ELSEVIER

European Journal of Operational Research 141 (2002) 440–453

EUROPEAN JOURNAL OF OPERATIONAL RESEARCH

www.elsevier.com/locate/dsw

Discrete Optimization

An improved algorithm for the packing of unequal circles within a larger containing circle

Huaiqing Wang [a,*], Wenqi Huang [b], Quan Zhang [a], Dongming Xu [a]

[a] *Department of Information Systems, City University of Hong Kong, Tat Chee Avenue, Kowloon, Hong Kong, China*
[b] *Department of Computer Science, Huazhong University of Science and Technology, Wuhan, China*

Received 7 June 1999; accepted 29 June 2001

Abstract

This paper describes an approved algorithm for the problems of unequal circle packing – the quasi-physical quasi-human algorithm. First, the quasi-physical approach for the general packing problems is described in solving the pure problems of unequal circle packing. The method is an analogy to the physical model in which a number of smooth cylinders are packed inside a container. A quasi-human strategy is then proposed to trigger a jump for a stuck object in order to get out of local minima. Our method has been tested in numerical experiments. The computational results are presented, showing the merits of the proposed method. Our algorithm can be thought as an adoptive algorithm of the Tabu search. © 2002 Published by Elsevier Science B.V.

Keywords: Optimisation; Packing; Modeling; Tabu search; Adoptive search

1. Introduction

The background of this research is based on two different issues. The first issue is industrial applications. A communication engineer may face a task to accommodate a bunch of optical fibers in a tube having a radius as small as possible. This can also be thought of packing a number of circular disks with different radii into a round tray having a smallest possible size. Another application is in the shipping industry. When transporting a batch of pipes of various sizes, one may want to insert as many small pipes as possible into a large one. This is generally referred to as packing of unequal circles (George et al., 1995).

The second issue is the general packing problem, in particular, the packing of irregular polygons on a plane. The problem is of theoretical importance as well as practical significance. Although the circle packing is less complicated than the packing of irregular polygons, they are both NP-hard. Results of the circle-packing problem will form a firm foundation for the further investigation of the more complicated irregular polygon problems.

* Corresponding author. Tel.: +852-2788-8491; fax: +852-2788-8694.

E-mail address: iswang@is.cityu.edu.hk (H. Wang).

0377-2217/02/$ - see front matter © 2002 Published by Elsevier Science B.V.
PII: S0377-2217(01)00241-7

At a first glance, packing of circles with unequal radii does not seem too difficult. This, in fact, is not true. The key lies in making the circumference of the round tray as small as possible. In doing so, a series of adjustments of the small inner disks have to be performed within the tray, resulting in a huge number of patterns. Only the pattern with a maximum condensation defines the minimum size of the circumferential outer circle. Since the number of possible combinations is enormous even if there are only a few inner circles, approaches fundamentally different from the classic methods must be sought.

Researches on packing equal circles into a rectangular bin have been documented. Iserman (1991) presented a series of heuristic solutions to the problems of equal-circle packing. Fraser and George (1994) focus on the stocking of cylindrical paper-rolls, in other words, putting a number of identical circular bins into a given rectangular box. Dowsland (1991) treated the problem from a different viewpoint, namely, finding the most suitable box in order to contain a given number of cylindrical objects. Under some simple conditions, the best solution to this type of problems is a pattern of square lattice. In more complicated cases, however, a denser pattern may be obtained by using some fast heuristic algorithms. Nurmela and Östergård (1997) have obtained the most condense possible pattern for packing 21–50 circles into a square.

There are fewer publications discussing the problems of unequal circle packing. In a discrete manner, Hochbaum and Maass (1985) considered the packing of objects with the same shape but different sizes. In their paper, the smallest spatial unit is a unit square. For example, a circle is approximated with a mosaic of many such small squares. They have discussed packing of objects with various shapes and sizes, including the unequal circle problems. The work of Hachbaum and Maass has an important theoretic significance as they have proved that, for the type of NP-hard problem such as packing of unequal circles, there exist highly accurate algorithms with polynomial complexity. Errors in the solutions obtained by using such algorithms as compared with rigorous solutions may be less than any given small positive number. However, as pointed out by the authors, their algorithm is only of conceptual significance. Even for a very small real-life example, the computation may take a prohibitively long time due to the high order of polynomials. Therefore, heuristic algorithms with a high efficiency must be sought.

George et al. (1995) studied the packing of unequal circles within a square with an application to the transport of tubes. Several practical requirements have been considered, including stability of the stock, operability, and the highest possible utilization of the space, but without mentioning the algorithmic efficiency. Similar to other methods involving genetic algorithm, their method cannot avoid the loss of information while passed from parent to offspring.

Lubachevsky and Graham (1997) studied the problem of finding a smallest possible round tray to contain a number of small circles having a unit radius. They were able to repeat even improve some previously obtained best patterns by using a billiard simulation algorithm that simulates elastic collision amongst smooth balls. Both the time intervals during which collisions occur and the deformation of the balls are assumed infinitesimal. For patterns of equally sized small circles within a large outer circle, they have proved that, in many circumstances, a seemingly ideal pattern of hexagon is in fact not the most condensed distribution.

In this paper, we present a new method that mimics human behavior to avoid being trapped into a local minimum, and call it the quasi-physical quasi-human (QuasiPQuasiH) algorithm. The paper is organized as follows. Section 2 gives some background and establishes a mathematical model in the form of nonlinear mathematical programming. Section 3 describes the quasi-physical (QuasiP) method, and the difficulties that arise when using this method. Section 4 describes the quasi-human (QuasiH) strategy. Sections 5 and 6 give mathematical descriptions to the QuasiP and QuasiPQuasiH methods, respectively. Section 7 presents the algorithm. Computation results are presented in Section 8. The final section concludes the paper.

2. Mathematical model of nonlinear programming

2.1. Preamble

Let us first consider the relationship between the problems from two different perspectives. Problem 1 is to find a packing circle of radius r_0 in order to contain a number of given circular bins with radii $r_1, r_2, \ldots, r_n$, respectively. Overlaps between the bins are not allowed. There may exist more than one solution in which r_0 is unique, while the coordinates of the bins are not. When a solution is obtained, the program will terminate. Problem 2 is simply the pure problem of circle packing, that is, given a round tray of radius r_0 and a number of circular bins with radii $r_1, r_2, \ldots, r_n$, respectively, find if and how these bins can be packed into the tray without overlap.

