17. Forestry Applications of Linear Programming

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Linear programming (LP) is a highly versatile mathematical optimization technique which has found wide use in management and economics. It is used both as a research technique and as a planning tool, particularly at the individual firm and industry level. In general, LP is designed to maximize or minimize a linear objective function subject to a set of linear constraints. Linear programming is one of a group of techniques which may be referred to as *mathematical programming*. Other related techniques are goal programming, mixed integer programming and quadratic programming. Some typical applications of linear programming include:

1) determining the most profitable combination of enterprise or activity levels for a business firm with limited supplies of various resources
2) determining the most profitable investment portfolio, given the amount of investment capital available, rates of return on various stocks, bonds and other 'paper assets', and limits on high-risk investments.
3) formulating mixtures to combine ingredients such that a required overall composition of the mix is satisfied at least cost. Important applications are fuel and fertilizer blending and determination of livestock rations or supplementary feeds.
4) scheduling the various tasks in a construction project so as to complete the overall project in minimal time or at minimal cost
5) determining the location and size of storage facilities and processing plants together with the distribution pattern, so as to minimize the total of transport, storage and processing costs.

As illustrated by this list, the range of applications of linear programming is indeed wide, and this is one of the most widely used operations research techniques. In this module, the algebraic formulation of LP models in explained, and a simplistic decision problem is formulated in a linear programming framework, and is then solved graphically to illustrate the basic principles of the technique. Computer solution of this problem using the *simplex* method is then demonstrated, with reference to setting up the model on a spreadsheet and interpreting the output. Further applications to transshipment modeling, capital budgeting and goal programming are then illustrated.

1. ALGEBRAIC FORMULATION OF LINEAR PROGRAMMING PROBLEMS

In mathematical terms, the purpose of linear programming is to optimize a linear *objective function* subject to a set of linear *constraints*. The objective function may, for example, express the profit from a combination of enterprises or activities, the *cost* of a combination of ingredients, or the *time* required to complete a series of tasks. In these contexts, optimization may mean either *maximization* of profits or *minimization* of costs or time. The nature of linear programming will be illustrated with reference to a profit maximization problem. A highly simplified example has been chosen deliberately, so as to illustrate the technique without undue computational distraction. More realistic applications will be developed later in this module.

Example 1

A cabinet-maker produces dining room suites and grandfather clocks out of Australian red cedar timber. He can obtain annual supplies of up to 600 linear metres (lm) of red cedar (one linear metre is equivalent to one metre in length by one

metre in width by one centimetre in depth). Up to 4000 hours of labour a year are available for operations such as sawing, joining and polishing. Each dining room suite requires 12 lm of timber, and each grandfather clock requires 7.5 lm of timber. One hundred hours of labour are required to produce a dining room suite, and 40 hours to produce a grandfather clock. The profit from each dining room suite is $900, and that from each grandfather clock is $450. The cabinet-maker wishes to know how many units of each furniture line to produce in order to maximize profits. Formulate this decision problem as a linear programming model.

Formulation of the model

Before this problem can be solved by graphical or other means, it must be expressed in algebraic form, i.e. as a model. The first step is to introduce an 'x' notation for the decision variables, here numbers of suites and clocks to be produced. Thus we let x_1 = number of dining room suites produced, and x_2 = number of grandfather clocks produced.

It is now possible to formulate an objective function which in this case is an equation defining total profit. The term 'profit' is used loosely here, in that the figures of $900 and $450 are more correctly called 'gross margins' for the two activities, i.e. they are returns net of variable but not fixed costs. In obtaining these figures, allowance is made for allocatable costs such as materials, labour and marketing costs, but not for overheads such as rent on premises or rates, depreciation of equipment, and accountancy. In this module the term net revenues will be used and the objective function will be referred to as a revenue function. If each dining room suite has a net revenue of $900, then the total of net revenues from producing (and selling) x_1 dining room suites will be 900 x_1 dollars. Similarly, the total net revenue from producing x_2 grandfather clocks will be 450 x_2. If the symbol Z is used to represent total net revenue, the objective function may be written as

$$Z = 900\ x_1 + 450\ x_2$$

The objective can now be identified more precisely as finding those values of x_1 and x_2 for which Z is a maximum, bearing in mind the restrictions on production imposed by limited supplies of timber and labour.

Resource restrictions also can be expressed in algebraic form. If x_1 dining room suites are produced, each requiring 12 lm of red cedar timber, then dining room suites will consume a total of 12 x_1 lm of timber. Similarly, if x_2 grandfather clocks are produced these will consume 7.5 x_2 lm of timber. The total amount of timber consumed cannot exceed the supply, so the production plan is constrained by the inequality expression

$$12\ x_1 + 7.5\ x_2 \leq 600$$

The left-hand-side of this expression indicates the amount of timber which will be used for any production policy (combination of x_1 and x_2 levels); the right-hand-side indicates timber supply. This timber constraint ensures that the demand for timber cannot exceed the supply; any production plan consuming more timber would violates this constraint and would therefore be *infeasible*.

Similar reasoning can be applied to derive a labour constraint. Since suites and clocks require 100 and 40 manhours of labour respectively, and since the labour supply is 4000 manhours, feasible levels of x_1 and x_2 are bounded by

$$100\ x_1 + 40\ x_2 \leq 4000$$

Two further constraints are necessary to define the decision problem fully. These are that the numbers of suites and clocks produced cannot be negative

i.e. $x_1 \geq 0$ and $x_2 \geq 0$

Non-negativity constraints may at first appear unnecessary in a practical sense; after all, it is not possible to produce negative numbers of suites or clocks. However, they must be included for mathematical completeness, to delineate fully the feasible region of production. The cabinet-maker's decision problem may now be summarized as a linear programming

model with an objective function, two activities (dining room suites and grandfather clocks), two resource constraints (timber and labour) and two non-negativity constraints, as follows:

maximize revenue $Z = 900 x_1 + 450 x_2$

subject to the resource constraints

$$12 x_1 + 7.5 x_2 \leq 600$$

$$100 x_1 + 40 x_2 \leq 4000$$

and non-negativity constraints

$$x_1 \geq 0, \; x_2 \geq 0$$

General algebraic formulation

For the more mathematically inclined, the above problem formulation may be generalized in abstract terms for greater numbers of activities and constraints. Suppose levels are to be determined for n *activities* ($x_1, x_2, \ldots x_n$) yielding individual *net revenues* $c_1, c_2, \ldots c_n$. Limited supplies are available of m *resources*, the supply levels being $b_1, b_2, \ldots b_m$. Each activity uses fixed amounts of resources; in particular, the requirement of resource i by one unit of activity j is a_{ij}. The elements of the matrix of a_{ij} values (for i=1 to m and j=1 to n) are known as *technical or input-output coefficients*. The linear programming model may now be written as

maximize $Z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$

subject to the linear resource constraints

$$a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n \leq b_1$$
$$a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n \leq b$$
$$:$$
$$a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n \leq b_m$$

and the non-negativity constraints

$$x_1 \geq 0, x_2 \geq 0, \ldots x_n \geq 0.$$

For readers familiar with matrix algebra, the decision problem may be expressed more compactly in matrix notation, as

maximize $Z = C^T X$

subject to $A X \leq B$ and $X \geq 0$

where C is a vector of activity net revenues (and C^T is the transpose of C) ;
X is a vector of activity levels;
A is a matrix of input-output coefficients;
B is a vector of resource supplies;
and 0 is the null vector.

