A Near-Optimal Linear Crosstalk Canceler for Upstream VDSL

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***Abstract*—Crosstalk is the major source of performance degradation in VDSL. Several crosstalk cancelers have been proposed to address this. Unfortunately, they suffer from error propagation, high complexity, and long latency. This paper presents a simple, linear zero-forcing (ZF) crosstalk canceler. This design has a low complexity and no latency and does not suffer from error propagation. Furthermore, due to the well-conditioned structure of the VDSL channel matrix, the ZF design causes negligible noise enhancement. A lower bound on the performance of the linear ZF canceler is derived. This allows performance to be predicted without explicit knowledge of the crosstalk channels, which simplifies service provisioning considerably. This bound shows that the linear ZF canceler operates close to the single-user bound. Therefore, the linear ZF canceler is a low-complexity, low-latency design with predictable near-optimal performance. The combination of spectral optimization and crosstalk cancellation is also considered. Spectra optimization in a multiaccess channel generally involves a complex optimization problem. Since the linear ZF canceler decouples transmission on each line, the spectrum on each modem can be optimized independently, leading to a significant reduction in complexity.**

***Index Terms*—Crosstalk cancellation, diagonal dominance, digital subscriber lines (DSL), dynamic spectrum management, linear, reduced complexity, vectoring.**

I. Introduction

NEXT-GENERATION DSL systems such as VDSL aim at providing extremely high data rates, up to 52 Mbps in the downstream. Such high data rates are supported by operating over short loop lengths and transmitting in frequencies up to 12 MHz. Unfortunately, the use of such high-frequency ranges causes significant electromagnetic coupling between neighboring twisted pairs within a binder. This coupling creates interference, referred to as crosstalk, between the DSLs within a network. Over short loop lengths crosstalk is typically 10–15 dB larger than the background noise and is *the* dominant source of performance degradation.

In upstream communications, the receiving modems are often colocated at the central office (CO) or at an optical network unit (ONU) located at the end of the street. This allows joint reception of the signals transmitted on the different lines, thereby enabling crosstalk cancellation.

Several crosstalk canceler designs have been proposed. A structure based on decision feedback, which we refer to as the *decision feedback canceler* (DFC), has been shown to achieve close to the theoretical channel capacity [1]. It should be noted, however, that the claim of near-optimal performance is based on the assumption of error-free decisions. For this to be valid a perfect channel code must be used, which has infinite decoying complexity and delay [2]. In practice a suboptimal code will be used, which can lead to decision errors, error propagation and poor performance. Furthermore, decoying of each user's codeword must be done before decisions are fed back. This leads to a high computational complexity and a latency that grows with the number of users in the binder. In VDSL systems, the codewords are interleaved across the entire DMT block to add robustness against deep frequency nulls, which result from line properties such as bridge taps. Furthermore, the codeword may be interleaved across several DMT blocks to add robustness against impulse noise [3]. This means that the codewords are already quite long, and the latency is typically at the limit required for most applications. Binders can contain hundreds of lines. As a result, it is difficult to apply the DFC in real-time application such as voice over IP or video conferencing.

Other cancellation techniques use turbo coding principles to facilitate cancellation [4], [5] or exploit the cyclostationarity of crosstalk [6], [7]. The advantage of these methods is that they do not require signal coordination, and can instead be applied independently on each modem. Unfortunately, these techniques are extremely complex and give poor performance when more than one crosstalker exists. Other techniques use joint linear processing at both the transmit and receive side of the link [8], [9]. This requires colocation of both CO and customer premises (CP) modems, which is typically not possible since different customers are situated at different locations. Furthermore, it has been shown that the theoretical channel capacity is achievable with receiver-side coordination only, so using coordination on both ends of the link does not improve performance [10].

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In this paper, we present a simple, linear zero-forcing (ZF) crosstalk canceler. This design has a low complexity and no latency and does not suffer from error propagation. Furthermore, since it is based on a ZF criterion, it removes all crosstalk. Despite these advantages, it is well known that ZF criteria can lead to severe noise enhancement in ill-conditioned channels.

To address this concern, this paper analyzes the performance of the linear ZF canceler in the VDSL environment. It is shown that due to the well-conditioned structure of the VDSL channel matrix, ZF designs cause negligible noise enhancement. As a result, this simple linear structure achieves near-optimal performance. We develop bounds to show that the linear ZF canceler operates close to the single-user bound in VDSL channels. These bounds allow the performance of the linear ZF canceler to be predicted without explicit knowledge of the crosstalk channels, which simplifies service provisioning significantly.

The rest of this paper is organized as follows. The system model for a network of VDSL modems transmitting to a single CO/ONU is given in Section II. A property of the upstream VDSL channel, known as *column-wise diagonal dominance* (CWDD), is explored. As described in Section III, from an information theoretical perspective, the upstream VDSL channel is a multiaccess channel (MAC). This allows the single-user bound to be applied to upper bound the capacity of the channel. To address the problems of the DFC, Section IV describes a much simpler linear design, the linear ZF canceler, that has a low complexity and no latency and is free from error propagation. Section IV uses the CWDD property to formulate a lower bound on the performance of the linear ZF canceler. This bound shows that the linear canceler operates close to the single-user bound. Section V describes power-loading algorithms for use with the linear canceler. Existing power-loading algorithms for the MAC are complex, having a polynomial complexity in the number of lines. Application of the linear canceler decouples the power allocation problem between lines. As a result the PSD for each line can be found through a low-complexity water-filling procedure and this simplifies power allocation significantly. Section VI compares the performance of the different cancelers based on simulations.

II. System Model

Assuming that the modems are synchronized and discrete multitone (DMT) modulation is employed, we can model transmission independently on each tone

$$\mathbf{y}_k = \mathbf{H}_k \mathbf{x}_k + \mathbf{z}_k. \tag{1}$$

Synchronization is straightforward to implement when the receiving modems are colocated, which is the assumption we make here. The vector $\mathbf{x}_k \triangleq [x_k^1, \cdots, x_k^N]^T$ contains transmitted signals on tone k, where the tone index k lies in the range $1 \ldots K$. There are N lines in the binder and x_k^n is the signal transmitted onto line n at tone k. The vectors $\mathbf{y}_k$ and $\mathbf{z}_k$ have similar structures. The vector $\mathbf{y}_k$ contains the received signals on tone k. The vector $\mathbf{z}_k$ contains the additive noise on tone k and is comprised of thermal noise, alien crosstalk, RFI etc. The $N \times N$ matrix $\mathbf{H}_k$ is the crosstalk channel matrix on tone k. The element $h_k^{n,m} \triangleq [\mathbf{H}_k]_{n,m}$ is the channel from TX m to RX n on tone k. The diagonal elements of $\mathbf{H}_k$ contain the direct-channels whilst the off-diagonal elements contain the crosstalk channels. We denote the transmit PSD of user n on tone k as $s_k^n \triangleq \mathcal{E}\{|x_k^n|^2\}$.

