Preprint of:
T. A. Nieminen, N. R. Heckenberg, and H. Rubinsztein-Dunlop
"Propagation of arbitrary non-paraxial beams by expansion in spherical functions"
in D. Neilson (ed)
Proceedings of the Australian Institute of Physics 15th Biennial Congress 2002
CD-ROM, Causal Productions (2002)
Presented at:
Fifteenth AIP Biennial Congress 2002/AOS2002
Sydney, Australia (2002)

Propagation of arbitrary non-paraxial beams by expansion in spherical functions

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Abstract

While the paraxial approximation ($kw \gg 1$, where w is the transverse width of the beam or system) is applicable to many, even most, optical systems, the highly non-paraxial regime, where the paraxial approximation and simple corrections to it fail, is becoming increasingly important with the development of intrinsically non-paraxial optical devices and structures such as nano/micro-cavities, photonic crystals, VCSELs, and others of sizes comparable to, or smaller than, the optical wavelength λ.

We calculate the propagation of a highly non-paraxial beam by expansion into spherical functions, which can be considered as fundamental non-paraxial modes. This method is applicable to arbitrary beams.

1 Propagation of beams

The propagation of arbitrary monochromatic electromagnetic radiation fields in isotropic homogeneous media is described by the vector Helmholtz equation:

$$\nabla \times \nabla \times \mathbf{E} + k^2 \mathbf{E} = 0 \tag{1}$$

where $\mathbf{E}$ is the complex amplitude of the electric field, and $k = 2\pi n/\lambda$ is the wavenumber. In principle, the propagation of such fields can be calculated by solving this differential equation, with suitable boundary conditions. In practice, analytical solutions are elusive, and direct numerical solution (such as by using finite-difference methods) are impractical for computational domains large compared to the wavelength.

Therefore, when considering the propagation of beams, that is, radiation fields with a general direction of propagation, and originating from a finite angular region, it is natural to consider theoretical simplifications. The usual choice is the use of the scalar paraxial wave equation, which is a close approximation as long as the wavevector at all points is directed almost parallel to the propagation direction of the beam. Analytical solutions can be given in terms of *beam modes*—eigenfunctions of the scalar paraxial wave equation—or propagation can be calculated using scalar diffraction theory.

However, the paraxial approximation requires that the cross-sectional width w of the beam be much greater than the wavelength ($kw \gg 1$), and becomes increasingly less applicable as the beam is more strongly focussed. For mildly non-paraxial beams, higher order corrections can be made (Barton and Alexander 1989, Davis 1979, Lax *et al.* 1975), but these are impractical in the highly non-paraxial regime.

2 Non-paraxial modes

Any plane wave is a solution to the vector Helmholtz equation, and the set of all possible plane waves forms a complete set of solutions. Therefore, any beam can be represented in terms of plane waves—the spatial Fourier transform of the beam. While this is theoretically simple, considerable practical and computational problems result if the plane wave expansion is used. While any particular beam can be written in terms of plane waves, the resulting expression is an integral—the plane wave basis is a continuous basis set. Computationally, the integral must be approximated as a sum. For this to be accurate, a large number of terms must be used, especially for strongly focussed beams. The discrete sum is not uniformly convergent over all space, so the approximation is only applicable within a finite region of space, making the calculation of the far field of the beam problematic. Therefore, we turn to a discrete basis set as a better choice from a computational perspective.

The geometry of strongly focussed beams suggests the use of vector spherical wavefunctions (VSWFs). The singularity-free regular VSWFs are:

$$\mathbf{RgM}_{nm}(k\mathrm{r}) = N_n j_n(kr)\mathbf{C}_{nm}(\theta,\phi) \tag{2}$$

$$\mathbf{RgN}_{nm}(k\mathrm{r}) = \frac{j_n(kr)}{krN_n}\mathbf{P}_{nm}(\theta,\phi) + N_n\left(j_{n-1}(kr) - \frac{nj_n(kr)}{kr}\right)\mathbf{B}_{nm}(\theta,\phi) \tag{3}$$

where $j_n(kr)$ are spherical Bessel functions, $N_n = 1/\sqrt{n(n+1)}$ are normalisation constants, and $\mathbf{B}_{nm}(\theta,\phi)$, $\mathbf{C}_{nm}(\theta,\phi)$, and $\mathbf{P}_{nm}(\theta,\phi)$ are the vector spherical harmonics:

$$\mathbf{B}_{nm}(\theta,\phi) = \mathbf{r}\nabla Y_n^m(\theta,\phi) = \nabla\times\mathbf{C}_{nm}(\theta,\phi) = \hat{\theta}\frac{\partial}{\partial\theta}Y_n^m(\theta,\phi) + \hat{\phi}\frac{im}{\sin\theta}Y_n^m(\theta,\phi), \tag{4}$$

$$\mathbf{C}_{nm}(\theta,\phi) = \nabla\times(\mathbf{r}Y_n^m(\theta,\phi)) = \hat{\theta}\frac{im}{\sin\theta}Y_n^m(\theta,\phi) - \hat{\phi}\frac{\partial}{\partial\theta}Y_n^m(\theta,\phi), \tag{5}$$

$$\mathbf{P}_{nm}(\theta,\phi) = \hat{\mathbf{r}}Y_n^m(\theta,\phi), \tag{6}$$

where $Y_n^m(\theta,\phi)$ are normalised scalar spherical harmonics. The usual polar spherical coordinates are used, where θ is the co-latitude measured from the $+z$ axis, and ϕ is the azimuth, measured from the $+x$ axis towards the $+y$ axis. $\mathbf{RgM}_{nm}$ and $\mathbf{RgN}_{nm}$ are TE and TM multipole fields, respectively.