2. GRAPHICAL SOLUTION OF RESOURCE ALLOCATION MODELS

Having expressed the production planning problem as an algebraic model, it is now time to derive optimal levels of the decision or policy variables. For simple problems such as this (with two activities), solution by graphical means is possible. Graphical presentation provides a number of practical insights into the nature of the decision problem and its solution. For the above example, a graph is set up as in Figure 1, in which the number of dining room suites (x_1) is measured on the horizontal axis and the number of grandfather clocks (x_2) is recorded on the vertical axis. The resource and non-negativity constraints are entered as straight lines on this graph, delineating the *feasible region*.

To fix the position of the timber constraint line, note that the maximal number of suites which could be produced if all timber were devoted to suites is the total timber supply divided by the per unit demand, i.e. $x_1 = 600/12$ or 50. No timber would then be available for production of clocks, i.e. $x_2 = 0$. On the other hand, if no timber were devoted to suites ($x_1 = 0$) then up to $x_2 = 600/7.5$ or 80 clocks could be produced before the supply of timber became exhausted. The two end points of the timber constraint – (0,80) and (50,0) – are drawn in Figure 1. If half the timber were used for suites and half for clocks then up to 25 suites and 40 clocks could be produced, corresponding to midway on the straight line joining the above end points. In fact, any point on the straight line between (0,80) and (50,0) corresponds to a production plan in which exactly 600 lm of timber are used. Any combination of x_1 and x_2 levels on or to the left of (below) this line is feasible in terms of the timber constraint $12x_1 + 7.5x_2 \leq 600$. Any point to the right would use more

than 600 lm of timber and violate this constraint.

Applying similar reasoning to that above, the labour constraint $100x_1 + 40x_2 \leq 4000$ can be drawn as a straight line connecting $x_1 = 4000/100 = 40$ and $x_2 = 0$ on the horizontal axis to $x_2 = 4000/40 = 100$ and $x_1 = 0$ on the vertical axis, as in Figure 2. Any production plan represented by an (x_1,x_2) pair on or to the left of this line is feasible in terms of labour use, while any production plan represented by a point to the right is infeasible. The combined effect of the timber and labour constraints is to restrict the feasible region to on or to the left of the line BCD in Figure 2.

It was noted earlier that in addition to resource constraints, non-negativity con-straints must be imposed on the feasible region, these being of the form $x_1 \geq 0$ and $x_2 \geq 0$. In Figure 2, the constraint $x_1 \geq 0$ confines solutions to on or to the right of the vertical axis, while $x_2 \geq 0$ means any point on or above the horizontal axis. The feasible region of production is now defined fully; it is the irregular area ABCD. Any numbers of suites and clocks corresponding to an (x_1,x_2) pair within or on the boundary of this region is feasible in that it will not use more timber or labour than is available. If a point corresponding to an (x_1,x_2) pair is not on the production frontier BCD then some timber and labour remain unused.

Having defined the feasible region it is now possible to determine the most profitable production policy, i.e. the policy for which the objective function Z takes as large a value as possible. Revenue considerations are introduced by drawing lines on the graph joining equally profitable (x_1,x_2) combinations, called *isorevenue lines*. The location of the initial isorevenue line is arbitrary. Suppose, for example, the policy of producing 30 dining room suites and no clocks were chosen; the total net revenue would then be

$$Z = 900(30) + 450(0) = 27,000 \text{ dollars}$$

This same level of revenue could be achieved by producing no suites and 27,000 / 450 or 60 clocks. In fact, any combination of (x_1, x_2) values along the line from (30,0) to (0,60) corresponds to a total net revenue of $27,000. The $27,000 isorevenue line is drawn as a broken line in Figure 3.

Various other isorevenue lines could be drawn, as illustrated in Figure 3. Each of these other lines is parallel to the $27,000 isorevenue line. The higher the revenue the further the line from the origin. For example, the $31,500 isorevenue line corresponds to the production of 35 suites or 70 clocks or any combination along the straight line joining these points. The most profitable plan would be indicated by the (x_1,x_2) pair touching the highest possible isorevenue line. A parallel shift to the right for the isorevenue line reveals that this must be at point C. Reading from the graph, the co-ordinates for point C correspond to an optimal plan of producing approximately 22 dining room suites and 44 grandfather clocks. The total net revenue from this plan is approximately 900(22) + 450(44) or 39,600 dollars.