Problem 2 may lead to Problem 1 in the following sense. If Problem 1 is solved, namely, a tray $C_{\min}$ with a minimum radius is found, then for any C that is smaller than $C_{\min}$, Problem 2 is insolvable. Otherwise, a solution to Problem 2 exists, and the pattern obtained in the solution to Problem 1 is a reasonable pattern for Problem 2. On the other hand, Problem 1 may also lead to Problem 2.

Having shown the interrelation between the two problem types, we will only concentrate on Problem 2, the pure problem of circle packing, in the following discussion.

We now give some initial QuasiH considerations. When packing a container, one always tends to fill in the corners and sides at an early stage, rather than put big objects in the middle. This process can be described in a mathematical language. An outer circle (the tray) of radius r_0 is first defined. Then, put the smaller circles (the bins) into it in a descending order in terms of size: $r_1, r_2, \ldots, r_n$. For brevity, the same symbol $r_i (i = 1, 2, \ldots, n)$ is used to represent the ith bin, as well as its circumference or radius when no confusion arises.

The first step is to lay r_1 on the bottom, as shown in Fig. 1. If $r_1 > r_0$, the problem is immediately insolvable. Assume k steps have been successfully taken, bins $r_1, r_2, \ldots, r_k$ are already in the tray without overlapping. The $(k+1)$th step is to put bin $r_{(k+1)}$ into a *reasonable angular position* as low and as close to the left as possible. The term *angular position* refers to the location at which bin $r_{(k+1)}$ is tangential with another two bins already in position. Being *reasonable* means that bin $r_{(k+1)}$ is inside the tray without overlapping its predecessors. As the number of such reasonable angular positions is less than or equal to $2C_{k+1}^2$, we choose the one to make bin $r_{(k+1)}$ at the lowest possible position.

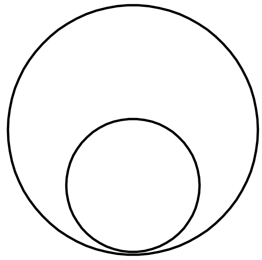

Fig. 1. A circle lays on the bottom.

If there are more than one such position, the left most one is chosen. Therefore the $(k+1)$th step leads to a unique result. In case a reasonable angular position is nonexistent, the algorithm fails. If, on the other hand, all the steps up to nth are successful, a solution is found.

However, this simple QuasiH treatment has its drawback, namely, the lack of completeness. The algorithm may fail for some solvable problems. Fig. 2 shows an example of bearing. If the biggest bin is first laid on the bottom, the rest cannot be accommodated.

The remedy is to combine the QuasiP method that is complete, with the QuasiH treatment that significantly improves the algorithmic efficiency.

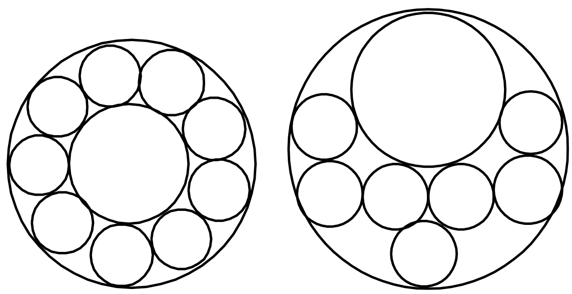

Fig. 2. Examples.

The QuasiPQuasiH algorithm has a much better completeness than the QuasiH method, and a much higher efficiency than QuasiP method.

2.2. A mathematical model of nonlinear programming

A general problem of unequal circle packing can be modeled as a nonlinear mathematical programming. We will show that one of its sub-models is a proper representation of the problem under study, i.e., finding if and how a given number of bins with different radii can fit into a given round tray.

The capital letter I represents the set of all bins, and i an element of I. The radius and coordinates of the ith bin are r_i, and (x_i, y_i), respectively. The Cartesian system is shown in Fig. 3, with the origin at the center of the tray. Consider the set of all different bin pairs, expressed as

$$H = \{(i,j) : i \in I, j \in I, j < i\}.$$

If the tray is also included in the set, and regarded as the 0th object that is part of the plane with the tray of radius r_0 removed, a new object set is obtained, denoted by

$$I^* = I \cup \{0\}.$$

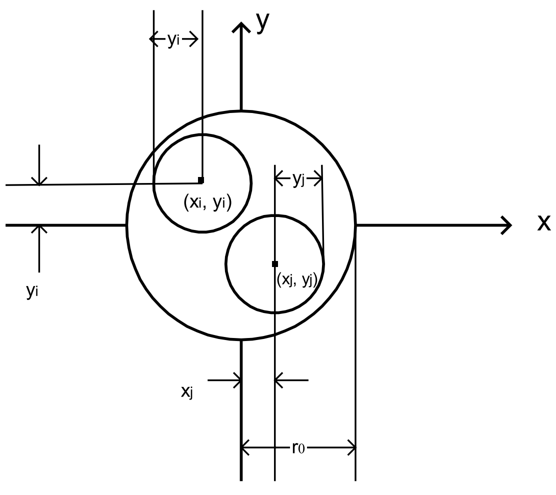

Fig. 3. Cartesian system.

Similarly, the set of all bin pairs in I^* is

$$H^* = \{(i,j) : i \in I^*, j \in I^*, j < i\}.$$

With the symbols defined and inspired by the study of George et al. (1995), the problem is now modeled as

$$\mathrm{Min}(r_0) \tag{1}$$

subject to

$$\sqrt{x_i^2 + y_i^2} \leqslant r_0 - r_i \quad \forall i \in I, \tag{2}$$

$$\sqrt{(x_i - x_j)^2 + (y_i - y_j)^2} \geqslant r_i + r_j \quad \forall (i,j) \in H, \tag{3}$$

$$-\infty < r_0 < +\infty, \tag{4}$$

$$-\infty < x_i < +\infty, \quad -\infty < y_i < +\infty \quad \forall i \in H. \tag{5}$$

In these expressions, r_i is a known real positive number, x_i, y_i, and r_0 are unknowns, $i \in I$.

In this model, there are $(2n+1)$ continuous deterministic variables and $n + n(n+1)/2$ constraints where n is the number of bins, $I = \{1, 2, \ldots, n\}$. Constraint (2) ensures that none of any part of each bin is outside the tray. Constraint (3) indicates no overlapping, namely, the distance between any two inner bins cannot be less than the sum of the two radii. Note that, since r_i in (2) is real and positive, we have $r_0 \geqslant 0$ in any set of solutions. If the problem is solvable, the minimum value of the object function r_0 is unique, but the corresponding (x_i, y_i) is not. One tray may allow many different packing patterns.