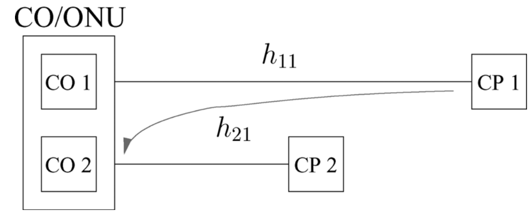

Fig. 1. Columnwise diagonal dominance $|h_{11}| \gg |h_{21}|$.

Since the receiving modems are colocated, the crosstalk signal transmitted from a disturber into a victim must propagate through the full length of the disturber's line. This is depicted in Fig. 1, where CP 1 is the disturber and CO 2 is the victim. The insulation between twisted pairs increases the attenuation. As a result, the crosstalk channel matrix $\mathbf{H}_k$ is CWDD, since on each column of $\mathbf{H}_k$ the diagonal element has the largest magnitude

$$|h_k^{n,m}| \ll |h_k^{m,m}|, \; \forall m \neq n. \tag{2}$$

CWDD implies that the crosstalk channel $h_k^{n,m}$ from a disturber m into a victim n is always weaker than the direct channel of the disturber $h_k^{m,m}$. The degree of CWDD can be characterized with the parameter α_k

$$|h_k^{n,m}| \leq \alpha_k \, |h_k^{m,m}|, \; \forall m \neq n. \tag{3}$$

Note that crosstalk cancellation is based on joint reception. As such it requires the colocation of receiving modems. So in all channels where crosstalk cancellation can be applied the CWDD property holds. CWDD has been verified through extensive measurement campaigns of real binders. In 99% of lines α_k is bounded

$$\alpha_k \leq K_{\text{xf}} \cdot f_k \cdot \sqrt{d_{\text{coupling}}} \tag{4}$$

where $K_{\text{xf}} = -22.5$ dB and f_k is the frequency on tone k in magahertz [11]. Here, d_{coupling} is the coupling length between the disturber and the victim in kilometers. The coupling length can be upper bounded by the longest line length in the binder. Hence

$$\alpha_k \leq K_{\text{xf}} \cdot f_k \cdot \sqrt{l_{\max}} \tag{5}$$

where $l_{\max}$ denotes the length of the longest line in the binder. To find a value for α_k that is independent of the particular binder configuration, $l_{\max}$ can be set to 1.2 km, which is the maximum deployment length for VDSL [11].[1] On typical lines α_k is then less than -11.3 dB. The following sections show that CWDD ensures a well-conditioned crosstalk channel matrix. This results in the near-optimality of the linear ZF canceler.

When VDSL modems are distributed from an ONU the noise on each line is typically spatially white, and we make this assumption for the remainder of this paper

$$\mathcal{E}\left\{\mathbf{z}_k\mathbf{z}_k^H\right\} = \sigma_k \mathbf{I}_N \tag{6}$$

where σ_k denotes the noise power on tone k. When VDSL modems are distributed from a CO the noise on each line may be correlated due to the presence of strong alien crosstalk. In this case, a noise prewhitening operation must be applied prior to crosstalk cancellation. This noise prewhitening may destroy the CWDD property of the channel matrix $\mathbf{H}_k$. In this case, the linear ZF canceler is no longer optimal, and more complex decision feedback structures must be employed [10]. Nevertheless, most VDSL deployments will occur from the ONU, where the assumption of spatially white noise is valid. The linear ZF canceler developed in this paper then provides a low-complexity, low-latency, near-optimal design.

III. Theoretical Capacity

Consider the *single-user bound*, which is the capacity achieved when only one user (CP modem) transmits and all receivers (CO modems) are used to detect that user. Using the single-user bound the capacity of user n on tone k is limited to

$$\begin{aligned} b_k^n &\le \Delta_f I\left(x_k^n;\mathbf{y}_k\right), \\ &= \Delta_f \log_2\left(1+\sigma_k^{-1}s_k^n \left\|\mathbf{h}_k^n\right\|^2\right) \end{aligned}$$

where $\mathbf{h}_k^n$ denotes the nth column of $\mathbf{H}_k$, Δ_f denotes the tone spacing, and s_k^n is the transmit power of user n on tone k. Here, we have assumed that the noise is Gaussian and spatially white and that the input distribution is Gaussian. Practical coding schemes are characterized by an signal-to-noise ratio (SNR)-gap to capacity Γ, which determines how closely the code comes to the theoretical capacity [13]. This limits the achievable data rate of user n on tone k to

$$b_k^n \le \Delta_f \log_2\left(1+\Gamma^{-1}\sigma_k^{-1}s_k^n \left\|\mathbf{h}_k^n\right\|^2\right).$$

The CWDD property (2) leads to the bound

$$\begin{aligned} \left\|\mathbf{h}_k^n\right\|_2^2 &= \left|h_k^{n,n}\right|^2 + \sum_{m\neq n}\left|h_k^{m,n}\right|^2, \\ &\le \left|h_k^{n,n}\right|^2\left[1+\alpha_k^2(N-1)\right]. \end{aligned} \tag{7}$$

[1]Standardization groups are currently considering the deployment of VDSL2 at lengths greater than 1.2 km [12]. However, at such distances, far-end crosstalk is no longer the dominant source of noise, and the benefits of far-end crosstalk cancellation are reduced considerably.

Hence, in CWDD channels, the achievable data rate[2] of user n on tone k, with a transmit power spectrum density (PSD) of s_k^n, is limited to

$$b_k^n \le b_{k,\text{bnd}}^n\left(s_k^n\right)$$

where

$$\begin{aligned} &b_{k,\text{bnd}}^n\left(s_k^n\right) \\ &\quad\triangleq \Delta_f \log_2\Big(1+\Gamma^{-1}\sigma_k^{-1}s_k^n\left|\mathbf{h}_k^{n,n}\right|^2\left[1+\alpha_k^2(N-1)\right]\Big). \end{aligned} \tag{8}$$

IV. Near-Optimal Linear Canceler

This section describes a simple linear crosstalk canceler. Unlike the DFC, this structure has low complexity and no latency and supports real-time applications. The structure is based on the ZF criterion, which leads to the following estimate of the transmitted vector:

$$\begin{aligned} \widehat{\mathbf{x}}_k &= \mathbf{H}_k^{-1}\mathbf{y}_k, \\ &= \mathbf{x}_k + \mathbf{H}_k^{-1}\mathbf{z}_k. \end{aligned} \tag{9}$$

Each user then experiences a crosstalk free channel, affected only by the filtered background noise.