It is to be noted that the above solution is approximate only, having been read imprecisely from a graph. Procedures are available for finding a more precise solution; this is to produce 44.44 clocks and 22.22 suites. However, in practice only whole numbers of suites and clocks can be produced, so these numbers could be rounded downwards to 44 and 22 respectively.
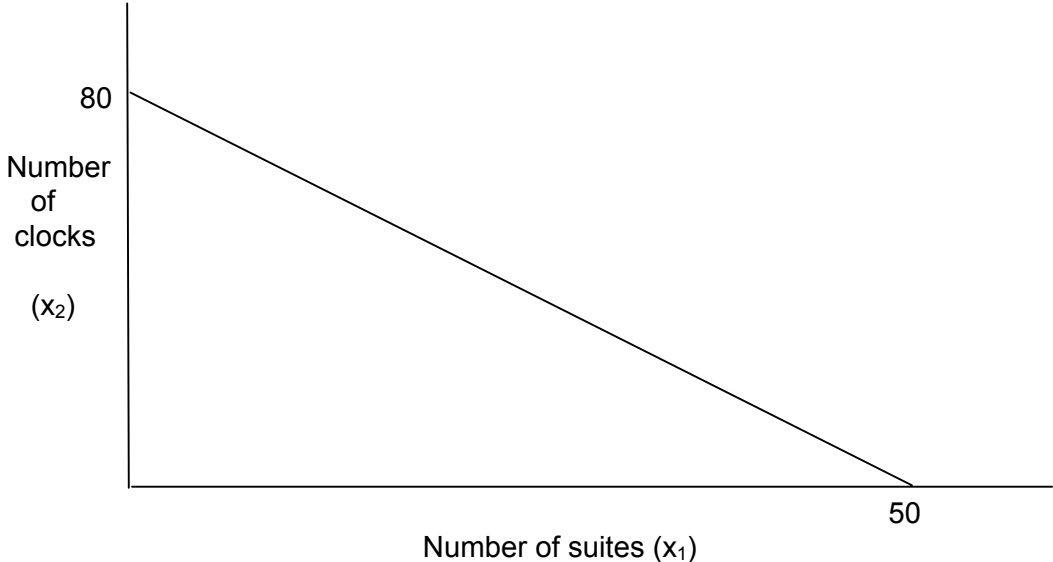
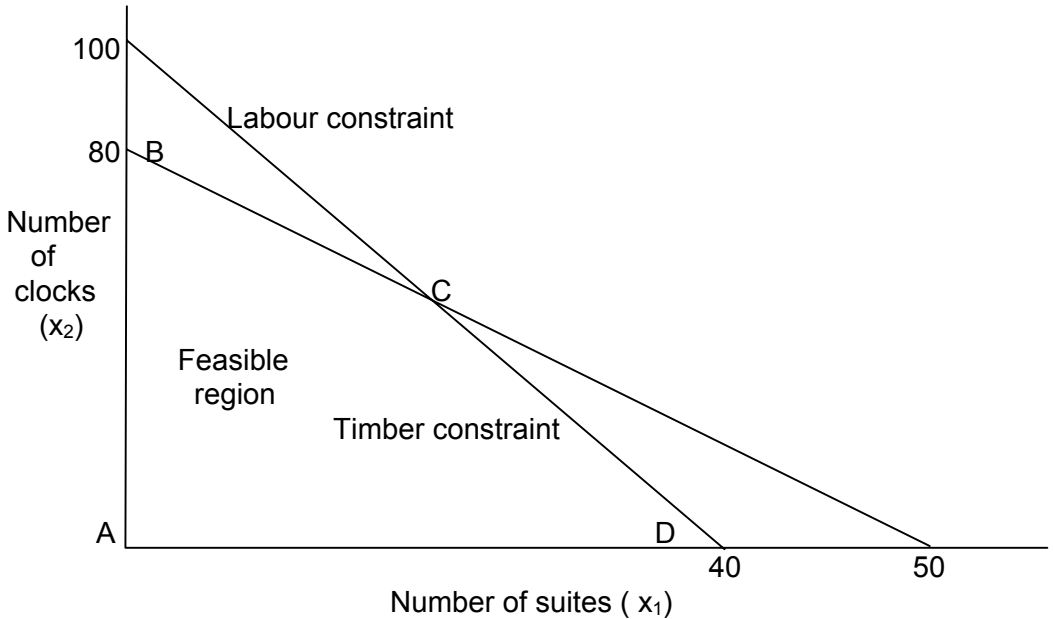Figure1. Graphical representation of timber constraint

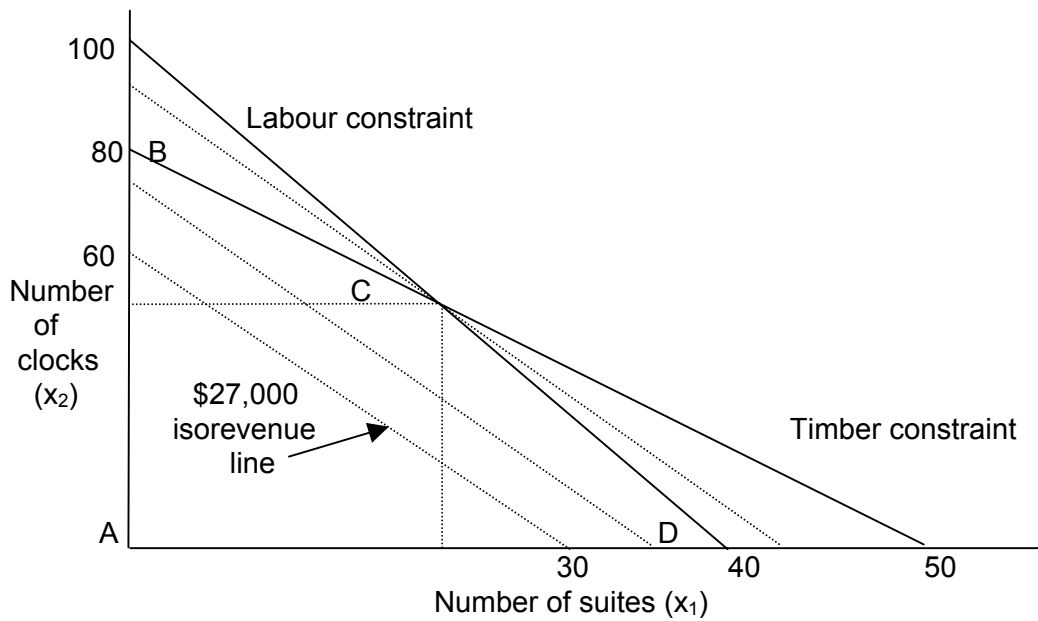Figure 2. Feasible region of production (ABCD)

Figure 3. Isorevenue lines and optimal plan

Sensitivity or stability analysis

The graphical solution procedure illustrated above has identified the most profitable combination of activity levels for given net revenues, resource supplies, and so on. However, in practice the concept of a single best policy is of limited value; it is desirable to know under what circumstances this policy remains optimal, and how it would change if any of the parameters were varied. Decision makers frequently require information about the sensitivity or stability of the optimal plan with respect to changes in net revenues, resource supplies or input-output coefficients. Two relatively simple but highly useful forms of sensitivity analysis are *net revenue ranging* and calculation of *shadow prices*.

Net revenue ranging

The fact that point C is optimal is due to the relative net revenues of clocks and suites. If the relativity between net revenues were to change then the slope of the isorevenue line would change, and a different plan could become optimal. For example, suppose the net revenue of suites remains at $900 per unit, but that for clocks rises to $600. The $27,000 isorevenue line will now run from (30,0) on the horizontal axis to (0,45) on the vertical axis, as in Figure 4. A parallel shift outwards in the isorevenue line

reveals that the optimal plan is now at point B, i.e. production of 80 clocks and no suites, with a total net revenue of 80 times $600 or $48,000.

By examining different slopes of isorevenue lines on a graph such as Figure 4, it is possible to determine within what range the net revenue of one activity can vary for a fixed net revenue of the other activity, while the current production plan remains optimal. The slope of any line on the (x_1,x_2) co-ordinate axis system is defined as the change in level of the variable on the vertical axis divided by the corresponding change in the variable on the horizontal axis. For the labour constraint, there is a decline in x_2 of 100 for an increase in x_1 of 40, and hence a slope of -100/40 or -2.5. Similarly, the slope of the timber constraint is -80/50 or -1.6. The slope of the isorevenue line is the negative of the ratio of net revenue of the activity on the horizontal axis to net revenue of the activity on the vertical axis, i.e. slope of isorevenue line