A set of real numbers r_0 and $x_1, y_1, \ldots, x_n, y_n$ that satisfy (2) and (3) is termed a feasible solution. It is noticed that for a feasible solution, r_0^* and x_1^*, $y_1^*, \ldots, x_n^*, y_n^*$, the minimum tray radius r_0 has a local minimum r_0^* at $x_1^*, y_1^*, \ldots, x_n^*, y_n^*$. That is, minor changes in the coordinates will result in an increase in r_0^*. Fig. 4 shows the patterns corresponding to global (left) and local (right) optima, respectively.

As showed in Fig. 4, for the problem considered in this paper, local minima phenomenon may occur, e.g., different circles of the same size are swapped, main circle is rotated, etc. Theoretical analysis and numerical experiences show that the model contains many local minima, and in many

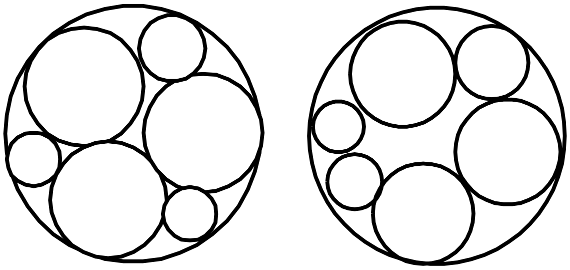

Fig. 4. Patterns.

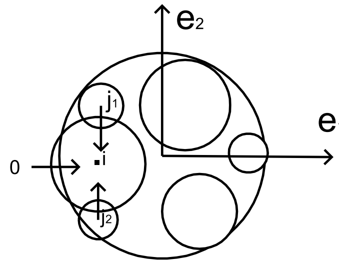

Fig. 5. The ith cylinder.

cases, the number of local minima is much greater than that of global minima. Furthermore, since the model is nonlinear, it is difficult to solve, even if the number of bins is quite small. Indeed, the problem has been proved to be NP hard (see, for example, Lenstra and Rinnooy Kan, 1979).

If (1) is removed from the nonlinear model, and let r_0 be an arbitrarily given real positive number, then (2)–(5) form a special type of nonlinear programming problems, that is, nonlinear constraint-satisfied problem. This is just the mathematical model for our pure problem of unequal circle packing. The positive number r_0 is the radius of the given tray.

3. Quasi-physical method and its drawback

We now describe a physical reality that is equivalent to a packing problem. Think of the bins as cylinders of unit height but with different diameters, and the tray as an infinite plate of unit thickness and with a round hole of radius r_0. Both the cylinders and the hole are smooth and elastic. Imagine that the cylinders are packed in the hole with all axes in parallel. Since all the objects have a tendency to restore their natural shapes and sizes, squeezing and collision occur. This leads to a series of complicated motion. In the end, everyone reaches a state of satisfaction, without extrusion and deformation.

It is obvious that equivalence exists between this physical experiment and our packing problems. Therefore a mathematical description for the movements of the cylinders will provide solutions to the packing problem.

Fig. 5 shows the situation in which the ith cylinder is deformed under the pressure from the 0th (the tray), j_1th, and j_2th objects:

$$\vec{F}_i = \vec{f}_{0i} + \vec{f}_{j_1 i} + \vec{f}_{j_2 i}. \tag{6}$$

Assume the positions of these objects are known. In Fig. 5, there is mutual embedding between the ith and the 0th, and between the j_1th and the j_2th objects. By loosely applying Hook's law without affecting the validity of the results in terms of packing, the three forces in the left side of Eq. (6) may be considered as proportional to the embedding depths. The directions of these forces are all toward the centers of the cylinders. So, the forces can be calculated from the geometry, assuming the elastic constant is 1.

$$\begin{aligned} \left|\vec{f}_{j_1 i}\right| &= \left|\sqrt{(x_i - x_{j_1})^2 + (y_i - y_{j_1})^2} - (r_i + r_{j_1})\right|, \\ \frac{\vec{f}_{j_1 i}}{\left|\vec{f}_{j_1 i}\right|} &= \frac{(x_i - x_{j_1})\vec{e}_1 + (y_i - y_{j_1})\vec{e}_2}{\left|(x_i - x_{j_1})\vec{e}_1 + (y_i - y_{j_1})\vec{e}_2\right|}, \end{aligned} \tag{7}$$

where $\vec{e}_1$ and $\vec{e}_2$ are unit vectors along the x and y axes, respectively, in the Cartesian system. Within a small time interval, the movement of the ith bin is a small step along the total force,

$$\left(x_i^{(t+1)}\vec{e}_1 + y_i^{(t+1)}\vec{e}_2\right) = \left(x_i^{(t)}\vec{e}_1 + y_i^{(t)}\vec{e}_2\right) + \vec{F}_i \cdot \varepsilon, \tag{8}$$

where ε is a small positive constant. The consequence of this small movement is a reduction of mutual squeezing. The "pain" is alleviated.

After every bin moves in this manner, an old pattern $(x_1^{(t)}, y_1^{(t)}, \ldots, x_n^{(t)}, y_n^{(t)})$ is said to have evolved

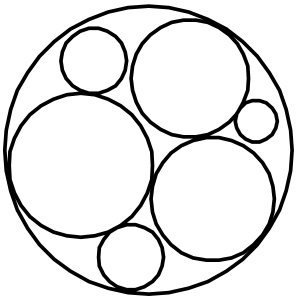

Fig. 6. A progressive evolution process.

into a new pattern $(x_1^{(t+1)}, y_1^{(t+1)}, \ldots, x_n^{(t+1)}, y_n^{(t+1)})$. Such a progressive evolution process simulates the movements of the cylinders. If ε is sufficiently small, a state illustrated in Fig. 6 will be reached, and the evolutionary movements will eventually cease.

Difficulties occur when the space is reduced. In this case, the objects may get stuck when the total forces exerted on the cylinders are zero,

$$\left(x_1^{(t+1)}, y_1^{(t+1)}, \ldots, x_n^{(t+1)}, y_n^{(t+1)}\right) = \left(x_1^{(t)}, y_1^{(t)}, \ldots, x_n^{(t)}, y_n^{(t)}\right).$$

This means that, although a number of cylinders are still in pain, they cannot move any further. The pain may even increase as the iteration continues. Thus, a solution cannot be found, which in fact exists.

Fig. 7 shows a jamming situation. Imagine that, in the left diagram, if one of the two small bins to the left can take a "big move" to the bottom-right corner, the process may continue, leading to a solution. This big move will be referred to as "jumping out of trap" or a *jump*. The bin jumps from a hopeless trap to find a more promising place, starting a new venture.