It is well known that ZF designs lead to severe noise enhancement when the channel matrix $\mathbf{H}_k$ is ill conditioned. Fortunately, CWDD ensures that the channel matrix is well-conditioned; so the linear ZF canceler leads to negligible noise enhancement and each user achieves a data rate close to the single-user bound (8). This observation is made rigorous in the following theorem. Before stating the theorem, we first define the some notation. We use the following iterative definition for $A_{\max}^{(m)}$ and $B_{\max}^{(m)}$

$$\begin{bmatrix} A_{\max}^{(m)} \\ B_{\max}^{(m)} \end{bmatrix} \triangleq \begin{bmatrix} 1 & \alpha_k(m-1) \\ \alpha_k & \alpha_k(m-1) \end{bmatrix} \begin{bmatrix} A_{\max}^{(m-1)} \\ B_{\max}^{m-1} \end{bmatrix} \tag{10}$$

with $A_{\max}^{(1)} \triangleq 1$ and $B_{\max}^{(1)} \triangleq 0$. Note that the matrix

$$\lim_{\alpha_k\to 0}\begin{bmatrix} 1 & (m-1)\alpha_k \\ \alpha_k & (m-1)\alpha_k \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \tag{11}$$

Recall that α_k measures the degree of CWDD of the crosstalk channel matrix $\mathbf{H}_k$ and extremely CWDD channels $\alpha_k \to 0$. From (11) it follows that

$$\lim_{\alpha_k\to 0} A_{\max}^{(m)} = 1,\ \forall m \tag{12}$$

$$\lim_{\alpha_k\to 0} B_{\max}^{(m)} = 0,\ \forall m. \tag{13}$$

[2]Here, we assume the use of a suboptimal, practical coding scheme with an SNR-gap of Γ. If an optimal code is used with no delay limit then Γ will be unity.

We also define

$$A_{\min}^{(m)} \triangleq A_{\min}^{(m-1)} - \alpha_k (m-1) B_{\max}^{(m-1)} \tag{14}$$

with $A_{\min}^{(1)} \triangleq 1$. Note that

$$\lim_{\alpha_k \to 0} A_{\min}^{(m)} = 1, \ \forall m. \tag{15}$$

Now define the function

$$f(N,\alpha_k) \triangleq \left(\frac{A_{\max}^{(N-1)}}{A_{\min}^{(N)}}\right)^2 + (N-1)\left(\frac{B_{\max}^{(N-1)}}{A_{\max}^{(N)}}\right)^2. \tag{16}$$

Observe that, from (12), (13), and (15)

$$\lim_{\alpha_k \to 0} f(N,\alpha_k) = 1. \tag{17}$$

We are now ready to state our theorem.

Theorem 1: If $A_{\min}^{(m)} \geq \alpha_k m B_{\max}^{(m)}$, $m = 1 \ldots N-1$, then the data rate achieved by the linear ZF canceler[3] can be lower bounded

$$R_n \geq \sum_k b_{k,\text{zf}-\text{bnd}}^n$$

where

$$b_{k,\text{zf}-\text{bnd}}^n \triangleq \Delta_f \log_2\left(1 + \Gamma^{-1}\sigma_k^{-1} s_k^n \left|h_k^{n,n}\right|^2 f^{-1}(N,\alpha_k)\right). \tag{18}$$

Furthermore, as α_k approaches zero, the linear ZF canceler achieves the single-user bound

$$\lim_{\alpha_k \to 0} b_{k,\text{zf}-\text{bnd}}^n = \lim_{\alpha_k \to 0} b_{k,\text{bnd}}^n. \tag{19}$$

So for CWDD channels, the linear ZF canceler is asymptotically optimal.

Proof: Equation (9) implies that, after application of the linear ZF canceler, the soft estimate of the transmitted symbol is

$$\widehat{x}_k^n = x_k^n + \left[\mathbf{H}_k^{-1}\right]_{\text{row } n} \mathbf{z}_k.$$

Hence, the postcancellation signal power is s_k^n, the postcancellation interference power is zero, and the postcancellation noise power is

$$\begin{aligned}\widetilde{\sigma}_{k,n} &\triangleq \mathcal{E}\left\{\left|\left[\mathbf{H}_k^{-1}\right]_{\text{row } n} \mathbf{z}_k\right|^2\right\}, \\ &= \left\|\left[\mathbf{H}_k^{-1}\right]_{\text{row } n}\right\|^2 \sigma_k \end{aligned} \tag{20}$$

[3]Again we assume the use of a suboptimal, practical coding scheme with an SNR-gap of Γ.

where (6) is applied in the second line. Hence, the data rate achieved by the linear ZF canceler is

$$b_{k,\text{zf}}^n(s_k^n) = \Delta_f \log_2\left(1 + \Gamma^{-1}\widetilde{\sigma}_{k,n}^{-1} s_k^n\right). \tag{21}$$

Define the matrix $\mathbf{G}_k \triangleq [g_k^{n,m}]$, where $g_k^{n,m} \triangleq h_k^{n,m}/h_k^{m,m}$. Now

$$\mathbf{H}_k = \mathbf{G}_k \text{diag}\left\{h_k^{1,1}, \ldots, h_k^{N,N}\right\}$$

hence

$$\mathbf{H}_k^{-1} = \text{diag}\left\{h_k^{1,1}, \ldots, h_k^{N,N}\right\}^{-1} \mathbf{G}_k^{-1} \tag{22}$$

and

$$\left[\mathbf{H}_k^{-1}\right]_{n,m} = \frac{1}{h_k^{n,n}} \left[\mathbf{G}_k^{-1}\right]_{n,m}. \tag{23}$$

Since the receivers are colocated at the CO, the upstream channel matrix is CWDD, as described by (3). This implies that $\mathbf{G}_k \in \mathbb{A}^{(N)}$, where $\mathbb{A}^{(N)}$ denotes the set of $N \times N$ diagonally dominant matrices, as defined in the Appendix. So Lemma 2 from the Appendix can be applied to bound the elements of $\mathbf{G}_k^{-1}$. This implies

$$\left|\left[\mathbf{H}_k^{-1}\right]_{n,m}\right| \leq \begin{cases} \left|h_k^{n,n}\right|^{-1} A_{\max}^{(N-1)}/A_{\min}^{(N)}, & n = m \\ \left|h_k^{n,n}\right|^{-1} B_{\max}^{(N-1)}/A_{\min}^{(N)}, & n \neq m \end{cases}$$

where $A_{\max}^{(N)}$ and $B_{\max}^{(N)}$ are defined in (10) and $A_{\min}^{(N)}$ is defined in (14). Hence

$$\left\|\left[\mathbf{H}_k^{-1}\right]_{\text{row } n}\right\|^2 \leq \left|h_k^{n,n}\right|^{-2} f(N,\alpha_k)$$

where $f(N,\alpha_k)$ is defined as in (16). Together with (20), this yields

$$\widetilde{\sigma}_{k,n} \leq \sigma_k \left|h_k^{n,n}\right|^{-2} f(N,\alpha_k).$$

Combining this with (21) leads to (18), which concludes the proof on the lower bound of the linear ZF canceler. To show that the linear ZF canceler achieves the single-user bound, first note that

$$\lim_{\alpha_k \to 0} b_{k,\text{bnd}}^n = \Delta_f \log_2\left(1 + \Gamma^{-1}\sigma_k^{-1} s_k^n \left|h_k^{n,n}\right|^2\right)$$

where the single-user bound $b_{k,\text{bnd}}^n$ is defined as per (8). Combining (17) with (18) implies

$$\lim_{\alpha_k \to 0} b_{k,\text{zf}-\text{bnd}}^n = \Delta_f \log_2\left(1 + \Gamma^{-1}\sigma_k^{-1} s_k^n \left|h_k^{n,n}\right|^2\right)$$

which concludes the proof. ■

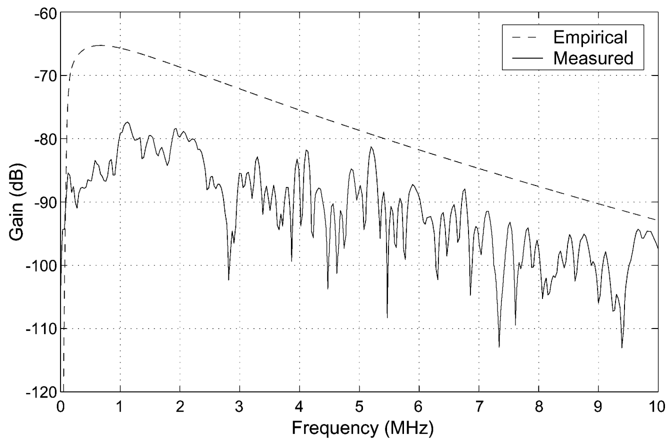

Fig. 2. Crosstalk channel transfer functions (1-km cable, 0.5-mm pairs).