$$= - \frac{\text{net revenue of suites}}{\text{net revenue of clocks}} \quad (\text{or} - c_1/c_2)$$
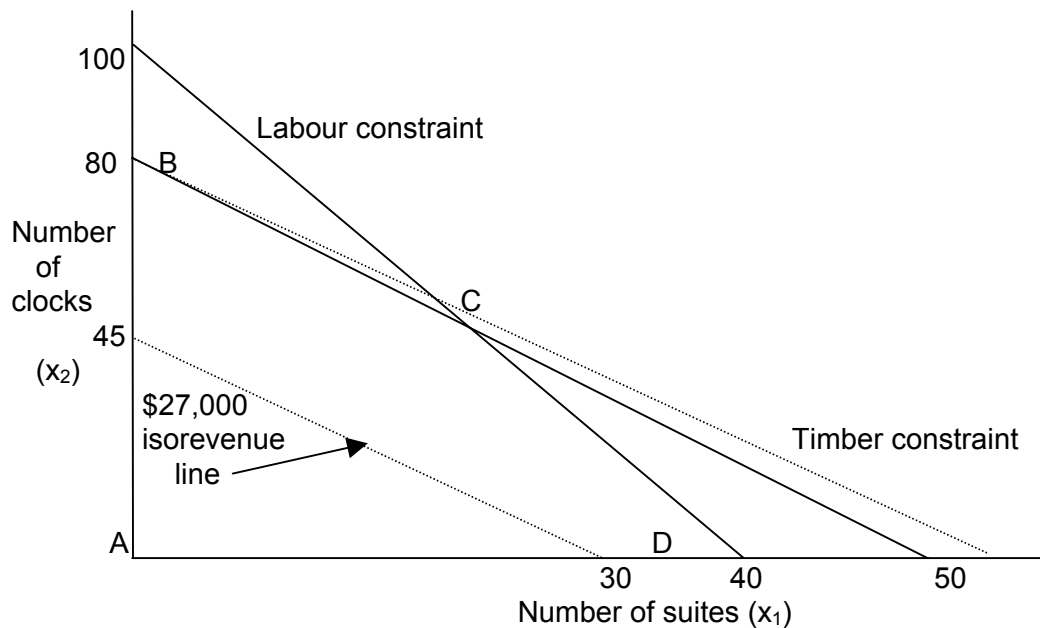
$$= - \$900 / \$450 = -2$$

Figure 4. Determination of the optimal plan when the net revenue of clocks is
increased to $600

Provided the slope of the isorevenue line remains between -2.5 and -1.6, the point furtherest from the origin touched by an isorevenue line will be point C. Now suppose the net revenue per suite is fixed by a sales contract at $900, but that for clocks is uncertain, depending on demand conditions. For point C to remain optimal, the net revenue of clocks must be such that the net revenue ratio remains between -2.5 and -1.6,

i.e. $-2.5 \leq -c_1 / c_2 \leq -1.6$

i.e. $2.5 \geq -c_1 / c_2 \geq 1.6$

Solving the two parts of this inequality expression for c_2, it is found that

$c_2 \geq c_1 / 2.5 = \$900/2.5 = \$360$, and

$c_2 \leq c_1 / 1.6 = \$900/1.6 = \$562.50$.

That is, the net revenue for clocks can vary in the range $360 to $562.50 without optimal activity levels changing.

A special case exists where the slope of the isorevenue line is exactly equal to that of a linear segment on the boundary of the feasible region. For example, if the net revenues of suites and clocks were $900

and $360 respectively then the slope of the isorevenue line would be -2.5, exactly matching that of segment CD on the boundary of the feasible region. In this case, any point along CD would be equally profitable; there is no unique optimal policy. Further, small variations in net revenues may result in large changes to the optimal policy. This, of course, is a an unusual situation in that it would only be by coincidence that the slopes of the isorevenue line and one of the constraint lines were exactly equal.

Shadow prices

The computer solution method for LP problems involves selecting activities to be brought into the basis (solution) only if their net revenue exceeds their opportunity cost. When an LP problem is solved, a number of activities may be absent from the optimal solution. These include real or production activities, and also what are called *disposal activities* or resource non-use activities. The latter are a device to convert inequalities into equations to provide an initial feasible solution for the simplex solution method. In this context, the constraints for the decision solved graphically above can be written as

$$12 x_1 + 7.5 x_2 + 1 x_3 = 600$$

$$100\ x_1 +\ 40\ x_2 + 1\ x_4 = 4000$$

where x_3 and x_4 are non-use or 'disposal' of timber and labour respectively.

In the case of a maximization problem, for a real activity, the shadow price is the amount by which the net revenue would be reduced if one unit of this activity were forced into the solution. Equivalently, it is the amount by which the net revenue would have to increase before this activity entered the optimal solution. For a disposal activity, the shadow price is the cost of forcing one unit of a resource into disposal or non-use, or equivalently the amount by which overall revenue would increase if another unit of the resource were available. The calculation of shadow prices will not be illustrated here, but the concept will be discussed further with respect to computer output.

3. APPLYING THE SIMPLEX METHOD TO RESOURCE ALLOCATION MODELS

The above method of solving linear programming problems is quick and simple when there are only two activities, even if there are many constraints. But if there are three or more activities, then the two-dimensional graphical approach breaks down. However, a mathematical procedure known as the simplex method has been devised which can be used to solve LP problems regardless of the numbers of activities and constraints (even when there are 1000 or more of each). The mathematical basis of the simplex solution *algorithm* involves advanced matrix algebra which will not be explained here. While it is desirable to have an understanding of the economic logic of the solution procedure, to use the technique it is only necessary to be able prepare the data (objective function, activities and constraints) for input to a computer package and to interpret the computer output.
Invariably, linear programming problems are solved using a computer package. A variety of computer programs are available for solving LP problems. Provided the problem is not too large – not more than a couple of hundred activities and constraints – Excel Solver may be used to obtain the optimal solution and shadow prices. For illustration purposes, an extension of the cabinet-maker's decision problem in which there are three activities and three constraints is presented below, and the method of using Excel Solver is demonstrated.

Example 2

Suppose the cabinet-maker of Example 1 can produce roll-top desks as well as dining room suites and grandfather clocks. Also, he wishes to distinguish between general labour (for sawing, joining and polishing) and specialized woodcarving labour used in decorating his furniture. Supplies of timber and general labour are as in Example 1, while 700 hrs per year of woodcarving labour are available. The net revenue from each desk is $600. Suites and clocks require timber and general labour as in Example 1, plus five and eight hours of carving labour respectively per unit. Each desk requires 8 lm of timber, 60 hrs of carpentry labour and 7 hrs of carving labour. Assuming that the cabinet maker's objective is to maximize net revenue, determine the optimal production plan, net revenue ranges and shadow prices.

Solution using Excel Solver

Table 1 presents the linear programming formulation for this problem, set up on a spreadsheet ready for solution using Solver. The activities are represented across the columns of the tableau (columns B to D), and the resources down the rows (rows 5 to 7), with the objective function as row 8. The *technical coefficients* form the body of the tableau (cell B5 through to D7). Column G lists the initial resource supplies or constraint right-hand-sides.