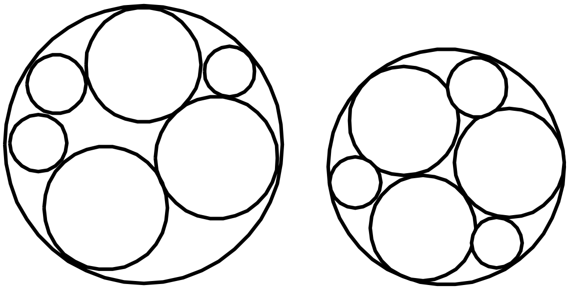

Fig. 7. A jamming situation.

Physically speaking, a trap is a state in which the system's potential energy is at a local minimum. It is clear that in the QuasiP algorithm, a big jump is impossible. The only way out is to start from scratch, and try again. Repeat the process over and over again, a solution can ultimately be obtained.

4. Quasi-human strategy

As has been shown, the purely physical laws such as Hook's do not provide a means for jumping out of a trap. We now turn our focus to human wisdom and experience in an attempt to seek a solution. In a society, people tend to change their position once they feel unsatisfied. The ones who are in the most miserable situation want to change most eagerly. On the other hand, the rich may also wish to spread their wealth to obtain some balance. We observe that changes of the former type happen more frequently than the latter.

Consider the bins in a tray as a "society". The situation of the stuck bins that are under heavy pressure from its neighbors is analogous to the poor. This implies that we can help the poorest by picking it out, and randomly putting it back to somewhere else within the tray. The same can be done to the rich in order to free some space so that others may share the resources. We call the former the strategy of *pain relief*, and the latter the strategy of *resource surrender*.

We now present the synthesis of the QuasiH strategy. The degree of mutual squeezing between two objects, including the tray and the bins, can be measured with the elastic potential. This potential is the integral of the elastic force along the embedding direction. From the previous discussion, the force is proportional to the embedding depth, therefore the potential proportional to the square of the embedding depth (see Fig. 8).

$$u_{ij} \approx d_{ij} \quad (i, j = 0, 1, 2, \ldots, n,\ i \neq j). \tag{9}$$

In (9), u_{0j} and d_{0j} represent the squeezing potential and embedding depth between the tray and the jth

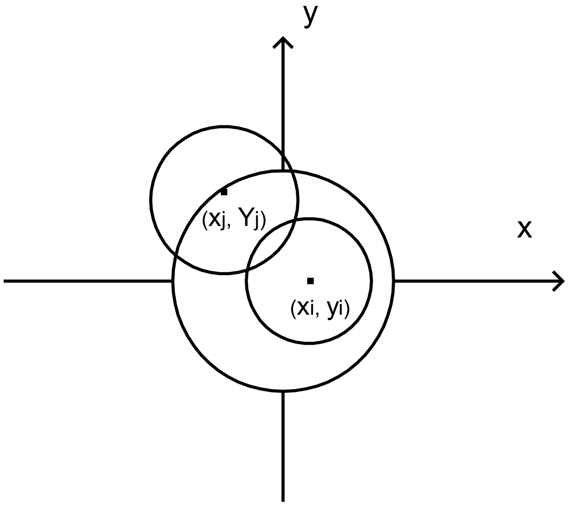

Fig. 8. The synthesis of the QuasiH strategy.

bin, respectively. Similarly u_{ij} and d_{ij} are the squeezing potential and the embedding depth between the ith and the jth bins, respectively. The depths can be calculated from

$$d_{0j} = \begin{cases} \sqrt{x_j^2 + y_j^2} + r_j - r_0 & \text{if } \sqrt{x_j^2 + y_j^2} + r_j > r_0, \\ 0 & \text{otherwise,} \end{cases} \tag{10}$$

$$d_{ij} = \begin{cases} r_i + r_j - \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2} \\ \quad \text{if } \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2} < r_i + r_j, \\ 0 \quad \text{otherwise.} \end{cases} \tag{11}$$

Ignoring the constants, the potential of the ith bin, either the tray or any of the bins, is

$$U_i = \sum_{\substack{\lambda=0 \\ \lambda \neq i}}^{n} d_{\lambda i}^2, \quad i = 0, 1, \ldots, n, \tag{12}$$

and the total potential of the system is

$$U = \sum_{i=0}^{n} U_i. \tag{13}$$

It is seen from (10)–(13) that the potential of each object and the total potential of the system are explicit functions of the pattern $(x_1, y_1, \ldots, x_n, y_n)$. U_i can be used to measure the degree of pain of the ith bin, DP_i:

$$\mathrm{DP}_i = U_i. \tag{14}$$

Although the embedding depth is the same for the two mutually embedded bins, it is considered that a smaller bin experiences more pain than a larger one. Therefore a concept of *relative pain*, RDP_i, is introduced:

$$\mathrm{RDP}_i = \frac{U_i}{r_i^2}. \tag{15}$$

From (10)–(12), RDP_i is dimensionless.

The first QuasiH strategy is formed when a jamming occurs in a pattern $(x_1, y_1, \ldots, x_n, y_n)$. The relative pains of all bins, $\mathrm{RDP}_1, \mathrm{RDP}_2, \ldots, \mathrm{RDP}_n$, are calculated. Choose the one having the maximum pain, and randomly put it into the tray. Suppose the ith is the one being singled out. The pattern

$$(x_1, y_1, \ldots, x_{i-1}, y_{i-1}, x_i, y_i, x_{i+1}, y_{i+1}, \ldots, x_n, y_n)$$

is modified into

$$(x_1, y_1, \ldots, x_{i-1}, y_{i-1}, \tilde{x}_i, \tilde{y}_i, x_{i+1}, y_{i+1}, \ldots, x_n, y_n),$$

where $\tilde{x}_i$ and $\tilde{y}_i$ are the coordinates of an arbitrary point within the tray. From this new pattern computation is carried on, and a jump is performed whenever a jamming occurs until a reasonable pattern is obtained as the solution:

$$(x_1, y_1, \ldots, x_n, y_n).$$

In the above, a combination of the pain relief strategy and the QuasiP algorithm has led to a simple QuasiPQuasiH algorithm. Experiments show that this algorithm considerably outperforms the purely quasi-physical method in terms of computation efficiency.

However, it has been found that, in some cases where the space is relatively compact, a same bin is always chosen to jump in consecutive quasi-human interventions, leading to an endless loop. Thus, only the pain relief strategy is obviously inadequate, and the resources surrender strategy may come to the rescue.

If a chosen miserable bin is the same as the previous one, the current choice is abandoned, but the current jamming pattern $(x_1, y_1, \ldots, x_n, y_n)$

retained. Find the least miserable bin instead, lay it randomly within the tray, and start over the QuasiP algorithm again. Thus, a QuasiPQuasiH algorithm based on two mutually supplementary strategies is established.

