is closed using the unification for the logic L, where $v \in \Phi^A_C$, and $u_i \in \Phi^A_V$. The intuition behind this definition is that A is a consequence of Γ when we take Γ as a set of global assumptions [7], i.e., true in every world in a Kripke model.

We now describe a systematic procedure for **KEM** by defining the following notions. Given a branch θ of a **KEM**-tree, we call an L-formula $X : u$ *E-analysed in* θ if either (i) X is of type α and both $\alpha_1 : t$ and $\alpha_2 : u$ occur in θ; or (ii) X is of type β and one of the following conditions is satisfied: (a) if $\beta_1^C : v$ occurs in θ and $[u;v]\sigma$, then also $\beta_2 : [u;v]\sigma$ occurs in θ, (b) if $\beta_2^C : v$ occurs in θ and $[u;v]\sigma$, then also $\beta_1 : [u;v]\sigma$ occurs in θ; or (iii) X is of type μ and $\mu_0 : (u', u)$ occurs in θ for some appropriate u' of the right type, not previously occurring in θ. We call a branch θ of a **KEM**-tree *E-completed* if every L-formula in it is E-analysed and it contains no complementary formulas which are not $\sigma_{\mathbf{L}}$-complementary. We say a branch θ of a **KEM**-tree *completed* if it is E-completed and all the L-formulas of type β in it either are analysed or cannot be analysed. We call a **KEM**-tree *completed* if every branch is completed.

The following procedure starts from the 1-branch, 1-node tree consisting of $\{X_1 : u, \ldots, X_m : v\}$ and applies the inference rules until the resulting **KEM**-tree is either closed or completed. At each stage of proof search (i) we choose an open non completed branch θ. If θ is not E-completed, then (ii) we apply the 1-premise rules until θ becomes E-completed. If the resulting branch θ' is neither closed nor completed, then (iii) we apply the 2-premise rules until θ becomes E-completed. If the resulting branch θ' is neither closed nor completed, then (iv) we choose an L-formula of type β which is not yet analysed in the branch and apply *PB* so that the resulting *LS*-formulas are $\beta_1 : u'$ and $\beta_1^C : u'$ (or, equivalently $\beta_2 : u'$ and $\beta_2^C : u'$), where $u = u'$ if u is restricted (and already occurring when $h(u) \in \Phi_C$), otherwise u' is obtained from u by instantiating $h(u)$ to a constant not occurring in u; (v) ("Modal *PB*") if the branch is not E-completed nor closed, because of complementary formulas which are not $\sigma_{\mathbf{L}}$-complementary, then we have to see whether a restricted label unifying with both the labels of the complementary formulas occurs previously in the branch; if such a label exists, or can be built using already existing labels and the unification rules, then the branch is closed, (vi) we repeat the procedure in each branch generated by *PB*.

It is possible to give termination conditions for **KEM**-trees resulting in canonical trees. Essentially a canonical tree will examine each combination of a formula and label only once, and it produces finitely many formulas and labels. Thus, if one proves that an unification for an axiom terminates and satisfies some reasonable algebraic properties, then the **KEM**-trees for that axiom terminate. Thus the proof search in a **KEM** tableau for a combination of logics $\mathbf{L}_1, \ldots, \mathbf{L}_n$ terminates if each $\mathbf{L}_i$ has a terminating **KEM** search procedure, and connecting axioms have unifications satisfying some safe conditions. A thorough analysis of the termination conditions for **KEM** and fibring is beyond the scope of this paper and it is left for future research. In particular we want to study the extent of the termination conditions for canonical trees and label structures developed in [12].