Table 1. Extended cabinet maker's problem in spreadsheet format

	A	B	C	D	E	F	G
1	Extended cabinet maker's decision problem						
2							
3	Constraint or objective	Dining room suites	Grandfather clocks	Roll-top desks	Resource use	Sign	Resource supply
4	Activity level	0	0	0			
5	Timber	12	7.5	8	0	$\leq$	600
6	General labour (hrs)	100	40	60	0	$\leq$	4000
7	Carving labour (hrs)	5	8	7	0	$\leq$	700
8	Net revenue	900	450	600	0		

Prior to using Excel Solver, it is necessary to introduce a row for activity levels (row 4); these activity levels are initially set at zero. It is also necessary to introduce a column for resource use (column E). As well, a column for signs of the constraints is introduced (column F), in this case containing only '$\leq$' signs.[1]

The most complex step in setting up the initial tableau is to enter formulae in the 'Resource use' column, i.e. column E:

1. the resource use for the timber constraint (cell E5) is entered as the formula '=SUMPRODUCT(B$4:D$4,B5:D5)'. The cell value initially takes a level of zero, because the activity levels in row 4 are zero. Note that absolute cell references are required for row 4 (represented by the dollar sign before the row number).
2. the contents of cell E5 are then copied to cell E6 through to cell E8. The coefficients in cell E5 to E7 represent 'resource use' with respect to the resource constraints, while the value in cell E8 is the level of the objective function. Initial values in these cells are again zero.

Once these data have been entered onto

the spreadsheet, Solver can be called up to further set up the problem for solution. Solver is to be found under the Tools menu of Excel. (If it is not currently available, seek assistance on how to access it.) Note that the general form of the Solver window is:

> Set Target Cell:
> Equal to:
> By Changing Cells:
> Subject to the Constraints:

First declare the target cell. This is the objective function, and is chosen by clicking in the total net revenue cell, here E8. Next check that the maximization option of the 'Equal to' row has been chosen, with a dot in the 'Max' circle. Cells B4 to D4 are selected as the changing cells, i.e. the levels of the decision or x variables.

It is next necessary to add constraints. First click on the 'Add' button. When adding constraints, select corresponding cells in columns E and G. In the 'Add constraints' dialogue box, under 'Cell reference', click on the cell range E5 to E7, then under 'Constraint' click on cells G5 to G7. The sign '$\leq$' is automatically selected as the default. Click on the Add or OK button to confirm that this constraint is to be added.

It is also necessary to add non-negativity constraints. Hence click on the Add button again, then select cells B4 to D4. In this case, the sign must be changed to '$\geq$', and the value of zero must be entered on the right-hand-side (under 'Constraint'). The OK button is now pressed to confirm addition of

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[1] If not available in the Excel version, these symbols may be copied and pasted from a wordprocessor file. These signs are included for readability of the tableau only, and could be replaced by text, e.g. 'le' for 'less than or equal to'.

this constraint. The constraints will now appear in the dialogue box. The Change button may be used to modify these constraints if there are any apparent errors.

The Solve option may now be used to obtain the optimal product mix. When clicking on the Solve button, the spreadsheet should change to that of Table 2. (If this is not the case, then there are errors in the tableau or Solver specifications such as errors in constraints which need to be tracked down.) The optimal solution is to produce 30.76 clocks and 46.15 desks (these levels could be truncated to whole numbers). The total net revenue is $41,538. All of the timber and general labour is used, but about 130 hours of carving labour are unused (or in disposal).

A little further information can be gleaned from through sensitivity or post-optimality analysis. This is not generated automatically – the 'Sensitivity' option must be chosen under the 'Reports' options in the 'Solver Results' dialogue box. It is possible to move between the spreadsheet and the reports by clicking on buttons near the bottom of the computer screen. Table 3 presents 'Sensitivity Report 1' for this decision problem. In the first section of this table, in the Reduced Gradient column, it is found that the net revenue of dining room suites would have to increase by a little over $69 for this activity to enter the basis or solution. In the second section, under Lagrange Multiplier, it is found that the reduction in total net revenue would be $23.08 if one unit of timber were removed and $6.92 if one hour of general labour were removed. These are the amounts the cabinet-maker could afford to pay for additional units of these resources and still be able to use them profitably.

As indicated by Example 2, inclusion of additional activities and constraints paves the way to more realistic formulation of decision problems. Of course, if there are large numbers of activities and constraints then there will be large numbers of columns and rows in the simplex tableau, and checking of the tableau becomes more critical. Even small linear programming problems are usually solved on a computer, and commercial computer packages of varying levels of sophistication are available. Some of these packages perform a more detailed post-optimality analysis than that presented above.

4. PRACTICAL DIFFICULTIES IN LINEAR PROGRAMMING ANALYSIS

The linear programming technique as outlined above is mathematically elegant, computer packages to derive optimal policies are readily available, and the technique is widely used. However, the user needs to be aware of a few theoretical limitations and occasional computational difficulties.

Correct perception of the role of linear programming

Linear programming is capable of simultaneously evaluating the profitability of large numbers of activities in the presence of numerous constraints. It can solve decision problems that are beyond the power of intuition, pencil-and-paper methods and formal budgeting. However, the role of this powerful decision aid is often poorly understood. The analyst carrying out a linear programming study in not usually the person responsible for making policy decisions and taking the consequences of these decisions. It should not be expected that the optimal plan derived from a single linear programming analysis will be implemented in precise detail by management. Decision-makers are likely to have particular hunches and preconceptions as to the best course of action. Results generated by a linear programming study typically are used to reinforce or challenge existing views and tentative plans, i.e. to provide decision support. Further, decision-makers often wish to ask a number of 'what if' type questions, e.g. 'What if suites can be produced with 90 rather than 100 hours of labour?' or 'What if the net revenue of clocks is $500 rather than $450?'.

Table 2. LP spreadsheet after solution by Solver

	A	B	C	D	E	F	G
1	Extended cabinet-maker's decision problem						
2							
3	Constraint or objective	Dining room suites	Grandfather clocks	Roll-top desks	Resource use	Sign	Resource supply
4	Activity level	0	30.769231	46.1538			
5	Timber	12	7.5	8	600	≤	600
6	General labour (hrs)	100	40	60	4000	≤	4000
7	Carving labour (hrs)	5	8	7	569.231	≤	700
8	Net revenue	900	450	600	41538.5		

Table 3. Sensitivity report for cabinet-maker's decision problem, as generated by Excel Solver

Cell	Name	Final Value	Reduced Gradient
B4	Activity level Dining room suites	0	-69.23083027
C4	Activity level Grandfather clocks	30.76923077	0
D4	Activity level Roll-top desks	46.15384615	0
Cell	Name	Final Value	Lagrange Multiplier
E5	Timber Resource use	600	23.07692308
E6	General labour (hrs) Resource use	4000	6.923076923
E7	Carving labour (hrs) Resource use	569.2307692	0

Results of post-optimality analysis and further computer solution runs will help to shed light on these questions. LP tends to be used interactively to explore a number of variations to a basic model in a single session on the computer, and to produce a good deal of information about optimal activity levels and sensitivity of profits or costs to variations in parameter levels (data estimates or assumptions). The broad picture which is built up of the alternatives and consequences can be of considerable help in guiding decision making.