Experiments show that the efficiency of the new algorithm is substantially higher not only than that of the QuasiP algorithm, but also the simple QuasiPQuasiH algorithm that merely incorporates the pain relief strategy. Endless loops have never happened in our experiments with the new method.

In simulated annealing (Kirkpatric et al., 1983), the object to be optimized also has a large number of local minima (traps). The philosophy behind this method and the strategy used are different from ours. Therefore the results and applicability with respect to specific applications are also different.

5. Mathematical description of the QuasiP method

From (1)–(13), in the problem of unequal circle packing, the total elastic potential of the system composed of a tray and several bins, U, is a known function of the system pattern with $2n$ independent variables:

$$U = U(x_1, y_1, \ldots, x_n, y_n). \tag{16}$$

U has the following properties: (i) it is defined on the entire $2n$-dimensional Euclidean space $(-\infty, +\infty)^{2n}$, smooth, continuous, and differentiable everywhere; (ii) it is nonnegative, namely, $U(x_1, y_1, \ldots, x_n, y_n) \geqslant 0$ in $(-\infty, +\infty)^{2n}$; (iii) if $U(x_1, y_1, \ldots, x_n, y_n) > 0$, $(x_1, y_1, \ldots, x_n, y_n)$ is not a solution and, if $U(x_1, y_1, \ldots, x_n, y_n) = 0$, $(x_1, y_1, \ldots, x_n, y_n)$ is a solution. Therefore the packing problem is converted to a problem of optimization of the total potential $U(x_1, y_1, \ldots, x_n, y_n)$. The aim is to find a minimum with a corresponding pattern $(x_1^*, y_1^*, \ldots, x_n^*, y_n^*)$. If the potential $U(x_1^*, y_1^*, \ldots, x_n^*, y_n^*) = 0$, $(x_1^*, y_1^*, \ldots, x_n^*, y_n^*)$ is a solution and, if $U(x_1^*, y_1^*, \ldots, x_n^*, y_n^*) > 0$, the problem is insolvable.

There exists an algorithm for the unconstrained optimization of smooth functions, i.e., the well-known method of gradient or steepest descends. It should be pointed out that the evolution of $(x_1, y_1, \ldots, x_n, y_n)$ in the gradient method is consistent with the successive updating of the patterns of the bins in a tray. The mathematical description of the gradient algorithm is as follows:

(1) Randomly define a number of initial points $(x_1^{(0)}, y_1^{(0)}), (x_2^{(0)}, y_2^{(0)}), \ldots, (x_n^{(0)}, y_n^{(0)})$ within a circle centered at the origin, and with a radius r_0. This gives an initial pattern $(x_1^{(0)}, y_1^{(0)}, x_2^{(0)}, y_2^{(0)}, \ldots, x_n^{(0)}, y_n^{(0)})$. Choose a positive number h as an initial step size, and a positive number less than 1, ξ, as a step shrinking factor. Choose two very small positive numbers, ε_1 and ε_2, as the criteria for the judgement of $\operatorname{grad} U$ being approximately zero.

(2) Evaluate the potential function $U(x_1^{(0)}, y_1^{(0)}, x_2^{(0)}, y_2^{(0)}, \ldots, x_n^{(0)}, y_n^{(0)})$. If

$$U\left(x_1^{(0)}, y_1^{(0)}, x_2^{(0)}, y_2^{(0)}, \ldots, x_n^{(0)}, y_n^{(0)}\right) < \varepsilon_1,$$

a solution is found, and the computation terminates. Proceed if

$$U\left(x_1^{(0)}, y_1^{(0)}, x_2^{(0)}, y_2^{(0)}, \ldots, x_n^{(0)}, y_n^{(0)}\right) \geqslant \varepsilon_1.$$

(3) Calculate vector $\operatorname{grad} U$ at $(x_1^{(0)}, y_1^{(0)}, x_2^{(0)}, y_2^{(0)}, \ldots, x_n^{(0)}, y_n^{(0)})$:

$$\operatorname{grad} U = \left(\frac{\partial U}{\partial x_1}, \frac{\partial U}{\partial y_1}, \ldots, \frac{\partial U}{\partial x_n}, \frac{\partial U}{\partial y_n}\right). \tag{17}$$

Calculate the absolute value of $\operatorname{grad} U$. $|\operatorname{grad} U| < \varepsilon_2$ indicates a jamming, and $(x_1^{(0)}, y_1^{(0)}, \ldots, x_n^{(0)}, y_n^{(0)})$ corresponds to a local minimum. In this case a new point is randomly chosen, and the computation resumes over again. If $|\operatorname{grad} U| \geqslant \varepsilon_2$, a new pattern is calculated following the gradient method:

$$\begin{aligned}
x_1^{(1)} &= x_1^{(0)} - \frac{\partial U}{\partial x_1} h_0, \\
y_1^{(1)} &= y_1^{(0)} - \frac{\partial U}{\partial y_1} h_0, \\
&\vdots \\
x_n^{(1)} &= x_n^{(0)} - \frac{\partial U}{\partial x_n} h_0, \\
y_n^{(1)} &= y_n^{(0)} - \frac{\partial U}{\partial y_n} h_0,
\end{aligned} \tag{18}$$

where the partial derivatives $\partial U / \partial x_i$, $\partial U / \partial y_i$, $i = 1, 2, \ldots, n$, are defined at $(x_1^{(0)}, y_1^{(0)}, \ldots, x_n^{(0)}, y_n^{(0)})$

in an $2n$-dimensional space. Using a vector representation,

$$\begin{aligned}&\left(x_1^{(1)}, y_1^{(1)}, \dots, x_n^{(1)}, y_n^{(1)}\right)\\&\quad = \left(x_1^{(0)}, y_1^{(0)}, \dots, x_n^{(0)}, y_n^{(0)}\right) - h_0 \,\mathrm{grad}\, U. \qquad (19)\end{aligned}$$

The geometrical implication of (19) is that pattern $(x_1^{(0)}, y_1^{(0)}, \dots, x_n^{(0)}, y_n^{(0)})$ becomes $(x_1^{(1)}, y_1^{(1)}, \dots, x_n^{(1)}, y_n^{(1)})$ after moving towards the opposite direction of the gradient by $h_0|\mathrm{grad}\, U|$.

If the new pattern is better than the previous one, the process is carried on from the new position. If not, the step size in (19) may have been too large. This is because $|\mathrm{grad}\, U| \geqslant \varepsilon_2$ indicates that the potential will decrease so long as the step is sufficiently small. Therefore, move back to $(x_1^{(0)}, y_1^{(0)}, \dots, x_n^{(0)}, y_n^{(0)})$, modify the step size in (19) into ξh_0, and start again. If still failing, reduce the step size again, until a better new pattern is obtained.