In practice, we have verified that the condition $A_{\min}^{(m)} \geq \alpha_k m B_{\max}^{(m)}$, $m = 1 \dots N-1$ holds for N up to 25 and for frequencies up to 12 MHz using standardized models for α_k, so the bound applies in most practical VDSL scenarios.

The function $f(N, \alpha_k)$ can be interpreted as an upper bound on the noise enhancement caused by the linear ZF canceler. In CWDD channels, α_k is close to zero, so (17) implies that $f(N, \alpha_k)$. is close to unity. Each modem then operates at a rate

$$b_{k,\mathrm{zf}}^n \simeq \Delta_f \log_2 \left(1 + \Gamma^{-1} \sigma_k^{-1} s_k^n \left|h_k^{n,n}\right|^2\right).$$

So the linear ZF canceler completely removes crosstalk with negligible noise enhancement. Since the linear ZF canceler operates close to the single-user bound (8) for CWDD channels, we can say that in CWDD channels it is a near-optimal design.

Note that the bound (18) can be used to predict and guarantee a data rate without explicit knowledge of the crosstalk channels. This is the case because the bound depends only on the binder size, direct channel gain, and background noise power. Good models for these characteristics exist based on extensive measurement campaigns. Crosstalk channels, on the other hand, are poorly understood and actual channels can deviate significantly from the few empirical models that exist. (See, for example, Fig. 2, which shows a measured crosstalk channel and the predicted crosstalk channel according to empirical models from standardization [3].) This can make provisioning of services difficult. Using the bound (18) allows us to overcome this problem. The bound tells us that the crosstalk channel gain is not important as long as CWDD is observed. CWDD is a well-understood and modeled phenomenon. As a result (18) allows provisioning to be done in a reliable and accurate fashion.

A note of explanation may be necessary at this point. It may seem that CWDD allows us to easily predict, or at least bound, the crosstalk power that a receiver experiences. This is *not* true. The crosstalk power that a receiver experiences depends on the magnitude of elements along a *row*, not *column*, of $\mathbf{H}_k$. This, in turn, depends on the configuration of the other lines within the binder, which varies dramatically from one scenario to another. For example, in the scenario in Fig. 5, the crosstalk from the 150-m line into the 1200-m line is stronger than the direct signal on the 1200-m line itself. So the crosstalk from the other lines into the 1200-m line cannot be bounded without knowledge of the entire binder configuration. This makes provisioning of services extremely difficult. CWDD, on the other hand, applies to all lines when receivers are colocated. No knowledge of the actual binder configuration is necessary. Using (18), the performance of a line can be estimated using only locally available information about the line itself, such as its direct channel attenuation and background noise.

The value for α_k from (5) is based on worst 1% case models [3]. Hence, for 99% of lines, α_k will be smaller. So in 99% of lines, a data rate above the bound (18) is achieved. The bound is thus a useful tool not just for theoretical analysis, but for provisioning of services as well.

Simulations in Section VI will use this bound to show that the linear ZF canceler operates close to the single-user bound, and hence is a near-optimal design.

V. Spectra Optimization

Whilst current VDSL standards require the use of spectral masks, there is growing interest in the use of adaptive transmit spectra, a technique known as dynamic spectrum management [14]. This section investigates the optimization of transmit spectra for use with the linear ZF canceler. Each transmitter is subject to a total power constraint

$$\Delta_f \sum_k s_k^n \leq P_n, \ \forall n. \tag{24}$$

The goal is to maximize a weighted sum of the data rates of the modems within the network

$$\max_{\mathbf{s}_1,\dots,\mathbf{s}_N} \sum_n w_n R_n \quad \text{s.t. } \Delta_f \sum_k s_k^n \leq P_n, \ \forall n \tag{25}$$

where the vector $\mathbf{s}_n \triangleq [s_1^n, \dots, s_K^n]$ contains the PSD of user n on all tones. The weights $w_1, \dots, w_N$ are used to ensure that each modem achieves its target data rate. The data rate R_n is a function of the transmit PSDs $\mathbf{s}_1, \dots, \mathbf{s}_N$ and also depends on the type of crosstalk canceler used. If an optimal decision-feedback-based canceler is used, the objective function becomes convex [15]. Solving (25) then requires the solution of a KN-dimensional convex optimization. Although the cost function is convex, no closed-form solution is known [15]. Numerical algorithms to solve this have been proposed and have a complexity $\mathcal{O}(N^4 K \log K)$, which can be prohibitively complex for large N [16], [17]. When the ZF DFC is applied, all crosstalk is removed, and the spectra optimization decouples into an independent power loading for each user. This reduces complexity to $\mathcal{O}(KN \log K)$, which is feasible for large N [1].

In this section, we show that the same approach can also be applied with the linear ZF canceler, reducing the complexity of power allocation considerably. Furthermore, Theorem 1 ensures that this approach leads to near-optimal performance, operating close to the single-user bound.

A. Theoretical Capacity

We start by extending the single-user bound from Section III to VDSL modems that may vary their transmit spectra under a total power constraint. The resulting upper bound is useful for evaluating crosstalk canceler performance with optimized spectra. Denote R_n as the data rate of user n. When the transmit PSD s_k^n is allowed to vary under a total power constraint (24), the achievable data rate for user n in a CWDD channel is bounded

$$R_n \leq \max_{\sum_k s_k^n \leq P_n} \sum_k b_{k,\text{bnd}}^n \left(s_k^n\right).$$

where $b_{k,\text{bnd}}^n(s_k^n)$ is defined (8). In this optimization, the objective function is concave, and the total power constraint forms a convex set. Hence, the Karush–Kuhn–Tucker (KKT) conditions are sufficient for optimality. Examining the KKT conditions leads to the following bound for CWDD channels:

$$R_n \leq \sum_k b_{k,\text{bnd}}^n \left(s_{k,\text{bnd}}^n\right) \tag{26}$$

where the single-user water-filling PSD is defined

$$s_{k,\text{bnd}}^n \triangleq \left[\frac{1}{\lambda_n} - \frac{\Gamma \sigma_k}{\left|h_k^{n,n}\right|^2 \left[1+\alpha_k^2(N-1)\right]}\right]^+ \tag{27}$$

the function $[x]^+ \triangleq \max(0,x)$, and λ_n is chosen such that power constraint on line n is tight, that is

$$\Delta_f \sum_k s_{k,\text{bnd}}^n = P_n. \tag{28}$$

B. Near-Optimal Linear Canceler

Transmit spectra optimization with the linear ZF canceler is now considered. Equation (21) implies that (25) is equivalent to

$$\max_{\mathbf{s}_1,\ldots,\mathbf{s}_N} \sum_n \sum_k w_n b_{k,\text{zf}}^n \left(s_k^n\right) \quad \text{s.t. } \Delta_f \sum_k s_k^n \leq P_n,\ \forall n. \tag{29}$$

Observe that, when using the linear ZF canceler, the data rate of each user depends only on its own transmit PSD. It is independent of the PSDs of the other users since all crosstalk will be removed. The optimization problem is now decoupled between users, allowing the optimal power allocation to be found independently for each user. This also implies that these PSDs are optimal regardless of the choice of weights w_n.