Assumptions of the technique

The linear programming model presented above has implicitly relied upon a number of assumptions, the acceptability of which may be questioned in particular applications.
Planning horizon. In production planning applications of linear programming, net

revenues make no allowance for overhead costs. That is, the firm is assumed to possess a fixed asset structure and the analysis concentrates on how current resources should be deployed to maximize profits in the short term. The examples which have been presented all rely on single-period or static models. In the longer term the firm may acquire more land, factories, machines and so on. Planning these capital acquisitions over time is a problem of a different order to maximizing gross margins for a single production period.

Single-valued expectations. It has been assumed that gross margins, resource supplies and input-output coefficients are known with certainty. In practice these are often estimated with a good deal of guesswork. Further, single point estimates are made when in reality it would be more

meaningful to specify a probability distribution of values, particularly in the case of net revenues.

Linear objective function. The objective function of the decision maker is assumed to be a linear function, and to depend on net revenues or input costs alone. Particularly in enterprise planning applications, the decision-maker may be concerned about both level of income and variability of income over time. Activities which have relatively low expected net revenues may be chosen because of low risk associated with this revenue. Suppose in Example 2 that suites have an assured market outlet at a pre-arranged price but clocks are sold directly by the manufacturer, at prices which can vary widely depending on the whim of the market. He may then produce more suites than indicated in the 'optimal' plan because of the lower uncertainty about the net revenue of suites.

Additivity within activities. The amount of each resource used per unit of each activity is assumed to be constant regardless of the level at which the activity is conducted. If one suite requires 12 lm of timber and 100 hrs of labour then 20 suites require 240 lm of timber and 2000 hrs of labour. In practice, increasing or decreasing returns to variable factors are common, e.g. as more suites are produced the labour input per suite may be reduced due to more streamlined production.

Fixed resource proportions. The input-output coefficients are constant within each activity vector, e.g. fixed proportions of timber and labour are used to produce each suite. It may be possible to use timber more efficiently, including using offcuts, if greater time is taken in carpentry and joining.

Independence of activities. It is assumed that the level at which any activity is conducted has no effect on input levels or revenue of any other activity, i.e. there is no complementarity between activities. If offcuts from production of suites could be used in production of clocks, then the timber requirement of the two activities would be less than that of either activity in isolation.

Divisibility. Resource inputs and activity levels are assumed to take a continuous range of fractional units. In Example 1, the 4000 hours of labour may be supplied by two full-time tradesmen, and any labour supply level which is not a multiple of 2000 hrs may not be practicable. Also, the solution obtained for this decision problem includes 30.77 clocks and 46.15 desks. However, clocks and desks can only be produced in whole numbers. An approximate integer solution can be obtained by truncating the final values of the decision variables, which here would yield 30 clocks and 46 desks. But we cannot be certain that this is the optimal integer plan.

In practice, procedures have been devised to overcome most if not all of the limitations implied by these assumptions, though at the cost of greater tableau complexity. Multi-period models may be constructed to allow for expansion in the fixed asset base over time. Variations in factor proportions may be readily accommodated, e.g. if labour and timber can be combined in different proportions to produce suites then two or more suite activities, each having different input-output coefficient vectors, may be defined. If complementarity exists between two activities then these may be combined into a single activity producing two products. Optimal integer levels may be obtained using the mixed integer programming technique. The point to be made here is that one should not be discouraged from using the linear programming by what appear superficially to limitations of the technique.

Some other potential problems

Sometimes a linear programming tableau is constructed which possesses either no solution or no unique optimal solution. For example, consider the following models:

Model 1

maximize $Z = x_1 + x_2$
subject to $2 x_1 + x_2 \geq 7$
$2 x_1 + x_2 \leq 6$
and $x_1 \geq 0, x_2 \geq 0$

Model 2

maximize $Z = x_1 + x_2$
subject to $2 x_1 + 3 x_2 \geq 7$
$4 x_1 + x_2 \geq 6$
and $x_1 \geq 0, x_2 \geq 0$

The constraints of Model 1 are contradictory since $2 x_1 + x_2$ cannot be at the same time greater than or equal to seven, and less than or equal to six, so this problem has no feasible solution. The objective function of Model 2 can be increased without limit, since no upper bounds are imposed on the values that x_1 and x_2 can take; this problem is said to be unbounded. The usual cause of these unsolvable problems is an error in specification of the model. For example, it is simply not possible to have unlimited resource supplies as implied by an unbounded problem.

Other problems can arise during solution such as rounding errors and 'degeneracy'. Specific procedures are available to overcome these. They will not be discussed further here.

5. OPTIMAL TRANSPORTATION AND LOCATIONAL EFFICIENCY MODELS

The purpose of a transportation model is usually to determine the transport or 'shipping' allocation or a commodity such that quantities are moved from various origins to various destinations at minimum cost.

Example 3

Suppose log timber is to be transported from various production regions to various markets in a small country, as in Table 4. The two domestic supply regions – central and islands – can produce an annual turnoff of 100,000 and 120,000 cubic metres (m^3) respectively, and up to 200,000 m^3 can be obtained annually as imports. Demands are 200,000 m^3 per year in the northern industrial region, 100,000 m^3/year in the western region and 70,000 m^3/year in the south and east region.

Solution

An initial transportation table is set up as in Table 4. This summarises the timber supplies from the various origins and demands at the various destinations. Transport costs from origin to destination in dollars per cubic metre are as indicated in the body of the table, e.g. \$40/m^3 for 'shipping' from the central region to the northern industrial region.

The optimal transport allocation can be obtained by linear programming. If we let

Z = total transport cost
x_{ij} = number of units transported from origin i to destination j ($1000 m^3$)
c_{ij} = cost of transport from origin i to destination j (\$/m^3)

then this decision problem may be represented algebraically as:

minimize overall transport cost

$$Z = 40x_{11} + 25x_{12} + 20x_{13} + 30x_{21} + \ldots + 60x_{33}$$

subject to supply and demand constraints

$$x_{11} + x_{12} + x_{13} \leq 100$$
$$x_{21} + x_{22} + x_{23} \leq 120$$
$$x_{31} + x_{32} + x_{33} \leq 200$$
$$x_{11} + x_{21} + x_{31} \geq 200$$
$$x_{12} + x_{22} + x_{32} \geq 100$$
$$x_{13} + x_{23} + x_{33} \geq 70$$

plus non-negativity constraints on all nine transport quantities ($x_{11} \geq 0, \ldots x_{33} \geq 0$).

Since the quantities are in terms of thousands of cubic metres, the objective function will be expressed in terms of \$1000s.

Table 4. Log supplies, demands and transport costs

Source	Destination			Supply (1000 m^3)
	Northern industrial region	Western region	South and east	
Central region	*40*	*25*	*20*	100
Islands	*30*	*20*	*35*	120
Imports	*100*	*100*	*60*	200
Demand (1000m^3)	200	100	70	

The initial spreadsheet for this model is set up as in Table 5. The transport quantities are initially set to zero (cells B6 to D8). A column for sums of supply allocations (column E) is added, with elements set as the supply sum for each origin, e.g. cell E6 is set equal to the sums of the values of cells B6 to D6. Similarly, a row for demand allocations (row 9) is added, with elements equal to the sums allocated for the various destinations, e.g. the value in cell B9 is set equal to the sum of values in cells B6 to B8. Transport costs are entered as a separate block (cells B12 to D14). A total transport cost (objective function) cell is set up) with the formula

'=SUMPRODUCT(B6:D8,B12:D14)'.