From the new pattern, repeat steps (2) and (3), until a solution is found, or a jamming, in which $|\mathrm{grad} U| < \varepsilon_2$, occurs. In the latter case, a completely new round of QuasiP computation should be initiated from a new initial condition.

Eqs. (6)–(8) describe the movement of the center of the ith bin, whereas Eq. (18) describes the movement of all bins as a whole. These equations are all well defined. Through rigorous yet not very complicated mathematical manipulations, it can be shown that these two methods are consistent except for a positive adjustable constant. The physical meaning of the negative gradient, $-\mathrm{grad}\, U$, in the gradient method is the generalized force in the system. $(-\partial U/\partial x_i, -\partial U/\partial y_i)$ represents the magnitude and direction of the total force exerted on the current ith bin.

6. Mathematical description of the QuasiPQuasiH algorithm: The natural language

We now synthesize the pain relief and resources surrender strategies into a unified QuasiH packing strategy, and give a precise description leading to the QuasiPQuasiH algorithm when incorporated into the QuasiP algorithm.

Definition 1. In a pure problem of circle packing, the coordinates of the centers of n bins, $x_1, y_1, \dots, x_n, y_n$, at any instant, is called a pattern of the system.

Definition 2. Under pattern $x_1, y_1, \dots, x_n, y_n$, the absolute degree of pain, DP_i of the ith bin, $i = 1, 2, \dots, n$, is the elastic potential energy in the bin:

$$\mathrm{DP}_i = U_i. \qquad (20)$$

Definition 3. Under pattern $x_1, y_1, \dots, x_n, y_n$, the relative degree of pain, RDP_i of the ith bin, $i = 1, 2, \dots, n$, is the elastic potential energy in the bin divided by the square of its radius:

$$\mathrm{RDP}_i = \frac{U_i}{r_i^2}. \qquad (21)$$

6.1. The quasi-human strategy

The first jump out of trap. Suppose the system reaches a local minimum, $P_{\mathrm{lm},1} = (x_1, y_1, \dots, x_n, y_n)$, for the first time. Identify the bin with the highest RDP, and call it the ith bin. Randomly relocate its center within the tray while keeping the rest of the bins at their current positions. Thus, the current state is changed from

$$(x_1, y_1, \dots, x_{i-1}, y_{i-1}, x_i, y_i, x_{i+1}, y_{i+1}, \dots, x_n, y_n)$$

to

$$(x_1, y_1, \dots, x_{i-1}, y_{i-1}, \tilde{x}_i, \tilde{y}_i, x_{i+1}, y_{i+1}, \dots, x_n, y_n).$$

From the new state, perform a new round of computation.

The $(k+1)$*th jump*. Suppose a local minimum, $P_{\mathrm{lm},1}$, is reached after the $(k+1)$th QuasiP computation. Identify the bin with the highest RDP, and call it the ith bin. Choose one of the following steps if corresponding condition holds:

(i) If the ith bin is not the one chosen in the kth jump with the highest relative RDP, the ith bin is relocated randomly with its center inside the tray.

(ii) If the ith bin is the one chosen in the kth jump with the highest RDP, then keep the pattern unchanged, find the bin with the least absolute pain, and call it the jth bin. Relocate the jth bin randomly with its center inside the tray.

(iii) If the kth jump is performed based on the least absolute DP, relocate the ith bin randomly with its center inside the tray.

After a choice is made in the $(k+1)$th jump, proceed the QuasiP computation again with the new starting pattern.

6.2. The quasi-physic quasi-human strategy

With the above-described QuasiH strategy and based on the QuasiP algorithm, the following QuasiPQuasiH algorithm is obtained:

At the beginning, the computation is performed using the QuasiP algorithm until a minimum is reached;

If $U = 0$, a solution is found, and computation terminates;

Otherwise, process jumps to a new position according to the QuasiH strategy and carries on computation with the QuasiP algorithm;

The process continues alternatively between QuasiP and QuasiH strategy;

The computation terminates when either a minimum with the total potential equal to zero is reached, hence a solution found, or a prescribed time limit is exceeded without finding a solution.

7. Algorithmic description of the QuasiPQuasiH algorithm: The formal language

In order to implement the QuasiPQuasiH algorithm on a PC, a program is written, which can be used to verify the proposed method. In the following discussion, pattern $(x_1, y_1, \ldots, x_n, y_n)$ is denoted by a vector $\mathbf{X}$ for brevity.

(1) Randomly define n points, $(x_1, y_1), \ldots, (x_n, y_n)$, in a round tray centered at the origin and with a radius r_0, hence an initial patter $\mathbf{X}$. Let $U_{\text{old}} = 0,\ t = 0,\ l_0 = 0,\ h = 1$;

(2) If $h < 10^{-30}$, go to (5);

(3) Calculate $U(\mathbf{X})$;

(4) If $U(\mathbf{X}) < 10^{-6}$, stop; otherwise calculate grad U,

(a) If $U(\mathbf{X}) < U_{\text{old}}$, $U_{\text{old}} \Leftarrow U(\mathbf{X})$, $\mathbf{X} \Leftarrow \mathbf{X} - (\operatorname{grad} U)h$, go to (2);

(b) If $U(\mathbf{X}) \geqslant U_{\text{old}}$, $U_{\text{old}} \Leftarrow U(\mathbf{X})$, $h \Leftarrow 0.8h$, $\mathbf{X} \Leftarrow \mathbf{X} - (\operatorname{grad} U)h$, go to (2);

(5) Find the bin with the highest RDP, and call it the lth bin, $1 \leqslant l \leqslant n$;

(6) If $l = l_0, t \Leftarrow t + 1$;

(7) If $t < 1$, change the component (x_l, y_l) in $\mathbf{X}$ into the current randomly defined coordinates, $(\tilde{x}, \tilde{y})$, and let $l_0 \Leftarrow l$, $h \Leftarrow 1$, then go to (2);

(8) If $t = 1$, find the bin with the least absolute DP, and call it the lth bin, $1 \leqslant l \leqslant n$, change the component (x_l, y_l) in $\mathbf{X}$ into the current randomly defined coordinates, $(\tilde{x}, \tilde{y})$, and let $t \Leftarrow 0$, $l_0 \Leftarrow 0$, $h \Leftarrow 1$, then go to (2).