Since the objective function is concave and the constraints form a convex set, the KKT conditions are sufficient for optimality. Examining these leads to the classic water-filling equation

$$s_{k,\text{zf}}^n = \left[\frac{1}{\lambda_n} - \Gamma \widetilde{\sigma}_{k,n}\right]^+. \tag{30}$$

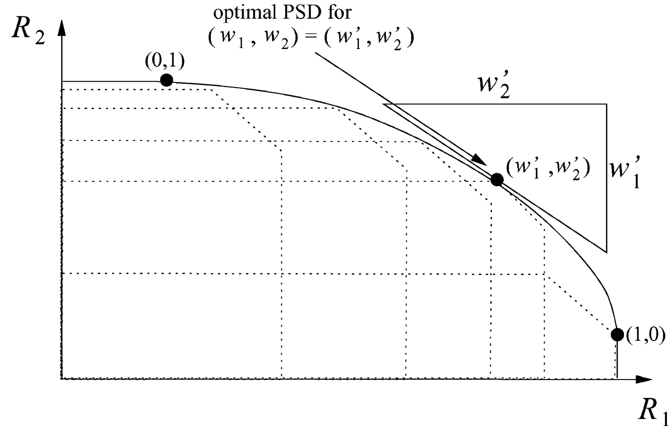

Fig. 3. Rate region in typical multiaccess channel.

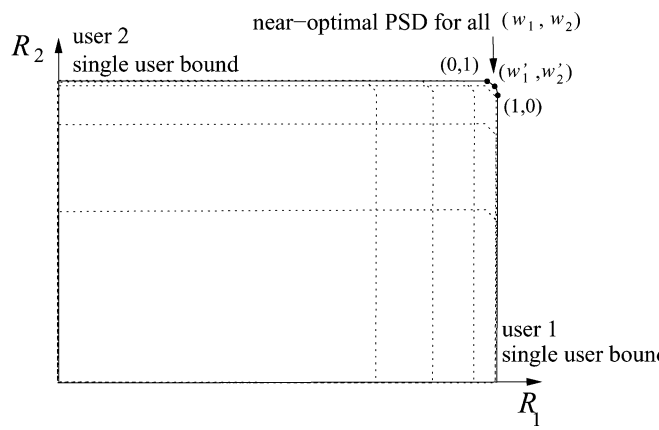

Fig. 4. Rate region in DSL channel.

The water-filling level λ_n must be chosen such that the total power constraint for user n is tight, that is $\Delta_f \sum_k s_{k,\text{zf}}^n = P_n$.

So the approach proposed in [1] for power allocation with the zero-forcing DFC is also valid here with the linear ZF canceler. Conventional water-filling algorithms can be applied to find the correct water-filling level with $\mathcal{O}(K \log K)$ complexity [18]. So the overall complexity of power allocation with the linear ZF canceler is $\mathcal{O}(NK \log K)$. This is a significant reduction when compared to existing power allocation algorithms for the general multiaccess channel, which have $\mathcal{O}(N^4 K \log K)$ complexity [16], [17].

In general, the rate region for a two-user multiaccess channel is nonrectangular as shown in Fig. 3. As we change the priorities between the users, the operating point on the rate region shifts. It is interesting to note that in DSL channels, due to CWDD, the rate region is close to rectangular as shown in Fig. 4. This is because the ZF canceller allows all users to operate close to their single-user bound simultaneously. As a result, during power allocation each user need only concern themselves with maximizing their own data rate. All crosstalk in the system will be completely removed at the receiver side with negligible impact on the direct channel gains. This can be clearly seen in (30) where the weights have no influence on the final power allocation. This implies that the same operating point is near-optimal regardless of the choice of priorities amongst the users, which simplifies power allocation.

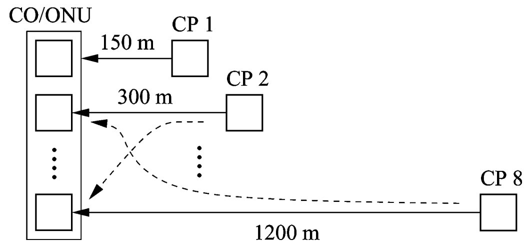

Fig. 5. Upstream VDSL scenario.

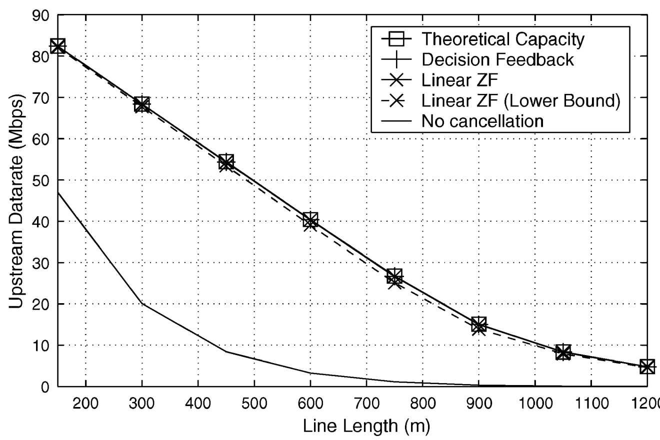

Fig. 6. Data rate with different cancelers.

Theorem 1 shows that, as a result of CWDD, the linear ZF canceler operates close to the single-user bound. So using the linear ZF canceler in combination with the power allocation (30) gives near-optimal performance. This is confirmed through simulation in the following section.

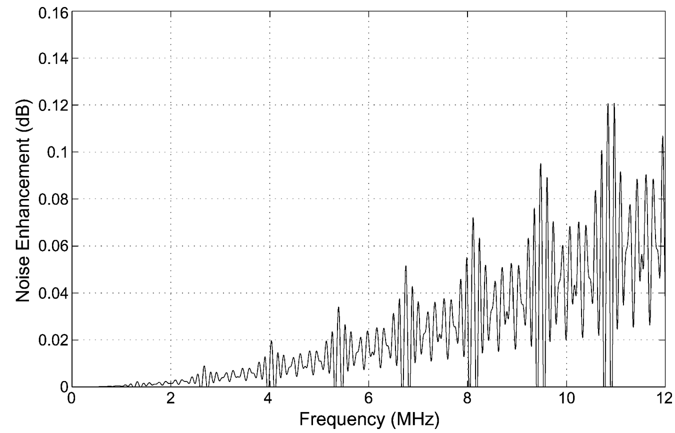

Fig. 7. Noise enhancement of ZF canceler on 600-m line.