Solver is now called up under the Tools menu, and the various entries are made in the dialog box. The target cell is set as cell C16, and the 'Min' option is chosen. The changing cells are selected as the range B6:D8. The constraints can now be entered. The column vector E6:E8 is set less than or equal to F6:F8 for supply constraints, the row vector B9:D9 is set greater than B10:D10 for the demand constraints, and the block of changing cells B6:D8 is set greater than or equal to zero.

On solving this LP problem, Table 6 is obtained. No timber is to be transported from the central region to the northern region (allocation in cell B6 = x_{11} = 0), 100 units (thousand cubic metres) are transported from the central region to the western region (allocation in cell C6 = x_{12} = 100) and so on. Although up to 200 units of timber may be imported, the demands are satisfied by imports of 150 units (cell E8).

A sensitivity table could be generated, and this would indicate for example the amount by which transport costs for each non-used shipment path would have to fall before transport through this path would become warranted (would reduce overall cost).

Transhipment models

The structure of transhipment models is similar to that of transportation models, except that intermediate destinations are added where product is stored or processed. For example, timber could be produced in a number of areas, processed into plywood at a specific location, and then transported to other areas for marketing. Changes in volume may take place with storage or processing, in which case it is necessary to standardize the volume units. In the case of processing, since the optimal shipment path will depend on processing cost at the various processing locations, but processing cost will vary with volume of throughput, it is usually necessary to adopt a trial-and-error solution approach.

6. CAPITAL BUDGETING OR PORTFOLIO SELECTION MODELS

Another useful application of linear programming is to assist in the selection of investment projects. In this application, the investment projects are treated as separate activities, and the objective function is defined in terms of net present values.

Table 5. Initial spreadsheet for timber transportation model

	A	B	C	D	E	F
1	Timber transportation model					
2						
3	Source		Destination			
4		Northern industrial region	Western region	South and east	Sum of allocations	Supply
5						
6	Central region	0	0	0	0	100
7	Islands	0	0	0	0	120
8	Imports	0	0	0	0	200
9	Sum of allocations	0	0	0		
10	Demand	200	100	70		
11						
12	Central region	40	25	20		
13	Islands	30	20	35		
14	Imports	100	100	60		
15						
16	Total transport cost =		0			

Table 6. Optimal log transport allocation

	A	B	C	D	E	F
1	Timber transportation model					
2						
3	Source		Destination			
4		Northern industrial region	Western region	South and east	Sum of allocations	Supply
5						
6	Central region	0	100	0	100	100
7	Islands	120	0	0	120	120
8	Imports	80	0	70	150	200
9	Sum of allocations	200	100	70		
10	Demand	200	100	70		
11						
12		40	25	20		
13		30	20	35		
14		100	100	60		
15						
16	Total transport cost =		18300			

Example 4

Three plantation type options are available for 200 ha of community land, viz. fast-growing 'softwood' species (e.g. Gmelina)[2], hardwoods (e.g. Mahogany) or mixed species planting. Resource constraints include land, labour, and capital in the first two years of plantation establishment. (The technical coefficients are provided in Table 7.) Over a 20-year period, two rotations of softwoods or one rotation of the other plantation types could be grown. $400,000 in capital is available for use in the first two years, and may be supplemented by borrowing of up to $50,000 in each of these years. The natural resource management agency wishes to determine what combination of the plantation types would provide the greatest aggregate NPV.

Solution

The initial spreadsheet for this decision problem is provided as Table 7. This is in a similar format to the earlier resource allocation model, except that a few features have been added:

1. Provision is made to transfer unused capital from the first year to the second. This takes up or uses capital in the first year, but makes capital available in the second year, which is achieved by having a +1 coefficient for the first year and a –1 coefficient for the second year.

2. Borrowing activities are introduced, with units of $1000. These are 'supply' activities, with negative coefficients (-1s) in the relevant capital rows.

3. The objective function coefficients for the borrowing activities are the loan interest and redemption costs, in year 20 currency equivalents. As an approximation, these are taken as negative NPVs equal to the amounts borrowed. [3]

4. Constraint rows are introduced for the borrowing activities, with +1 coefficients in the columns of the borrowing activities, so as to limit the amounts borrowed.

When this problem is solved, the spreadsheet of Table 8 is obtained. The optimal investment is to choose softwoods only, plant all 200 ha, and borrow $50,000 in each year. Not all of the available labour is used. An aggregate NPV of $5.9M is predicted.

Extensions to this portfolio selection model include specifying dependent or mutually exclusive projects, and specifying that if a project is to be introduced then the level must be sufficiently high to warrant the tooling up, i.e. imposing a minimum threshold level. For example, it might be decided that at most two of the three plantation type options considered above can be adopted, and that the minimum area for any plantation type is 50 ha. These requirements can easily be built into the model if mixed-integer programming – available within Excel Solver – is used.

Portfolio selection models can also be designed to take risk on asset returns into account. For example, the variance as well as the expected payoff for each investment may be estimated, and quadratic programming used to determine the optimal investment portfolio given the decision-maker's degree of risk aversion.

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[2] Gmelina is not technically a softwood, but has similar wood properties.

[3] Calculation of objective function coefficient for the borrowing activity is rather complex. This is the present value of the interest plus redemption payments at the end of 20 years,

and will depend on when the repayment is made and what is the inflation rate. If it is assumed that the loan interest rate is equal to the inflation rate, then a simple approximation may be made. If a loan were taken out at the beginning of a year, at say 15%, and paid at the end of the year with interest (i.e. $1.15 repaid for each dollar valued), the present value of the repayment would be $1.15/1.15 or $1. Similarly, if a loan were repaid after 20 years, and interest were compounded, the repayment would have a present value of $1.

Table 7. Initial tableau for forestry investment portfolio decision problem

	A	B	C	D	E	G	H	H	I	J
1	Portfolio selection model									
2										
3	Constraint or objective	Soft-woods	Natives	Mixed species	Transfer capital, yr 1->2	Borrow capital, year 1 ($1000)	Borrow capital, year 2 ($1000)	Res-ource use	Sign	Resource supply
4										
5	Activity level	0	0	0	0	0	0			
6	Land (ha)	1	1	1				0	$\leq$	200
7	Labour (person days)	45	30	35				0	$\leq$	10000
8	Capital, year 1 ($1000)	2	2.5	2.3	1	-1		0	$\leq$	400
9	Capital, year 2 ($1000)	0.5	0.5	0.7	-1		-1	0	$\leq$	0
10	Max. loan, year 1					1		0	$\leq$	50
11	Max. loan, year 2						1	0	$\leq$	50
12	NPV ($1000)	30	28	24		-1	-1	0		