Note that, in the program, two minor modifications to the QuasiP algorithm described in Section 5 have been made for the sake of convenience. First, when a new position is arrived by a small step using the gradient method, if it is not better than the previous one, one should go back to the previous position according to the algorithm in Section 5. But here, only the step size is reduced without going back. Secondly, the criterion for the total potential used here is $h < 10^{-30}$ rather than $|\operatorname{grad} U| \leqslant \varepsilon_2$.

8. Numerical experiments

8.1. Measures of algorithmic performance

The proposed algorithm does not possess a complete computability, but only a partial computability. That is to say, if the pure problem of circle packing is solvable for a particular instance, $r_0; r_1, r_2, \ldots, r_n$, the algorithm will produce a solution in a sufficiently long time; if, on the other hand, the problem is insolvable, the algorithm cannot provide such a decision as it being insolvable. For any given instance $r_0; r_1, r_2, \ldots, r_n$, it is difficult for us to tell whether it is solvable. However, we have a method feasible to some degree. Firstly, take the instance as solvable and conduct QuasiPQuasiH algorithm on it. If the QuasiPQuasiH algorithm produces a solution, we can say the instance is solvable of course. Otherwise, in a sufficiently log time, if the QuasiPQuasiH algorithm could not produce a solution, we say the instance is insolvable.

Therefore, the computational examples considered here are confined to solvable instances.

For a given computational example, one may ask if it is a hard example, namely an example that is difficult to solve. In contrast to intuition, examples with a large n are not necessarily hard. In other words, the number of bins is not the decisive parameter. For example, packing of small square blocks in a large rectangular container is a trivial problem. Similar cases exist in circle packing.

A hard example in the pure problem of circle packing has the following three characteristics: (i) it is solvable; (ii) the tray size is marginal, namely, a slight reduction makes the problem insolvable; and (iii) the total elastic potential $U(x_1, y_1, \ldots, x_n, y_n)$ has many local minima at which U is not zero.

It is generally accepted that one criterion for the evaluation of algorithm performance is the time taken when solving a given instance, $r_0; r_1, r_2, \ldots, r_n$: the shorter, the better.

In this paper, r_0, together with the pattern of the bins, $r_1, r_2, \ldots, r_n$, forms an approximate solution to the circle packing problem. The approximate solution r_0 to the problem of circle packing $r_1, r_2, \ldots, r_n$, is likely to be somewhat greater than the theoretical minimal radius r_0^*. Nonetheless, the results are usually satisfactory if positive error $r_0 - r_0^*$ is small enough. For an approximate solution r_0, the magnitude of it is an important criterion in judging the algorithm performance.

Therefore, in comparing different algorithms for solving problems of circle packing, both speed and quality of solution are important measures of their performances. The difference $r_0 - r_0^*$ represents the precision of an algorithm for the instance $r_1, r_2, \ldots, r_n$.

In order to evaluate the precision of different algorithms, hard examples are used. However, rigorous solutions (r_0^*) are usually unavailable for such examples. At present, even some seemingly simple problems do not have rigorous solutions. An example is the packing of 19 circles with equal radius. Since a compact and symmetrical pattern is available, we believe that the minimum radius of the outer circle is $r_0^* = 4.863703\ldots$, assuming the radius of the 19 identical circles is 1. However, it still lacks a proof (see, for example, Lubachevsky and Graham, 1997). Therefore, instead of r_0^*, we can only use the smallest possible radius, $\tilde{r}_0$, that is obtainable manually by average human beings, as a reference in our quality evaluation.

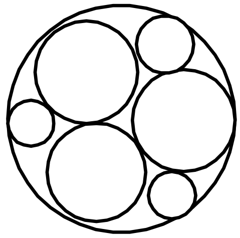

Fig. 9. An example.

In the mean time, there are known special patterns that are compact and symmetrical. These provide some useful instances with rigorous solutions. Fig. 9 shows such an example, in which $r_1 = r_2 = r_3 = 100$, and $r_4 = r_5 = r_6 = 48.26$. The solution is $r_0^* = 215.47$, which can be shown using elementary geometry. These examples may also be used in the evaluation of the algorithm.

To our knowledge, no systematic discussions on the evaluation of algorithmic performance for the pure problems of circle packing, including both speed and solution quality, have been documented in the literature. Therefore, we can only compare the proposed QuasiPQuasiH algorithm with the previous QuasiP in order to show the merits of the QuasiH strategy. Nonetheless, a reference is made to the theoretical formulas on the computational complexity in Hochbaum and Maass (1985).

8.2. Results

A total of 20 examples are used in this work for the purpose of algorithm evaluation, in terms of both speed and quality. These include four equal circle problems and 16 unequal circle problems. The number of bins ranges from 3 to 50, and the ratio between the smallest and the largest bins from 1 to 15. It is believed that these examples are representative in the problems under study since the circle packing is tackled as a series pure problems of circle packing.

Table 1
Computation results of Fig. 10

M	r_0	r_i	Execution time (in seconds)	
			QuasiP	QuasiPQuasiH
7	60	$r_1 = r_2 = \cdots = r_7 = 20$	1.87, 0.05, 0.05, 0.11, 0.05	0.04, 0.04, 0.04, 0.04, 0.02
	159.32	$r_1 = r_2 = \cdots = r_{50} = 20$	3315, 1998, 1392, 81, 1011	148, 877, 100, 431, 284
50				
	215.47	$r_1 = r_2 = r_3 = 100$	4294, 1889, 1862,	34, 13, 14, 87, 94
12				
		$r_4 = r_5 = r_6 = 48.26$	5000(failed), 5000(failed)	
		$r_7 = r_8 = \cdots = r_{12} = 23.72$		
15	39.37	$r_1 = 1$	3000(failed), 3000(failed),	512, 269, 26, 55, 60
		$r_2 = 2$	3000(failed), 2431, 540	
		$r_3 = 3$		
		$\dots$		
		$r_{15} = 15$		

Three algorithms are compared. They are (1) Hochbaum and Maass (1985), (2) quasi-physical, and (3) quasi-physical quasi-human.

The Hochbaum–Maass algorithm is a purely theoretical method that is complete. It has a definite measure of complexity in terms of time:

$$t \sim \left(\frac{2}{\varepsilon}\right)^2 \cdot |x|^{(2/\varepsilon)^2},$$

where $|x|$ is the length of solutions, ε is the allowed relative error. We were unable to conduct a numerical experiment using their algorithm because t becomes prohibitively large for a not-too-small ε (for example, 0.1).

For each of the examples, a series of computations were carried out until a satisfactory r_0 is obtained. The satisfaction is based on two criteria: (i) a better result is impossible with a manual method, and (ii) neither can QuasiPQuasiH proceed any further for a smaller r_0.