Fig.7. shows a **KEM** tableaux proof using the inference rules in section 4.3 and following the proof search mentioned above to solve the first conjunct of (2). The proof goes as follows; 1. is the negation of the formula to be proved. The formulas in 2–5 are the global assumptions of the scenario and accordingly they must hold in every world of every model for it. Hence we label them with a variable W_0 that can unify with every

1. $\mathbf{F}\Box_j\Box_p appt$	w_0	9. $\mathbf{T}(place \wedge time \rightarrow appt)$	(W_1^j, W_1^p, w_0)
2. $\mathbf{T}\Box_p\Box_j(place \wedge time \rightarrow appt)$	W_0	10. $\mathbf{F} place \wedge time$	(w_1^p, w_1^j, w_0)
3. $\mathbf{T}\Box_w(\Box_p time \rightarrow \Box_j time)$	W_0	11. $\mathbf{T}\Box_p time \rightarrow \Box_j time$	(W_1^w, w_0)
4. $\mathbf{T}\Box_p\Box_j place$	W_0	12. $\mathbf{T}\Box_j place$	(W_2^p, w_0)
5. $\mathbf{T}\Box_p time$	W_0	13. $\mathbf{T} place$	(W_2^j, W_2^p, w_0)
6. $\mathbf{F}\Box_p appt$	(w_1^j, w_0)	14. $\mathbf{F} time$	(w_1^p, w_1^j, w_0)
7. $\mathbf{F} appt$	(w_1^p, w_1^j, w_0)	15. $\mathbf{T}\Box_p time$	(w_1^j, w_0)
8. $\mathbf{T}\Box_j(place \wedge time \rightarrow appt)$	(W_1^p, w_0)	16. $\mathbf{T} time$	(W_3^p, w_1^j, w_0)
		$\times$	

Fig. 7. Proof of $\Box_j\Box_p$ using **KEM** representation.

other label. This is used to derive 12. from 11. and 5. using a β-rule, and for introducing 15.; 6. is from 1., and 7. from 6. by applying π rule. Similarly we get 8. from 2., 9. from 8. using ν rule. 10. comes from 9. and 7. through the use of modus tollens. Applying ν rule twice we can derive 11. from 3. as well as 13. from 12. Through propositional reasoning we get 14. from 10. and by using ν rule on 15. we get 16. (14. and 16.) are complementary formulas and this results in a closed tableaux because the labels in 14. and 16. unify, denoting that the contradiction holds *in the same world*.

5 Concluding Remarks

In this paper we have argued that multimodal logics of agents (**MMA**) can be explained in terms of fibring as combination of simpler modal logics. Then we have outlined three labelled tableaux systems (path, graph and unification). For each of the method we have seen how they can deal with the Friend's puzzle as a way to evaluate their features.

In the path approach, as mentioned earlier, we need to use specific ν-rule for each logic whereas **KEM** uses only one ν-rule and unification is logic dependent. The graph approach on the other hand does not require, in general, any new rule, since it uses the semantic structure to propagate formulas to the appropriate labels. It is then suitable for an approach based on fibring, since the relationships between two labels can be given in terms of fibring. But then the advantage of **KEM** over the graph approach is in the full flexibility of the application of the rules. In the graph based approach one need to apply the π-rules (or the ρ-rule) before the ν-rules whereas in **KEM** no such restrictions exist. Also **KEM** is more suited for fibring because the mechanism it uses to check and manipulate labels during model generation is close to semantic fibring.

KEM, in general similar to the graph approach, does not need logic dependent rules, however, similar to the path approach, it needs logic dependent label unifications. We have seen that the label algebra can be seen as a form of fibring [10], thus simple fibring does not require special attention in **KEM**; therefore it allows for a seamless composition of (sub)tableaux for modal logics. The label algebra contrary to the graph reasoning mechanism is not based on first order logic and thus can deal with complex structure and is not limited to particular fragment. Indeed **KEM** has been proved able to deal with complex label schema for non-normal modal logics in a uniform way [14] as well as other intensional logics such as conditional logics [2]. For these reasons we believe that **KEM** offers a suitable framework for constructing decision procedures for multi-modal logic for multi-agent systems. As we only described the static fragment of **MMA** logics, (no temporal evolution was considered), the future work is to extend the tableaux framework so as to accommodate temporal modalities.

