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Minimum Specific Energy and Critical Flow Conditions in Open Channels

by H. Chanson[1]

Abstract : In open channels, the relationship between the specific energy and the flow depth exhibits a minimum, and the corresponding flow conditions are called critical flow conditions. Herein they are re-analysed on the basis of the depth-averaged Bernoulli equation. At critical flow, there is only one possible flow depth, and a new analytical expression of that characteristic depth is developed for ideal-fluid flow situations with non-hydrostatic pressure distribution and non-uniform velocity distribution. The results are applied to relevant critical flow conditions : e.g., at the crest of a spillway. The finding may be applied to predict more accurately the discharge on weir and spillway crests.

Keywords: Open channel flow; Critical flow; Minimum specific energy; Discharge coefficient; Weirs and spillways.

INTRODUCTION

Considering an open channel flow, the free-surface is always at atmospheric pressure, the driving force of the fluid motion is gravity, and the fluid is incompressible and Newtonian. Newton's law of motion leads to the Navier-Stokes equations. The integration of the Navier Stokes equations along a streamline, assuming that the fluid is frictionless, the volume force potential (i.e. gravity) is independent of the time, for a steady flow (i.e. $\partial V/\partial t = 0$) and an incompressible flow (i.e. ρ = constant), yields :

$$\frac{P}{\rho} + g * z + \frac{V^2}{2} = \text{constant} \tag{1}$$

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where ρ is the fluid density, g is the gravity acceleration, z is the elevation aligned along the vertical direction and positive upwards), P is the pressure, v is the velocity (Henderson 1966, Liggett 1993, Chanson 1999,2004). Equation (1) is the local form of the Bernoulli equation.

In this study, the singularity of the depth-averaged Bernoulli principle for open channel flow is detailed : i.e, the critical flow conditions. Detailed expressions of the critical flow properties are derived for the general case of non-hydrostatic and non-uniform velocity distributions. The results are then applied to the rating curve of weir crest acting as discharge meter.

Application to open channel flows

In open channels, it is common to use the depth-averaged Bernoulli equation within the frame of relevant assumptions (e.g. Liggett 1993) :

$$H = \frac{1}{d} * \int_0^d \left(\frac{v(y)^2}{2 * g} + z(y) + \frac{P(y)}{\rho * g} \right) * dy = \beta * \frac{V^2}{2 * g} + \Lambda * d + z_o = \text{constant} \qquad (2)$$

where H is the depth-averaged total head, z_o is the bottom elevation, d is the flow depth, β is the momentum correction coefficient (or Boussinesq coefficient), V is the depth-averaged velocity :

$$V = \frac{1}{d} * \int_0^d v * dy \qquad (3)$$

y is the distance normal to the channel bed and Λ is a pressure correction coefficient defined as :

$$\Lambda = \frac{1}{2} + \frac{1}{d} * \int_0^d \frac{P(y)}{\rho * g * d} * dy \qquad (4)$$

For a flat channel assuming a hydrostatic pressure distribution, the pressure correction coefficient Λ is unity and the depth-averaged total head H equals :

$$H = \beta * \frac{V^2}{2 * g} + d + z_o = E + z_o \qquad (5)$$

where E is the depth-averaged specific energy.

CRITICAL FLOW CONDITIONS

Considering a short smooth transition, and assuming a constant flow rate, the relationship between the specific energy ($E = H - z_o$) and flow depth exhibits a characteristic shape (e.g. Fig. 1). For a given cross-section shape, the specific energy is minimum for flow conditions (d_c, V_c) called the critical flow conditions. The concept of critical flow conditions was first developed by Bélanger (1828) as the location where $d = Q^2/(g*A^2)$ for a flat channel, where Q is the flow rate and A is the flow cross-section area. It was associated with the idea of minimum specific energy by Bakhmeteff (1912,1932). Both Bélanger and Bakhmeteff developed the concept of critical flow in relation with the singularity of the backwater equation for $d = Q^2/(g*A^2)$ (i.e. critical flow conditions).