In the computation, an approximate solution r_0 is first sought for each example. Thus a solvable pure problem of circle packing, $r_0; r_1, r_2, \dots, r_n$, is established. On each example, both the QuasiP and QuasiPQuasiH methods are tested five times, respectively. The total number of computation is then $20 \times 2 \times 5 = 200$. Every time an initial condition is randomly set.

The computer used in the experiment was a Pentium 233 MHz. The programming language C was used. Table 1 gives the results of four representative instances. These are chosen because they are the relatively difficult ones in which slight reduction of r_0 would lead to insolvable instances.

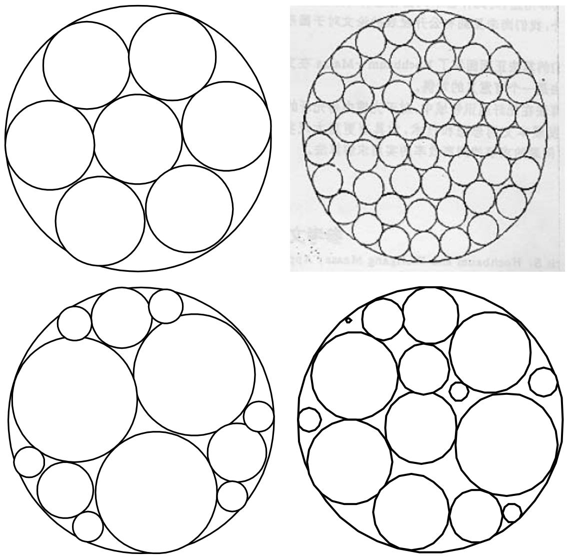

Fig. 10. Four examples of results ($M = 7$, 50, 12, and 15).

Fig. 10 and Table 1 show the geometry, each being a result of a successful computation with the proposed QuasiPQuasiH algorithm.

9. Conclusions

A novel algorithm, QuasiPQuasiH, based on physical analogy together with human behavior has been introduced. The key of the method lies in a powerful means for getting out of local minima. This is achieved through the incorporation of a QuasiH strategy that results in a significant improvement to the previous QuasiP algorithm.

The effectiveness of the algorithm is verified in the numerical experiments. Results are presented to show the practicality of the new method. Since systematic reports on the evaluation of the algorithm for problems under study, in terms of both speed and quality, are unavailable in the literature, a comparative study with other methods is not possible. The computations were conducted mainly in comparison with the previously published QuasiP approach.

The experimental results are exceptionally good as shown in Table 1. A substantial reduction in computing time has been achieved, especially for those difficult examples. The quality of the results has exceeded the best possible obtained by skillful manual drawing.

Hochbaum and Maass (1985) work has theoretical importance since it provides a complete and rigorous solution with a polynomial complexity. However, the orders of the polynomials are very high, therefore the required computing time is prohibitively long. Actual computation using their method is impractical.

Regarding the question raised by George et al. (1995) as how to nest pipes inside one another optimally, the proposed algorithm may provide a useful answer. The problem discussed here is in fact the first sub-problem of the cylindrical bin packing put forward by George et al.

Tabu search is a metaheuristics and applied successfully in various combinatorial optimization (Glover and Manuel, 1997; Aarts and Lenstra, 1997) and continuous optimization problems (Battiti and Tecchiolli, 1996; Chelouah and Siarry, 2000). Tabu search has a huge range of sophistication in many of its applications. A distinguishing feature of Tabu search is represented by its exploitation of adaptive forms of memory (Glover and Manuel, 1997), i.e., the tabu list is used to prevent cycling back to some previous points; in quasi-human strategy, the memory is used to check whether the two most miserable bins successively selected (i.e., the current and the previous most miserable bins) are the same. If they are, the pain relief strategy is switched to resources surrender strategy, the current choice is cancelled and the least miserable bin is selected instead for moving. Thus, the QuasiPQuasiH strategy can be thought as an adoptive algorithm of the Tabu search.

As the main focus of this paper is on the effectiveness of the QuasiH strategy in conjunction with the QuasiP algorithm, the QuasiP itself is implemented with the simplest gradient method. In fact, with more sophisticated methods such as Fletcher–Power–Davidon's quasi-Newton method and the well-known Newton–Raphson method, further improvement to the QuasiPQuasiH algorithm is possible. This is one of the areas we are currently working on.

The research was partly motivated by industrial applications such as stacking of tubes and optimization of optical fiber cable designs. It is hoped that the new method will provide answers to some questions raised by practical workers.

Acknowledgements

The authors would like to thank the anonymous referees for their constructive comments and suggestions.

References

Aarts, E., Lenstra, J.K., 1997. Local Search in Combinatorial Optimization. Wiley, New York.

Battiti, R., Tecchiolli, G., 1996. The continuous reactive tabu search: Blending combinatorial optimization and stochastic search for global optimization. Annals of Operations Research 63, 53–188.

Chelouah, R., Siarry, P., 2000. Tabu search applied to global optimization. European Journal of Operational Research 123, 256–270.

Dowsland, K.A., 1991. Palletisation of cylinders in cases. Operations Research Spektrum 13, 171–172.

Fraser, H.J., George, J.A., 1994. Integrated container loading software for pulp and paper industry. European Journal of Operational Research 77, 466–674.

George, J.A., George, J.M., Lamar, B.W., 1995. Packing different-sized circles into a rectangular container. European Journal of Operational Research 84, 693–712.

Glover, F., Manuel, L., 1997. Tabu Search. Kluwer Academic Publishers, Dordrecht.

Hochbaum, D.S., Maass, W., 1985. Approximation schemes for covering and packing problems in image processing and VLSI. Journal of the Association for Computing Machinery 32, 130–136.

Iserman, H., 1991. Heuristiken zur lösung des zweidimensionalen packproblems für rundgefäße. Operations Research Spektrum 13, 213–223.

Kirkpatric, S., Gellatt Jr., C.D., Vecchi, M.P., 1983. Optimization by simulated annealing. Science 220, 671–680.

Lenstra, J.K., Rinnooy Kan, A.H.G., 1979. Complexity of packing, covering, and partitioning problems. In: Schrijver, A. (Ed.), Packing and Covering in Combinatorics. Mathematisch Centrum, Amsterdam, pp. 275–291.

Lubachevsky, B.D., Graham, R.L., 1997. Curved hexagonal packings of equal disks in a circle. Discrete and Computational Geometry 18, 179–194.

Nurmela, K.J., Östergård, P.R.J., 1997. Packing up to 50 equal circles in a square. Discrete and Computational Geometry 18, 111–120.