Since the regular VSWFs are a complete set of solutions to the vector Helmholtz equation (1) in a source/scatter-free region, any beam can be written in terms of *expansion coefficients* (also called *beam shape coefficients*) a_{nm} and b_{nm} as

$$\mathbf{E}_{\mathrm{inc}}(\mathrm{r}) = \sum_{n=1}^{\infty}\sum_{m=-n}^{n} a_{nm}\mathbf{RgM}_{nm}(k\mathrm{r}) + b_{nm}\mathbf{RgN}_{nm}(k\mathrm{r}). \tag{7}$$

In practice, the VSWF expansion will be terminated at some $n = N_{\max}$. For the case of multipole fields produced by an antenna that is contained within a radius a, $N_{\max} = ka$ is usually adequate, but $N_{\max} = ka + 3\sqrt[3]{ka}$ is advisable if higher accuracy is needed (Brock 2001). This can also be used as a guide for choosing $N_{\max}$ for beams—if the beam waist is contained in a radius a, this can be used to choose $N_{\max}$. It appears that $a = w_0$ generally gives adequate results. Therefore, this method is ideal for strongly focussed beams, since a narrow beam waist gives good convergence for small $N_{\max}$.

Once the expansion coefficients for any given beam are known, the full vector electric and magnetic fields associated with the beam can be calculated at any point in space. The expansion coefficients can be found by a variety of methods (Nieminen 2002). The irradiance and field components in the focal plane are shown in figure 1 for plane and circularly polarised LG_{03} beams. Non-paraxial effects such as non-zero E_z and the non-axisymmetric shape of the plane polarised beam are clearly visible.

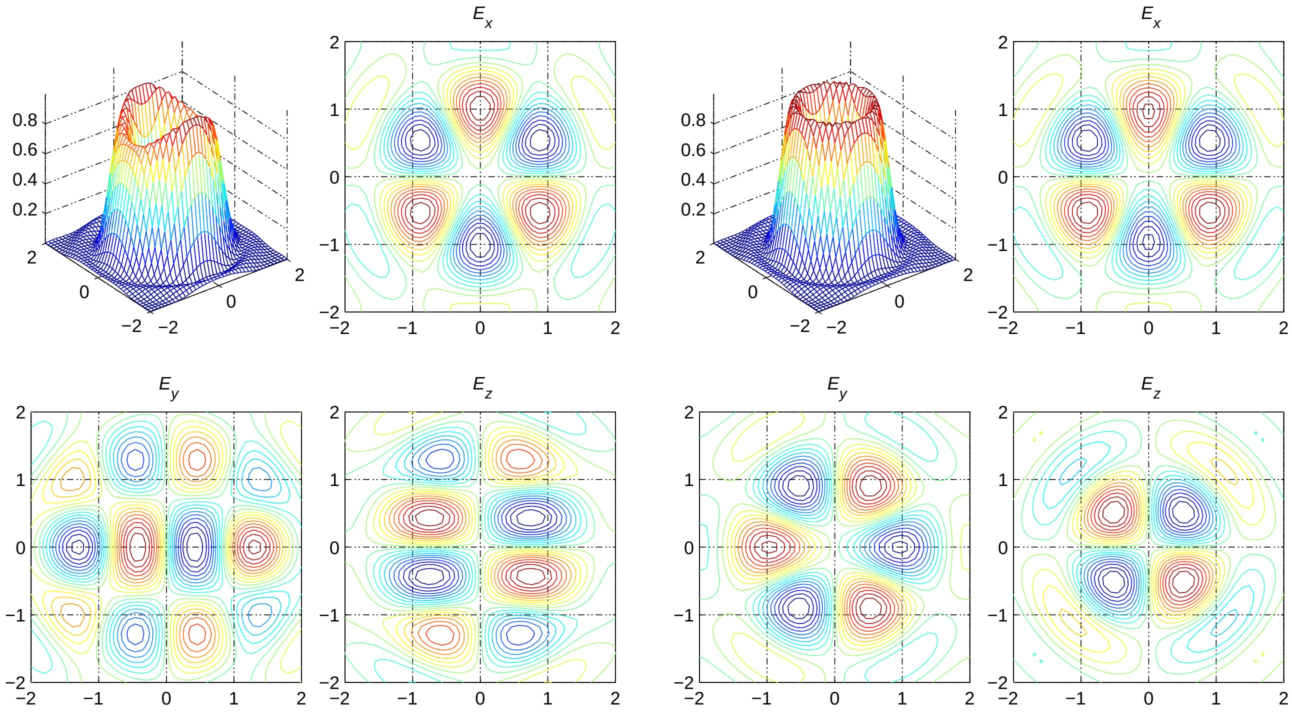

Figure 1. Focal plane irradiance and electric field components for plane polarised (left) and circularly polarised (right) LG_{03} beams of waist radius $w_0 = 0.5\lambda$. All distances are in units of the wavelength.

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Nieminen, T.A., Rubinsztein-Dunlop, H., and Heckenberg, N.R. 2002. Multipole expansion of strongly focussed laser beams. To appear in *J. Quant. Spectrosc. Radiat. Transfer*.