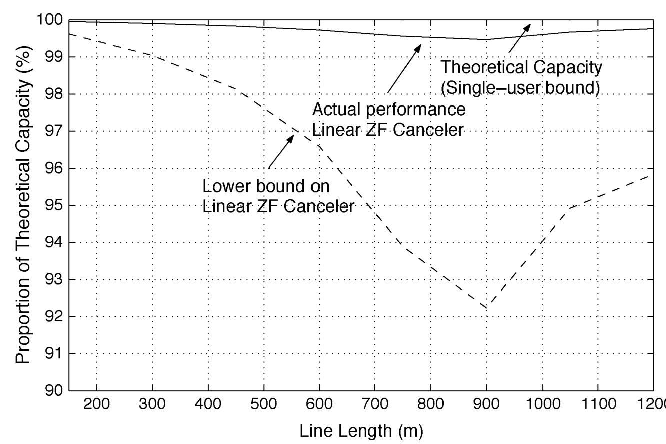

Fig. 8. Proportion of single-user bound achieved by ZF canceler.

VI. PERFORMANCE

This section evaluates the performance of the linear ZF canceler in a binder of eight VDSL lines. Performance is compared with the DFC and the single-user bound. Performance is evaluated in terms of achievable data rate using the SNR-gap to capacity approach [13]. The line lengths range from 150 to 1200 m in 150-m increments, as shown in Fig. 5. For all simulations, the line diameter is 0.5 mm (24-AWG). Direct and crosstalk channels are generated using semiempirical models [11]. The target symbol error probability is 10^{-7} or less, the coding gain is set to 3 dB, and the noise margin is set to 6 dB, which results in an SNR-gap f 12.9 dB. As per the VDSL standards, the tone spacing Δ_f is set to 4.3125 kHz [3], [11]. The modems use 4096 tones and the 998 FDD bandplan. Background noise is generated using ETSI noise model A [11].

A. Fixed Transmit Spectra

Current VDSL standards require that modems transmit under a spectral mask of −60 dBm/Hz [3], [11]. This section evaluates the performance of the linear ZF canceler when all modems are operating at this mask.

Fig. 6 shows the data rate achieved by each of the lines with the different crosstalk cancelers. The linear ZF canceler achieves substantial gains, typically 30 Mb/s or more over conventional systems with no cancellation. As can be seen the linear ZF canceler achieves near-optimal performance, operating close to the single-user bound. This is a direct result of the CWDD of $\mathbf{H}_k$, which ensures that the linear ZF canceler causes negligible noise enhancement, that is $f(N, \alpha_k)$, as defined in (16), is close to unity. The noise enhancement caused by the linear ZF canceler on the 600-m line is plotted for each tone in Fig. 7. As can be seen the noise enhancement is less than 0.16 dB, which has negligible impact on performance.

Fig. 8 shows the data rate achieved by the linear ZF canceler as a percentage of the single-user bound. Performance does not drop below 99% of the single-user bound. The lower bound on the performance of the linear ZF canceler (18) is also included for comparison. As can be seen, the bound is quite tight and guarantees that the linear ZF canceler will achieve at least 92% of the single-user bound.

It is interesting to note in Fig. 8 that the bound drops to its lowest value at 900 m. The reason for this is as follows. On short lines, the coupling length d_{coupling}, as defined in (4), is short. This results in a low value for α_k, and as a result, the linear ZF canceler causes negligible noise enhancement. On longer lines α_k is larger so we should expect to see the noise enhancement increase, as shown in Fig. 7. However, as the line length increases, the direct channel attenuation becomes so bad in the high frequencies that these tones are shut off. The majority of data transmission then occurs in the low frequencies, where the

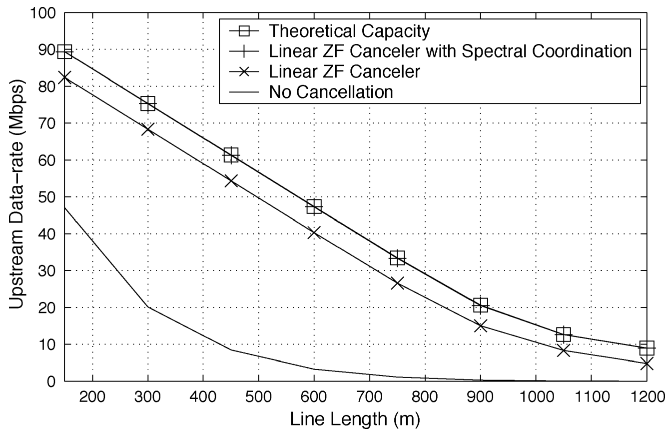

Fig. 9. Data rate with optimized spectra.

crosstalk coupling and α_k are low. So on long lines, the noise enhancement at the higher frequencies has negligible impact. It is thus on the intermediate-line lengths, such as 900 m, where the noise enhancement of the linear ZF canceler will result in the largest performance degradation, as seen in Fig. 8.

B. Optimized Transmit Spectra

This section investigates the performance of the linear ZF canceler with optimized spectra (30). A total power constraint of 11.5 dBm/Hz is applied to each modem as per the VDSL standards [3], [11]. Spectral mask constraints are not applied. Fig. 9 shows the data rates achieved on each line. The use of optimized spectra yields a gain of 5–8 Mb/s. The benefit is more substantial on the longer lines, where a 5-Mb/s gain can double the data rate.

Fig. 9 shows that spectra optimization gives maximum benefit on long lines. This is to be expected since on long lines the direct channel gain decreases more rapidly with frequency. Note that the benefit of adaptive spectra, when crosstalk has already been cancelled, comes primarily from the modem loading power in the best parts of the channel, which are typically in the lower frequencies.

VII. Conclusion

This paper investigated the design of crosstalk cancelers for upstream VDSL. Existing designs based on decision feedback suffer from error propagation, high complexity and long latency. A linear ZF canceler is proposed, which has a low complexity and no latency.

An oft-cited problem with the ZF design is that it leads to severe noise enhancement in ill-conditioned channels. Fortunately, VDSL channels with colocated receivers are columnwise diagonal dominant. This ensures that the VDSL channel is well conditioned, and noise enhancement caused by the ZF design is negligible.

An upper bound on the capacity of the multiuser VDSL channel was derived. This single-user bound shows that spatial diversity in the VDSL environment is negligible. Therefore, the best outcome that a crosstalk canceler can achieve is the complete suppression of crosstalk without noise enhancement.

A lower bound on the performance of the linear ZF canceler was derived. This bound depends only on the binder size, direct channel gain, and background noise for which reliable models and statistical data exist. As a result, the performance of the linear ZF canceler can be accurately predicted, which simplifies service provisioning considerably. This bound shows that the linear ZF canceler operates close to the single-user bound. So the linear ZF canceler is a low-complexity, low-latency design with predictable, near-optimal performance.