References

1. A. Artosi, P. Benassi, G. Governatori, and A. Rotolo. Shakespearian modal logic: A labelled treatment of modal identity. In *Advances in Modal Logic*, volume 1. CSLI, 1998.
2. A. Artosi, G. Governatori, and A. Rotolo. Labelled tableaux for non-monotonic reasoning: Cumulative consequence relations. *Journal of Logic and Computation*, 12(6):1027–1060, 2002.
3. M. Baldoni. *Normal Multimodal Logics: Automatic Deduction and Logic Programming Extension*. PhD thesis, Universita degli Studi di Torino, Italy, 1998.
4. P. Blackburn and M. de Rijke. Zooming in, zooming out. *Journal of Logic, Language and Information*, 1996.
5. A. S. d' Avila Garcez and D. M. Gabbay. Fibring neural networks. In *AAAI-2004*, pages 342–347. AAAI/MIT Press, 2004.
6. R. Fagin, J. Y. Halpern, Y. Moses, and M. Y. Vardi. *Reasoning About Knowledge*. The MIT Press, 1995.
7. M. Fitting. *Proof Methods for Modal and Intuitionistic Logics*. Reidel, Dordrecht, 1983.
8. D. M. Gabbay. *Fibring Logics*. Oxford University Press, Oxford, 1999.
9. D. M. Gabbay and G. Governatori. Dealing with label dependent deontic modalities. In *Norms, Logics and Information Systems. New Studies in Deontic Logic*. IOS Press, 1998.
10. D. M. Gabbay and G. Governatori. Fibred modal tableaux. In *Labelled Deduction*. Kluwer academic Publishers, 2000.
11. G. Governatori. Labelled tableau for multi-modal logics. In *TABLEAUX*, volume 918, pages 79–94. Springer, 1995.
12. G. Governatori. *Un modello formale per il ragionamento giuridico*. PhD thesis, CIRSFID, University of Bologna, 1997.
13. G. Governatori. On the relative complexity of modal tableaux. *Electronic Notes in Theoretical Computer Science*, 78:36–53, 2003.
14. G. Governatori and A. Luppi. Labelled tableaux for non-normal modal logics. In *AI*IA 99: Advances in AI*, LNAI-1792, pages 119–130, Berlin, 2000. Springer.
15. G. Governatori, V. Padmanabhan, and A. Sattar. On Fibring Semantics for BDI Logics. In *Logics in Artificial Intelligence: (JELIA-02), Italy*, LNAI-2424. Springer, 2002.
16. M. Kracht and F. Wolter. Properties of independently axiomatizable bimodal logics. *The Journal of Symbolic Logic*, 56(4):1469–1485, 1991.
17. J. W. Llyod. Modal higher-order logic for agents. http://users.rsise.anu.edu.au/ jwl/beliefs.pdf, 2004.
18. A. Lomuscio. *Information Sharing Among Ideal Agents*. PhD thesis, School of Computer Science, University of Brimingham, 1999.
19. Vineet Padmanabhan. *On Extending BDI Logics*. PhD thesis, School of Information Technology, Griffith University, Brisbane, Australia, 2003.
20. A. S. Rao and M. P. Georgeff. Formal models and decision procedures for multi-agent systems. Technical note 61, Australian Artificial Intelligence Institute, 1995.
21. A. Sernadas, C. Sernadas, and A. Zanardo. Fibring modal first-order logics: Completeness preservation. *Logic Journal of the IGPL*, 10(4):413–451, 2002.
22. F. Wolter. Fusions of modal logics revisited. In *Advances in Modal Logic*, volume 1. CSLI Lecture notes 87, 1997.

23. M. Wooldridge. *Reasoning about Rational Agents*. The MIT Press, 2000.
24. A. Zanardo, A. Sernadas, and C. Sernadas. Fibring: Completeness preservation. *Journal of Symbolic Logic*, 66(1):414–439, 2001.