A typical situation with minimum specific energy is shown on Figure 2 where critical flow conditions occur at the point of maximum invert elevation : i.e., the weir crest. Assuming a smooth frictionless overflow, the depth-averaged Bernoulli equation states :

$$H = (z_o)_{crest} + E_{min} = (z_o)_{crest} + \Lambda_{crest} * d_c + \beta_{crest} * \frac{V_c^2}{2 * g} \qquad (6a)$$

where H is the upstream total head, $(z_o)_{crest}$ is the crest elevation, E_{min} is the minimum specific energy (Fig. 2), β_{crest} and Λ_{crest} are respectively the momentum and pressure correction coefficients at the crest. Note that, at the crest, the y-direction is exactly vertical, the streamlines are curved, the pressure distribution is not hydrostatic and the velocity distribution is not uniform.

For a rectangular channel, the continuity and Bernoulli equations give two equations in terms of the critical flow depth and depth-averaged velocity :

$$q = V_c * d_c = C_D * \sqrt{g} * \left(\frac{2}{3} * E_{min}\right)^{3/2} \qquad \text{Continuity equation (7)}$$

$$E_{min} = \Lambda_{crest} * d_c + \beta_{crest} * \frac{V_c^2}{2 * g} \qquad \text{Bernoulli equation (6b)}$$

where q is the discharge per unit width and C_D is a dimensionless discharge coefficient.

If the minimum specific energy E_{min} and flow rate per unit width q are known parameters, the combination of Equations (7) and (6b) gives a third order polynomial equation in terms of the dimensionless flow depth at crest (i.e. d_c/E_{min}) :

$$\left(\frac{d_c}{E_{min}}\right)^3 - \left(\frac{d_c}{E_{min}}\right)^2 * \frac{1}{\Lambda_{crest}} + \frac{1}{2} * \frac{\beta_{crest} * C_D^2}{\Lambda_{crest}} * \left(\frac{2}{3}\right)^3 = 0 \qquad (8)$$

Equation (80) has one, two or three real solutions depending upon the sign of the discriminant Δ (see App. I) :

$$\Delta = \left(\frac{1}{3 * \Lambda_{crest}}\right)^6 * 4 * \beta_{crest} * C_D^2 * \Lambda_{crest}^2 * \left(\beta_{crest} * C_D^2 * \Lambda_{crest}^2 - 1\right) \qquad (9)$$

The solution of Equation (8) gives an expression of the dimensionless critical depth as a function of the pressure correction coefficient, momentum correction coefficient and discharge coefficient. Further the Bernoulli equation implies for an overflow :

$$0 \le \Lambda_{crest} * \frac{d_c}{E_{min}} \le 1 \qquad (10)$$

The detailed solutions of Equation (8) are developed in Appendix I. Meaningful solutions exist only for $\Delta \le 0$. These solutions are plotted in Figure 3 as $d_c/E_{min}*\Lambda_{crest}$ versus $\beta_{crest}*C_D^2*\Lambda_{crest}^2$. The analytical results are compared with the re-analysis of experimental data (Fawer 1937, Vo 1992), flow net analysis (Fawer 1937) and detailed analytical solution (Fawer 1937). The experimental flow conditions are listed in Table 1 and one complete data set is presented in Figure 4.

Overall the data (Fig. 3) follow closely the solution S3 :

$$\frac{d_c}{E_{min}} = \frac{2}{3 * \Lambda_{crest}} * \frac{1}{2} * \left(1 - \cos(\delta/3) + \sqrt{3 * (1 - (\cos(\delta/3))^2)}\right) \qquad \text{Solution S3} \quad (11)$$

where :

$$\cos\delta = 1 - 2 * \beta_{crest} * C_D^2 * \Lambda_{crest}^2 \qquad (12)$$

It is unclear why experimental data do not follow the solution S1, although it is conceivable that S1 might be an unstable solution.

For a hydrostatic pressure distribution ($\Lambda_{crest} = 1$) and an uniform velocity distribution ($\beta_{crest} = 1$), the discharge coefficient is unity and the flow depth at crest equals :

$$\frac{d_c}{E_{min}} = \frac{2}{3} \qquad (13)$$

For example, an ideal fluid flow above a broad-crested weir.

DISCUSSION

The analysis of Equation (8) yields basic conclusions. First the product $\beta_{crest}*C_D{}^2*\Lambda_{crest}{}^2$ must be less or equal than unity :

$$\beta_{crest} * C_D{}^2 * \Lambda_{crest}{}^2 \le 1 \tag{14}$$

As the momentum correction coefficient β_{crest} is equal to or larger than unity, Equation (14) implies that discharge coefficients larger than unity may be obtained only when the crest pressure distribution is less than hydrostatic.