The combination of spectral optimization and crosstalk cancellation was considered. Spectra optimization in a multiaccess channel generally involves a highly complex optimization problem. Since the linear ZF canceler decouples transmission on each line, the spectrum on each modem can be optimized independently, leading to a significant reduction in complexity.

Appendix
Bounds on Diagonally Dominant Matrices

Define the set $\mathbb{A}^{(N)}$ of $N \times N$ matrices, such that for any $\mathbf{A}^{(N)} \in \mathbb{A}^{(N)}$, it holds that

$$|a_{n,n}| = 1 \tag{31}$$
$$|a_{n,m}| \le \alpha_k, \ \forall n \neq m \tag{32}$$

where $a_{n,m} \triangleq [\mathbf{A}^{(N)}]_{n,m}$. Define the set $\mathbb{B}^{(N)}$ of $N \times N$ matrices, such that for any $\mathbf{B}^{(N)} \in \mathbb{B}^{(N)}$, it holds that

$$\begin{aligned} |b_{n,n}| &= 1, \ \forall n < N \\ |b_{N,N}| &\le \alpha_k \\ |b_{n,m}| &\le \alpha_k, \ \forall n \neq m \end{aligned} \tag{33}$$

where $b_{n,m} \triangleq [\mathbf{B}^{(N)}]_{n,m}$.

Lemma 1: Consider any $\mathbf{A}^{(N)} \in \mathbb{A}^{(N)}$ and $\mathbf{B}^{(N)} \in \mathbb{B}^{(N)}$. The magnitude of the determinants of $\mathbf{A}^{(N)}$ and $\mathbf{B}^{(N)}$ can be bounded as follows:

$$\left|\det\left(\mathbf{A}^{(N)}\right)\right| \le A_{\max}^{(N)}, \tag{34}$$
$$\left|\det\left(\mathbf{B}^{(N)}\right)\right| \le B_{\max}^{(N)}, \tag{35}$$

where $A_{\max}^{(N)}$ and $B_{\max}^{(N)}$ are defined in (10). Furthermore, if

$$A_{\min}^{(m)} \ge \alpha_k m B_{\max}^{(m)}, \ m = 1 \ldots N-1 \tag{36}$$

then the following bound also holds:

$$\left|\det\left(\mathbf{A}^{(N)}\right)\right| \ge A_{\min}^{(N)} \tag{37}$$

where $A_{\min}^{(N)}$ is defined in (14). Note that $|\cdot|$ denotes the absolute value operator, whilst $\det(\cdot)$ denotes the determinant operator.

Proof: The proof is based on induction. Begin by assuming that the bounds (34) and (37) hold for any $(N-1) \times (N-1)$

matrix $\mathbf{X}$ in the set $\mathbb{A}^{(N-1)}$, for some specific value of N. That is

$$|\det(\mathbf{X})| \leq A_{\max}^{(N-1)},\ \forall \mathbf{X} \in \mathbb{A}^{(N-1)} \quad (38)$$
$$|\det(\mathbf{X})| \geq A_{\min}^{(N-1)},\ \forall \mathbf{X} \in \mathbb{A}^{(N-1)}. \quad (39)$$

Also assume that (35) holds for any $(N-1)\times(N-1)$ matrix $\mathbf{X}$ in the set $\mathbb{B}^{(N-1)}$. That is

$$|\det(\mathbf{X})| \leq B_{\max}^{(N-1)},\ \forall \mathbf{X} \in \mathbb{B}^{(N-1)}. \quad (40)$$

Now consider any matrix $\mathbf{A}^{(N)} \in \mathbb{A}^{(N)}$. Decompose $\mathbf{A}^{(N)}$ as

$$\mathbf{A}^{(N)} = \begin{bmatrix} & \mathbf{A}^{(N-1)} & & a_{1,N} \\ & & & \vdots \\ & & & a_{N-1,N} \\ a_{N,1} & \cdots & a_{N,N-1} & 1 \end{bmatrix}$$

where $a_{n,m} \triangleq [\mathbf{A}^{(N)}]_{n,m}$ and $\mathbf{A}^{(N-1)}$ is the submatrix containing the first $N-1$ rows and columns of $\mathbf{A}^{(N)}$. By expanding the determinant along the last row of $\mathbf{A}^{(N)}$, it can be seen that

$$\begin{aligned} \left|\det\left(\mathbf{A}^{(N)}\right)\right| &= \left|\det\left(\mathbf{A}^{(N-1)}\right) + \sum_{m=1}^{N-1}(-1)^{N-m}a_{N,m} \times \det\left(\left[\overline{\mathbf{A}}_m^{(N-1)} \quad \mathbf{a}_N\right]\right)\right| \quad (41) \\ &\leq \left|\det\left(\mathbf{A}^{(N-1)}\right)\right| + \sum_{m=1}^{N-1}\alpha_k\left|\det\left(\left[\overline{\mathbf{A}}_m^{(N-1)} \quad \mathbf{a}_N\right]\right)\right| \quad (42) \end{aligned}$$

where $\overline{\mathbf{A}}_m^{(N-1)}$ is the submatrix formed by removing column m from $\mathbf{A}^{(N-1)}$ and $\mathbf{a}_N \triangleq [a_{1,N} \ldots a_{N-1,N}]^T$. The second line makes use of (32). Define the permutation matrix

$$\Pi_m \triangleq [\mathbf{e}_1 \cdots \mathbf{e}_{m-1}\mathbf{e}_{m+1} \ldots \mathbf{e}_{N-1}\mathbf{e}_m]$$

where $\mathbf{e}_m$ is defined as the mth column of the $(N-1)\times(N-1)$ identity matrix. Note that

$$\Pi_m^T\left[\overline{\mathbf{A}}_m^{(N-1)}\ \mathbf{a}_N\right] \in \mathbb{B}^{(N-1)}.$$

Using the fact that row permutations have no effect on the magnitude of a determinant, together with (40), implies

$$\sum_{m=1}^{N-1}\alpha_k\left|\det\left(\left[\overline{\mathbf{A}}_m^{(N-1)} \quad \mathbf{a}_N\right]\right)\right| \leq \alpha_k(N-1)B_{\max}^{(N-1)}. \quad (43)$$

Combining this with (42) and (38) yields

$$\left|\det\left(\mathbf{A}^{(N)}\right)\right| \leq A_{\max}^{(N-1)} + \alpha_k(N-1)B_{\max}^{(N-1)}.$$

Note that from (10) by definition

$$A_{\max}^{(N)} = A_{\max}^{(N-1)} + \alpha_k(N-1)B_{\max}^{(N-1)} \quad (44)$$

hence

$$\left|\det\left(\mathbf{A}^{(N)}\right)\right| \leq A_{\max}^{(N)}. \quad (45)$$