Table 8. Final tableau for forestry investment portfolio decision problem

	A	B	C	D	E	G	H	H	I	J
1	Portfolio selection model									
2										
3	Constraint or objective	Soft-woods	Natives	Mixed species	Transfer capital, yr 1->2	Borrow capital, year 1 ($1000)	Borrow capital, year 2 ($1000)	Res-ource use	Sign	Resource supply
4										
5	Activity level	200	0	0	50	50	50			
6	Land (ha)	1	1	1				200	$\leq$	200
7	Labour (person days)	45	30	35				9000	$\leq$	10000
8	Capital, year 1 ($1000)	2	2.5	2.3	1	-1		400	$\leq$	400
9	Capital, year 2 ($1000)	0.5	0.5	0.7	-1		-1	0	$\leq$	0
10	Max. loan, year 1					1		50	$\leq$	50
11	Max. loan, year 2						1	50	$\leq$	50
12	NPV ($1000)	30	28	24		-1	-1	5900		

7. GOAL PROGRAMMING

A more systematic way of dealing with multiple goals is through *goal programming* (GP). Here, a number of goals are identified, and a target or *aspiration level* is specified for each. *Deviational activities* are then introduced in the tableau to allow for underachievement or overachievement of goals relative to the aspiration levels. In particular, an *underachievement activity* is introduced for each aspiration floor, and an *overachievement activity* is introduced for each aspiration ceiling. A series of constraints is then set up for the goals. In the objective function, the coefficients are not the activity payoffs. Rather, they are the costs of underachievement and overachievement of goals relative to the aspiration levels. The objective function now states that the sum of shortfalls and over-runs, with appropriate coefficients for each, be minimized.

Because of the presence of deviational variables in goal programming, the constraints are referred to as *soft constraints*, c.f. the *hard constraints* for resource limits. Even if some or all the aspiration levels cannot be met, the problem will not be found to be infeasible (with an error message when an attempt is made to find the optimal solution). Rather, levels of the deviational variables in the solution will indicate the extent to which it is not possible to meet the goal targets.

The question arises as to what coefficients or weights to place on the deviational activities. Shortfalls can be expressed in a variety of different units, such as dollars, rate of return as a percentage, jobs and hectares. One approach would be to place a weight on each deviational variable to represent the cost of goal under-achievement or overachievement. Applying a system of weights to these is known as *weighted goal programming*. In practice, expressing the deviations in a common unit – say dollars – may be difficult and unnecessary.

Often a priority ordering can be established between goal dimensions, referred to as *lexicographic* or *preemptive* goal programming. For example, the requirement may be to achieve the target dividend rate first, then NPV payoff, then job creation, then the environmental requirement. This could be achieved by placing weights of different orders of magnitude on the shortfall activities. However, software has been developed in which the priority ordering is stated as input data and the solution algorithm enforces this priority ordering of goals. Often, it is possible to solve lexicographic goal programming problems by using an appropriate system of weighting on deviational variables (e.g. $1M per unit shortfall for top priority goals, $0.1M for second priority goals, and so on.)

Example 5

A government wishes to support local development projects with employment and environmental benefits. Three projects are identified which will contributed to these goals, viz. road construction, tree planting and public housing. Each kilometre of road construction provides 200 jobs (person months), has net carbon release of 2 tonnes, and requires expenditure of $100,000. Each hectare of tree planting generates 3 jobs, sequesters 5 tonnes of carbon per year and requires $2000 of capital. Each house constructed provides 20 jobs and requires expenditure of $30,000. Determine the government program which meets these goals most closely, if equal weights are attached to the three goals.

Solution

The initial tableau is set out as Table 9. Note that shortfall activities are included for jobs and carbon sequestration (with coefficients of +1 in the relevant constraint rows), and an expenditure over-run activity is included (with coefficient of +1 in the expenditure row). The objective is to minimize the sum of deviations. Note that the constraints are in inequality form, rather than forcing exact equalities. The constraints are set up in the usual way, i.e. the formula

  '=SUMPRODUCT(B$4:G$4,B5:G5)

is entered in cell H5, and copied into cells H6 to H8. For the objective function, this in effect means summing the numbers of units for the three deviation variables (values in cells E4 to G4). The objective function is one of minimization.

The solution to this problem is reported in the spreadsheet of Table 10. The recommended portfolio includes about 3.5 km of road construction and a little over 100 ha of tree planting. The jobs and carbon sequestration goals are met, but there is an expenditure over-run of a little over $50,000.

f any goal were considered more important than another, this could be reflected by weights of different orders of magnitude in the objective function. For example, if the expenditure control was a critical goal, a weight of say 1000 could be entered in cell G8.

Table 9. Initial tableau of local government jobs and environment program

	A	B	C	D	E	F	G	H	I	J
1	Goal programming example									
2										
3	Constraint or activity level	Road constn. (km)	Tree planting (ha)	House constn. (houses)	Job short-fall	Carbon seqn. shortfall	Expen diture over-run	Res. use	Sign	Resource supply
4	Activity level	0	0	0	0	0	0			
5	Jobs (person months)	200	3	20	1			0	$\geq$	1000
6	Carbon sequestration (tonnes)	-2	5	0		1		0	$\geq$	500
7	Expenditure ($1000)	100	2	30			-1	0	$\leq$	500
8	Deviation				1	1	1	0		

Table 10. Final tableau for local government jobs and environment program

	A	B	C	D	E	F	G	H	I	J
1	Goal programming example									
2										
3	Constraint or activity level	Road constn. (km)	Tree planting (ha)	House constn. (houses)	Job short-fall	Carbon seqn. shortfall	Expend iture over-run	Res. use	Sign	Resource supply
4	Activity level	3.48	101.39	0	0	0	50.70			
5	Jobs (person months)	200	3	20	1			1000	$\geq$	1000
6	Carbon sequestration (tonnes)	-2	5	0		1		500	$\geq$	500
7	Expenditure ($1000)	100	2	30			-1	500	$\leq$	500
8	Deviation				1	1	1	50.70		

8. CONCLUDING COMMENTS

In this module, the highly versatile technique of linear programming has been introduced with reference to a simple profit maximization problem involving two production activities. This decision problem can be expressed as an algebraic model, and solved graphically, providing insights into the nature of the resource allocation problem. The simplex method, which is quite general in that it can solve linear programming problems containing any numbers of activities and constraints, and is readily available in computer packages. Sensitivity analysis has been shown to provide further information about solution stability and opportunities for short-run expansion of production.

Extensions to the linear programming technique to handle more complex applications include transportation modeling, portfolio selection and goal programming. The technique has great potential for modeling decision situations in forest management and policy.

FURTHER READING

Dayananda, D., Irons, R., Harrison, S., Herbohn, J. and Rowland, P. (in press), *Capital Budgeting: Financial Appraisal of Investment Projects*, Cambridge University Press, Cambridge.

Dent, J.B., Harrison, S.R. and Woodford, K.B. (1986), *Farm Planning with Linear Programming*: *Concept and Practice*, Butterworth, Sydney.

Pannell, D. (1997), *Introduction to*

Practical Linear Programming, Wiley, New
York.