Second, for transition from sub- to super-critical flow, experimental results (Fig. 3) indicate that:

$$\frac{d_c}{E_{min}} * \Lambda_{crest} \le \frac{2}{3} \tag{15}$$

That is, the dimensionless critical flow depth equals the solution S3 (App. I). That the streamline curvature implies usually ($\Lambda_{crest} < 1$) at a weir crest. Therefore Equation (15) does not imply $d_c/E_{min} < 2/3$. The result is well-known for overflow circular weirs (e.g. Vo 1992, Chanson and Montes 1998).

Third the pressure and velocity distributions at the crest can be predicted using ideal fluid flow theory (i.e. potential flow theory). Hence β_{crest} and Λ_{crest} may be calculated theoretically because the entire flow field may be predicted assuming an ideal fluid with irrotational flow motion. Assuming a two-dimensional flow, the vertical distributions of pressure and velocity may be accurately determined numerically by a computational method, graphically by a flow net analysis, or analytically for simple geometries (e.g. Fawer 1937, Rouse 1946, Jaeger 1956, Vallentine 1969). Then the relevant parameters become d_c/E_{min} and C_D, or q, d_c and E_{min}.

Application : spillway crest as a discharge meter

If the momentum and pressure correction coefficients may be predicted theoretically, the continuity and Bernoulli equations imply that a spillway crest may be used as an accurate discharge meter using the solution of Equation (8). In practice the upstream head above spillway crest (i.e. E_{min}) is known and the unknown is the flow rate q. If the flow depth at the crest (i.e. the critical flow depth d_c) is measured, Equation (11) and Figure 3 provide the value of the discharge coefficient C_D satisfying Equation (8), and the flow rate is deduced from Equation (7). In contrast with the free overfall, a spillway crest is a better discharge meter because the correction coefficients and the discharge coefficient are close to unity.

Considering a circular weir (R = 4.5 m) with an upstream vertical weir in a rectangular channel, the upstream head above crest is 10.6 m and the measured depth on the crest is 7.4 m. Compute the flow rate. Fawer (1937) derived the flow net solution of this case. His graphical result based upon 8 stream tubes predicted: $\beta_{crest} = 1.0$ and $\Lambda_{crest} = 0.5$. For these values, Equation (11) and Figure 3 imply that $C_D = 1.44$. Fawer conducted the corresponding experiment that yielded : $C_D = 1.4$, while his detailed potential flow solution gave $C_D = 1.38$. Both results are close to the analytical prediction (Eq. (11) and Fig. 3).

CONCLUSION

Critical flow conditions in open channel are re-analysed using the depth-averaged form of the Bernoulli equation. At critical flow, a new analytical expression of the critical flow depth is derived for ideal-fluid flows. The result is applied to critical flow situations with non-hydrostatic pressure distribution and non-uniform velocity distribution. It yields pertinent information on the flow properties at a weir crest. The findings may be applied to predict accurately the discharge at the crest at spillways and weirs, by combining Equation (11) and Figure 3 with simple ideal fluid flow theory (e.g. flow net analysis).

In real-fluid flows, boundary friction induces a flow region affected by shear and momentum exchange: i.e., a developing boundary layer. Converging and accelerating flow situations (e.g. spillway intake) have generally thin boundary layers. Present ideal fluid results may be applied to short transitions of real fluids to a satisfactory degree of approximation, but they are not applicable to long waterway : e.g., undular flow in culvert barrel.

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APPENDIX I. CRITICAL FLOW DEPTH

The continuity and Bernoulli equations give an expression for the critical flow depth as the solution of:

$$\left(\frac{d_c}{E_{min}}\right)^3 - \left(\frac{d_c}{E_{min}}\right)^2 * \frac{1}{\Lambda_{crest}} + \frac{1}{2} * \frac{\beta_{crest} * C_D^2}{\Lambda_{crest}} * \left(\frac{2}{3}\right)^3 = 0 \qquad \text{(I-1)}$$

Equation (I-1) has one, two or three real solutions depending upon the sign of the discriminant Δ :

$$\Delta = \frac{1}{\Lambda_{crest}^6} * \frac{4}{3^6} * \beta_{crest} * C_D^2 * \Lambda_{crest}^2 * \left(\beta_{crest} * C_D^2 * \Lambda_{crest}^2 - 1\right) \qquad \text{(I-2)}$$

