Labelled Tableaux for Non-Normal Modal Logics

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Abstract. In this paper we show how to extend KEM, a tableaux-like proof system for normal modal logic, in order to deal with classes of non-normal modal logic, such as monotonic and regular, in a uniform and modular way.

1 Introduction

Non-normal modal logics have a long tradition, however, despite their heritage thay have been the subject of a very few recent attempts of mechanization [6, 9, 8]. One of the main reasons for this underdeveloppement is that modern automated proof techniques are mainly semantic based, and non-normal modal logic have more complex structures than normal modal logics. Nevertheless Hilbert systems for non-normal modal logic are very close to those for normal modal logics: they lack the axiom $\Box\top$ or the equivalent rule of necessitation ($A/\Box A$).

The second objection to non-normal modal logic we would like to answer to is that concerning their possible applications. The necessitation rule is a very strong inference rule and it comports significant consequences; for example under the epistemic interpretation of the modal operators, it implies omniscience: the agent must be an ideal agent, i.e., it must be a perfect reasoner and it must have unlimited computational ability. This seems to be a very unrealistic assumption so some scholars (see, among others, [4, 12, 11, 13] suggested to use non-normal modal logics to model epistemic reasoning. On the other hand one could argue this is not the case with more exact discipline such as mathematics. However this is not the case: it is well known that provability in Peano arithmetic can be represented with the normal modal logic GL, but some classes of arithmetic formulas (i.e., Σ_1-sentences) are represented by a non-normal modal logic [2].

It is not the aim of this work to investigate applications of non-normal modal logics. Instead we want to present a tableau-like proof system (called KEM) for classes of non-normal modal logics, namely: regular and monotonic. The main feature of KEM is its label formalism studied to simulate the semantics of modal logics. The differences between the various classes of modal logics are embedded in the definition of the basic unification; however the various extensions (in each class) arising from modal axioms are dealt in a uniform way wrt the various classes.

In Section 1 we shall resume briefly the basic of non-normal modal logic, then in the next sections we shall describe KEM in details. More precisely in Section 4 we introduce the label formalism, then in Section 5 we describe the unification mechanism for dealing with the various classes of non-normal modal logic, and in Section 6 we present KEM inference rules. Finally in Section 7 we outline the soundness and completeness proofs.

2 Non-Normal Modal Logics

We shall consider only modal logics extending classical propositional logic, and where the modal operators $\Box$ and $\Diamond$ are the dual of each other (i.e., $\Box \leftrightarrow \neg\Diamond\neg$).

The rules we use to extend classical propositional logic are:

$$\frac{\vdash (A_1 \wedge \cdots \wedge A_n) \to A}{\vdash (\Box A_1 \wedge \cdots \wedge \Box A_n) \to \Box A}\ n \geq 0 \tag{RK}$$

and, in particular, we shall consider

$$\frac{\vdash A}{\vdash \Box A}\ (RK, n = 0) \tag{Nec}$$

$$\frac{\vdash A \to B}{\vdash \Box A \to \Box B}\ (RK, n = 1) \tag{RM}$$

$$\frac{\vdash (A \wedge B) \to C}{\vdash (\Box A \wedge \Box B) \to \Box C}\ (RK, n = 2) \tag{RR}$$

We can now classify modal logics according to their deductive power.

Definition 1. *A modal logic Σ is:*

1. monotonic *iff it is closed under RM;*
2. regular *iff it is closed under RR;*
3. normal *iff it is closed under RK.*

We can now formulate the relationships between the various classes of modal logics

Theorem 1.

1. *Every regular logic is monotonic;*
2. *Every normal logic is regular, and therefore monotonic.*

Proof. For the proof see [3, 235].

According to [3] the smallest regular logic is called R, the smallest monotonic logic M, and the smallest normal logic K.

The semantic of non-normal modal logic is given in terms of neighborhood semantics. A model is a structure

$$\mathscr{M} = \langle W, N, v \rangle$$

where W is a set of possible worlds, N is a function from W to $\mathscr{P}(\mathscr{P}(W))$ and v is an evaluation function: $v : \mathit{WFF} \times W \mapsto \{T, F\}$, where WFF is the set of well-formed formulas.