Now consider any matrix $\mathbf{B}^{(N)} \in \mathbb{B}^{(N)}$. Decompose $\mathbf{B}^{(N)}$ as

$$\mathbf{B}^{(N)} = \begin{bmatrix} & \mathbf{C}^{(N-1)} & & b_{1,N} \\ & & & \vdots \\ & & & b_{N-1,N} \\ b_{N,1} & \cdots & b_{N,N-1} & b_{N,N} \end{bmatrix}$$

where $b_{n,m} \triangleq [\mathbf{B}^{(N)}]_{n,m}$ and $\mathbf{C}^{(N-1)}$ is the submatrix containing the first $N-1$ rows and columns of $\mathbf{B}^{(N)}$. By expanding the determinant along the last row of $\mathbf{B}^{(N)}$, it can be seen that

$$\left|\det\left(\mathbf{B}^{(N)}\right)\right| = \left|b_{N,N}\det\left(\mathbf{C}^{(N-1)}\right) + \sum_{m=1}^{N-1}(-1)^{N-m}b_{N,m}\det\left(\left[\overline{\mathbf{C}}_m^{(N-1)} \quad \mathbf{b}_N\right]\right)\right| \quad (46)$$

where $\overline{\mathbf{C}}_m^{(N-1)}$ is the submatrix formed by removing column m from $\mathbf{C}^{(N-1)}$ and $\mathbf{b}_N \triangleq [b_{1,N} \ldots b_{N-1,N}]^T$. Note that $\mathbf{C}^{(N-1)} \in \mathbb{A}^{(N-1)}$ and

$$\Pi_m^T\left[\overline{\mathbf{C}}_m^{(N-1)} \quad \mathbf{b}_N\right] \in \mathbb{B}^{(N-1)}.$$

Using the fact that row permutations have no effect on the magnitude of a determinant, together with (38), (40), and (46) now yields

$$\left|\det\left(\mathbf{B}^{(N)}\right)\right| \leq \alpha_k A_{\max}^{(N-1)} + \alpha_k(N-1)B_{\max}^{(N-1)}.$$

Note that from (10) by definition

$$B_{\max}^{(N)} = \alpha_k A_{\max}^{(N-1)} + \alpha_k(N-1)B_{\max}^{(N-1)} \quad (47)$$

hence

$$\left|\det\left(\mathbf{B}^{(N)}\right)\right| \leq B_{\max}^{(N)}. \quad (48)$$

We now proceed with the inductive proof. First, note that from (31) $|\mathbf{A}^{(1)}| = 1$ and from (33) $|\mathbf{B}^{(1)}| \leq \alpha_k$, so (38) and (40) hold for $N=2$. Hence through induction, (45) and (48) imply that (34) and (35) must hold for all N. This concludes the proof for the upper bounds (34) and (35).

We now turn our attention to the lower bound (37). We will make use of the following property, which can be readily proven. For any real scalars x and y, it holds that

$$|x+y| \geq ||x|-|y||.$$

Furthermore, if $|x| \geq |y|$, then

$$|x+y| \geq |x|-|y|. \tag{49}$$

Now let

$$x = \det\left(\mathbf{A}^{(N-1)}\right)$$
$$y = \sum_{m=1}^{N-1}(-1)^{N-m} a_{N,m} \det\left(\left[\overline{\mathbf{A}}_m^{(N-1)} \quad \mathbf{a}_N\right]\right).$$

Now a summation will have the largest absolute value if all terms inside the summation have the same sign. This observation leads to the bound

$$\begin{aligned}|y| &\leq \sum_{m=1}^{N-1}\left|a_{N,m}\det\left(\left[\overline{\mathbf{A}}_m^{(N-1)} \quad \mathbf{a}_N\right]\right)\right| \\ &\leq \sum_{m=1}^{N-1}\alpha_k\left|\det\left(\left[\overline{\mathbf{A}}_m^{(N-1)} \quad \mathbf{a}_N\right]\right)\right| \\ &\leq \alpha_k(N-1)B_{\max}^{(N-1)}\end{aligned} \tag{50}$$

where (32) is applied in the second line, and (43) is applied in the third line. Equation (39) implies

$$|x| \geq A_{\min}^{(N-1)}. \tag{51}$$

Combining this with (36) yields

$$\begin{aligned}|x| &\geq \alpha_k(N-1)B_{\max}^{(N-1)}, \\ &\geq |y|,\end{aligned}$$

where (50) is applied in the second line. From (41) it is clear that

$$\left|\det\left(\mathbf{A}^{(N)}\right)\right| = |x+y|.$$

Since $|x| \geq |y|$, (49) can now be applied, resulting in the following bound

$$\begin{aligned}\left|\det\left(\mathbf{A}^{(N)}\right)\right| &\geq |x|-|y|, \\ &\geq A_{\min}^{(N-1)} - \alpha_k(N-1)B_{\max}^{(N-1)}\end{aligned}$$

where (50) and (51) are applied in the second line. Note that from (14) by definition

$$A_{\min}^{(N)} \triangleq A_{\min}^{(N-1)} - \alpha_k(N-1)B_{\max}^{(N-1)}$$

hence

$$\left|\det\left(\mathbf{A}^{(N)}\right)\right| \geq A_{\min}^{(N)}. \tag{52}$$

Now note that $|\mathbf{A}^{(1)}| = 1$ and $A_{\min}^{(1)} = 1$, so (39) holds for $N = 2$. Hence through induction, (52) implies that (37) holds for all N. This concludes the proof for the lower bound (37). ■

Lemma 2: If $\mathbf{G} \in \mathbb{A}^{(N)}$ and $A_{\min}^{(m)} \geq \alpha_k m B_{\max}^{(m)}$, $m = 1 \ldots N-1$; then the magnitude of the elements of $\mathbf{G}^{-1}$ can be bounded

$$\left|[\mathbf{G}^{-1}]_{n,m}\right| \leq \begin{cases} A_{\max}^{(N-1)}/A_{\min}^{(N)}, & n = m \\ B_{\max}^{(N-1)}/A_{\min}^{(N)}, & n \neq m. \end{cases} \tag{53}$$

Proof: By definition of the matrix inverse

$$\left|[\mathbf{G}^{-1}]_{n,m}\right| = \left|\det(\overline{\mathbf{G}}^{m,n})\right| / |\det(\mathbf{G})| \tag{54}$$

where $\overline{\mathbf{G}}^{m,n}$ is the submatrix formed by removing row m and column n from $\mathbf{G}$. Now $\mathbf{G} \in \mathbb{A}^{(N)}$ so from Lemma 1

$$|\det(\mathbf{G})| \geq A_{\min}^{(N)}. \tag{55}$$

If $m = n$ then $\overline{\mathbf{G}}^{m,n} \in \mathbb{A}^{(N-1)}$ and from Lemma 1

$$\left|\det(\overline{\mathbf{G}}^{m,m})\right| \leq A_{\max}^{(N-1)}, \ \forall m. \tag{56}$$

If $m \neq n$ then $\Pi_n^T \overline{\mathbf{G}}^{m,n} \Pi_m \in \mathbb{B}^{(N-1)}$ and from Lemma 1

$$\left|\det(\overline{\mathbf{G}}^{m,n})\right| = \left|\det\left(\Pi_n^T \overline{\mathbf{G}}^{m,n} \Pi_m\right)\right| \leq B_{\max}^{(N-1)}, \ \forall m \neq n. \tag{57}$$

Combining (54), (55), (56) and (57) yields (53), which concludes the proof. ■

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