For $\Delta > 0$, Equation (I-1) has only one real solution :

$$\frac{d_c}{E_{min}} * \Lambda_{crest} = \sqrt[3]{\frac{1}{27} * \left(1 - 2*\beta_{crest}*C_D^2*\Lambda_{crest}^2\right) + \sqrt{\Lambda_{crest}^6 * \Delta}} + \sqrt[3]{\frac{1}{27} * \left(1 - 2*\beta_{crest}*C_D^2*\Lambda_{crest}^2\right) - \sqrt{\Lambda_{crest}^6 * \Delta}} + \frac{1}{3} \qquad \text{(I-3)}$$

For $\Delta = 0$, the following condition holds :

$$\beta_{crest} * C_D^2 * \Lambda_{crest}^2 - 1 = 0 \qquad \text{(I-4)}$$

The only physical solution of Equation (I-1) is :

$$\frac{d_c}{E_{min}} * \Lambda_{crest} = \frac{2}{3} = \frac{2}{3} * \sqrt{\beta_{crest}} * C_D * \Lambda_{crest} \qquad \text{(I-5)}$$

For $\Delta = 0$, the second real solution is negative : $d_c/E_{min} = -1/(3*\Lambda_{crest})$.

For $\Delta < 0$ there are three real solutions :

$$\left(\frac{d_c}{E_{min}} * \Lambda_{crest}\right)_1 = \frac{2}{3} * \left(\frac{1}{2} + \cos(\delta/3)\right) \qquad \text{Solution S1 (I-6A)}$$

$$\left(\frac{d_c}{E_{min}} * \Lambda_{crest}\right)_2 = \frac{2}{3} * \frac{1 - \cos(\delta/3) - \sqrt{3 * (1 - (\cos(\delta/3))^2)}}{2} \qquad \text{Solution S2 (I-6B)}$$

$$\left(\frac{d_c}{E_{min}} * \Lambda_{crest}\right)_3 = \frac{2}{3} * \frac{1 - \cos(\delta/3) + \sqrt{3 * (1 - (\cos(\delta/3))^2)}}{2} \qquad \text{Solution S3 (I-6C)}$$

where :

$$\cos\delta = 1 - 2 * \beta_{crest} * C_D^2 * \Lambda_{crest}^2 \qquad \text{(I-7)}$$

Note that $\Delta < 0$ implies : $\beta_{crest}*C_D^2*\Lambda_{crest}^2 < 1$.

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APPENDIX III. NOTATION.

The following symbols are used in this paper :

A = flow cross-section area (m^2);

C_D = discharge coefficient;

d = flow depth (m) measured perpendicular to the channel bottom;

d_c = critical flow depth (m) : i.e., flow depth at minimum specific energy;

E = specific energy (m);

E_{min} = minimum specific energy (m);

g = gravity constant;

H = depth-averaged total head (m);

P = pressure (Pa);

Q = water discharge (m^3/s);

q = water discharge per unit width (m^2/s);

R = radius (m) or curvature of spillway crest;

t = time (s);

V = depth-averaged velocity (m/s);

V_c = critical flow velocity (m/s) : i.e., depth-averaged flow velocity at minimum specific energy;

v = local velocity (m/s);

W = channel width (m);

y = distance measured perpendicular to the channel bottom (m);

z = elevation (m) taken positive upwards;

z_o = bed elevation (m) taken positive upward;

β = momentum correction coefficient;

ρ = water density (kg/m^3);

Λ = pressure correction coefficient;

Subscript

c = critical flow conditions (i.e. at minimum specific energy);

crest = conditions at crest.

Table 1. Summary of re-analysed experimental flow measurements

Reference (1)	Configuration (2)	Measurements (3)	Remarks (4)
Fawer (1937)	Circular weir (R = 0.0325 m) Vertical upstream wall 3:2 downstream slope Weir height: 0.0315 and 0.3325 m	Invert pressure distributions. Vertical distributions of pressure and velocity (Pitot tube).	2.5 m long 0.303 m wide upstream flume.
Vo (1992)	Circular weir (R = 0.0095 to 0.1516 m) Upstream slope: 90º, 75º, 60º Downstream slope: 75º, 60º, 45º	Invert pressure distributions. Vertical distributions of pressure and velocity (LDV).	1.8 m long 0.254 m wide upstream flume. (Also Ramamurthy et al. 1992.)