Before providing the evaluation clauses for the formulas we need to define the notion of truth set.

Definition 2. *Let $\mathscr{M}$ be a model and A be a formula. The truth set of A wrt to $\mathscr{M}$, $\|A\|^{\mathscr{M}}$ is thus defined:*

$$\|A\|^{\mathscr{M}} = \{w \in W : v(A, w) = T\}$$

The evaluation clauses for atomic and boolean formulas are as usual while those for modal operators are given below.

Definition 3. *Let w be a world in $\mathscr{M} = \langle W, N, v\rangle$*

1. $w \models \Box A \iff \|A\|^{\mathscr{M}} \in N_w$.
2. $w \models \Diamond A \iff W - \|A\|^{\mathscr{M}} \notin N_w$.

It is natural to add some conditions on the function N in neighborhood models. The conditions relevant for the present work are given in the following definition.

Definition 4. *Let $\mathscr{M}$ be a model. For every world $w \in W$ and every proposition A, and B.*

(m) *If* $\|A\| \cap \|B\| \subseteq N_w$, *then* $\|A\| \in N_w$ *and* $\|B\| \in N_w$
(c) *If* $\|A\| \in N_w$ *and* $\|B\| \in N_w$, *then* $\|A\| \cap \|B\| \in N_w$.
(n) $W \in N_w$.

According as N in a neighborhood model satisfies condition (m), (c), or (n), we shall say that the model is *supplemented*, is *closed under intersections*, or *contains the unit*. When a model is both supplemented and closed under intersections then we shall call it a *quasi-filter*; when a quasi-filter contains the unit it is a *filter*.

We are now able to state the correspondence theorem for non-normal modal logics.

Theorem 2.

1. M *is characterized by the class of supplemented models;*
2. R *is characterized by the class of quasi-filters;*
3. K *is characterized by the class of filters.*

Proof. For the proof see [3, 257]

From now on we shall use $\models_{\Sigma} A$ to denote that A is valid in the class of model characterizing Σ.

3 KEM

KEM (see [1, 10]) is a labelled analytic proof system based on a combination of tableau and natural deduction inference rules which allows for a suitably restricted ("analytic") application of the cut rule; the label scheme arises from an alphabet of constant and variable "world" symbols. A "world" label is a world-symbol or a "structured" sequence of world-symbols we call a "world-path". Constant and variable world-symbols denote worlds and set of neighbors respectively (in a neighborhood model), while a world-path conveys information about access between the worlds in it. We attach labels to signed formulas (i.e. formulas prefixed with a "T" or "F") to yield *labelled signed formulas* (*LS-formulas*). A LS-formula TA, i (FA, i) means that A is true (false) at the (last) world (on the path) i. In the course of proofs labels are manipulated in a way closely related to the modal semantics and "matched" using (specialized, logic-dependent) unification algorithms.

4 Label Formalism

The set $\Im$ of labels arises from two (non empty) sets $\Phi_C = \{w_1, w_2, \dots\}$ (the set of *constant world symbols*), and $\Phi_V = \{W_1, W_2, \dots\}$ (the set of *variable world symbols*) through the following

Definition 5.

$$\Im = \bigcup_{1 \le i} \Im_i \textit{ where } \Im_i \textit{ is}:$$
$$\Im_1 = \Phi_C \cup \Phi_V;$$
$$\Im_2 = \Im_1 \times \Phi_C;$$
$$\Im_{n+1} = \Im_1 \times \Im_n,\ n > 1.$$

That is, a world-label is either (i) an element of the set Φ_C, or (ii) an element of the set Φ_V, or (iii) a path term (k', k) where (iiia) $k' \in \Phi_C \cup \Phi_V$ and (iiib) $k \in \Phi_C$ or $k = (i', i)$ where (i', i) is a label. From now on we shall use $i, j, k, \dots$ to denote arbitrary labels.