Fig. 1 - Relationship between dimensionless specific energy E/d_c and dimensionless flow depth d/d_c in a smooth rectangular channel assuming a hydrostatic pressure distribution

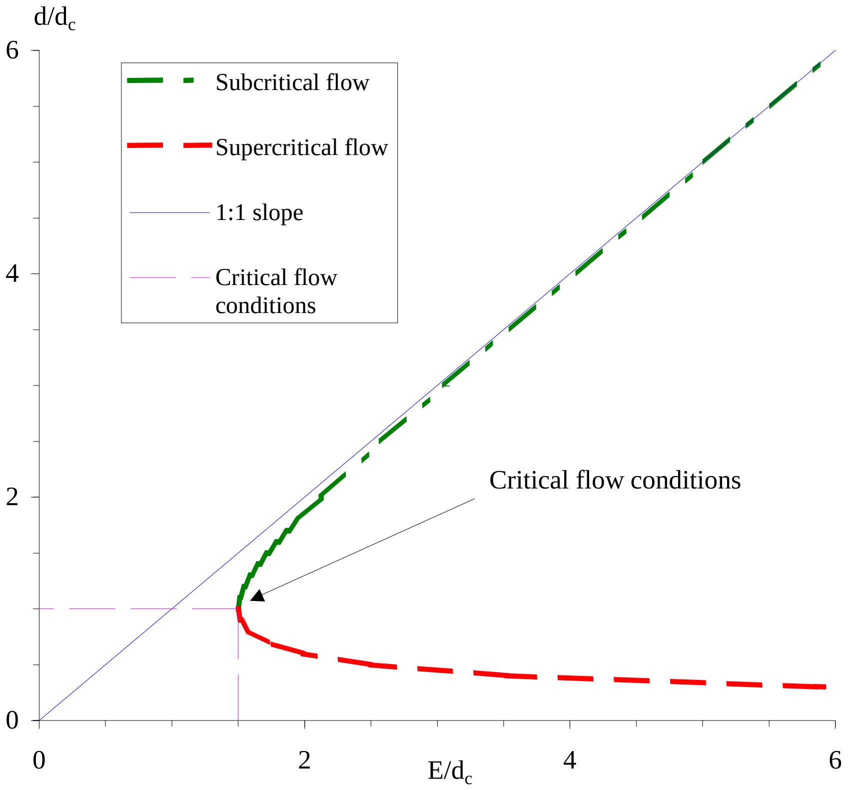

Fig. 2 - Critical flow conditions at a weir crest - Definition sketch

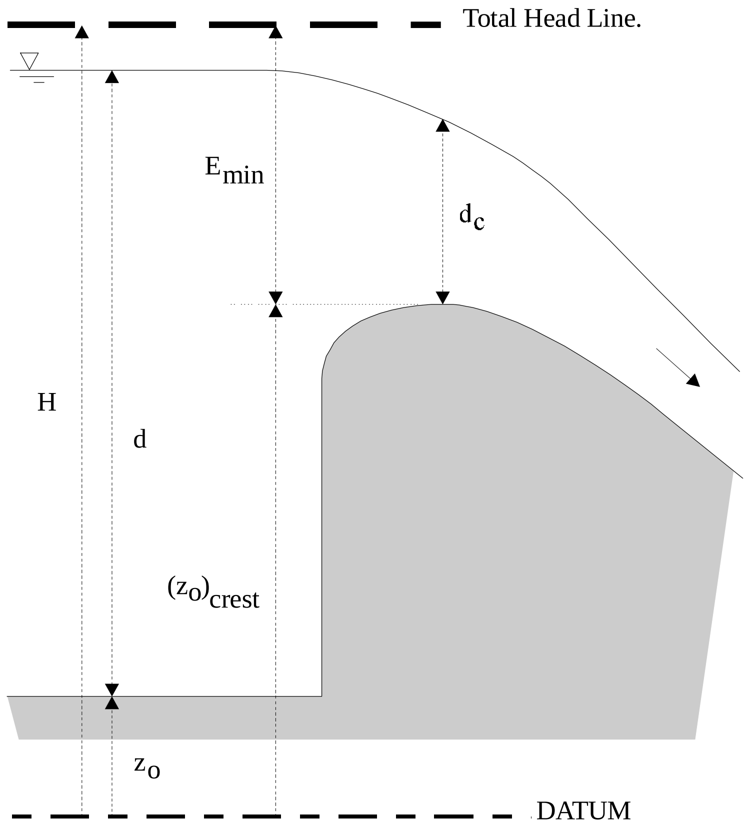

Fig. 3 - Dimensionless critical flow depth at the crest of a weir : $d_c/E_{min}*\Lambda_{crest}$ versus $\beta_{crest}*C_D^2*\Lambda_{crest}^2$ - Comparison with experimental data (Fawer 1937, Vo 1992), a flow net analysis with 8 stream tubes (Fawer 1937) and a detailed analytical solution (Fawer 1937)

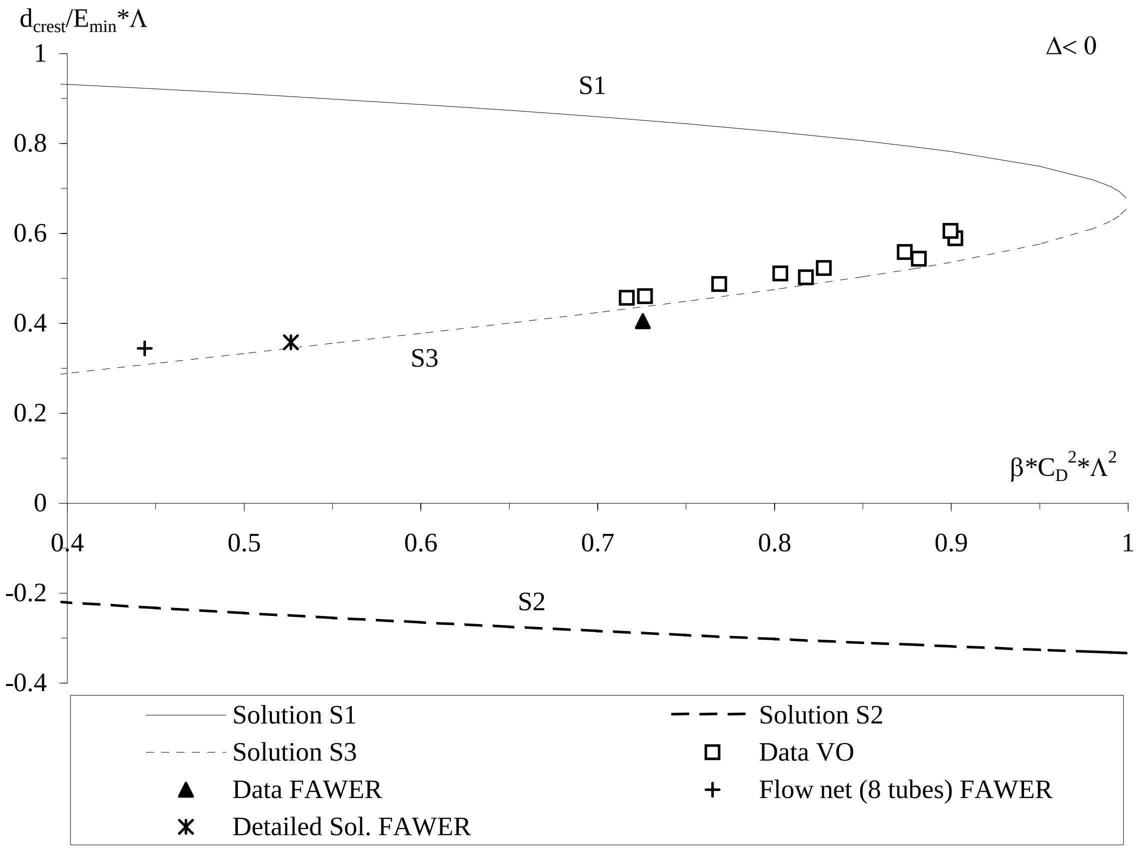

Fig. 4 - Dimensionless pressure and velocity distributions at the crest of a weir : $P/\rho*g*d_c$ and $V/\sqrt{g*E_{min}}$ versus d_c/E_{min} (Fawer 1937, d_c = 0.0537 m, E_{min} = 0.0768 m, R = 0.0325 m) - Comparison between experimental data, flow net analysis (8 stream tubes) and detailed analytical solution (ideal fluid)