For any label $i = (k', k)$ we shall call k' the *head* of i, k the *body* of i, and denote them by $h(i)$ and $b(i)$ respectively. Notice that these notions are recursive (they correspond to projection functions): if $b(i)$ denotes the body of i, then $b(b(i))$ will denote the body of $b(i)$, $b(b(b(i)))$ will denote the body of $b(b(i))$; and so on. We call each of $b(i)$, $b(b(i))$, etc., a *segment* of i. Let $s(i)$ denote any segment of i (obviously, by definition every segment $s(i)$ of a label i is a label); then $h(s(i))$ will denote the head of $s(i)$.

For any label i, we define the length of i, $\ell(i)$, as the number of world-symbols in i, i.e., $\ell(i) = n \Leftrightarrow i \in \Im_n$. $s^n(i)$ will denote the segment of i of length n, i.e., $s^n(i) = s(i)$ such that $\ell(s(i)) = n$. We shall use $h^n(i)$ as an abbreviation for $h(s^n(i))$.

For any label $i, \ell(i) > n$, we define the *countersegment-n* of i, as follows:

$$c^n(i) = h(i) \times (\cdots \times (h^k(i) \times (\cdots \times (h^{n+1}(i), w_0))))(n < k < l(i))$$

where w_0 is a dummy label. In other words the countersegment-n of a label i is the label obtained from i by replacing $s^n(i)$ with a dummy world symbol.

5 Unifications

The key feature of KEM is that in the course of proof labels are manipulated in a way closely related to the semantics of modal operators and "matched" using a specialized unification algorithm. That two labels i and k are unifiable means, intuitively, that the set of worlds they "denote" have a non-null intersection. The basic element of the unification is the substitution function which maps each variable in labels to a label, and each constant to itself.

The label unification is the core of KEM. The unifications for the various logics, as usual, are defined from a substitution and imposing conditions on the substitution produces the basic unification for the classes of logics we are dealing with.

Let $\rho^{\mathcal{C}}$ be a substitution function defined on labels. We first build a basic unification for the classes of logics, then we define the unifications corresponding to the various modal axioms relying on basic unifications, finally we compose the axiom unifications into the unifications for the corresponding logics.

For two labels i and j, and a substitution ρ, if ρ is a unifier of i and j then we shall say that i, j are $\sigma^{\mathcal{C}}$-*unifiable*. We shall use $(i,j)\sigma^{\mathcal{C}}$ to denote both that i and j are $\sigma^{\mathcal{C}}$-unifiable and the result of their unification. In particular

$$\forall i,j,k \in \Im, (i,j)\sigma^{\mathcal{C}} = k \text{ iff } \exists \rho^{\mathcal{C}}(\rho^{\mathcal{C}}(i) = \rho^{\mathcal{C}}(j) \text{ and } \rho^{\mathcal{C}}(i) = k)$$

On this basis we may define several specialised, logic-dependent notions of σ-unification characterizing the various modal logic. The first step in order to define the unifications characterizing the various modal logic is to define unifications (axiom unifications) corresponding to the modal axioms. Then in the same way a modal logic is obtained by combining several axiom we define combined unifications, that, when applied recursively produce the logic unifications.

The general form of a $\sigma^{\mathcal{C}A}$ unification is:

$$(i,j)\sigma^{\mathcal{C}A} \iff (f_A(i), g_A(j))\sigma_{\mathcal{C}} \text{ and } C^A$$

where f_A and g_A are given logic-dependent functions from labels to labels and C^A is a set of constraints (see [10, 1, 7] for example of logic unifications).

A combined unification $\sigma^{\mathcal{C}A_1\cdots A_n}$ is generally defined as the combination of the axiom unifications for the axioms characterizing the logic

$$(i,j)\sigma^{\mathcal{C}A_1\cdots A_n} \iff \begin{cases} (i,j)\sigma^{\mathcal{C}A_1} & C^{A_1} \\ \vdots & \vdots \\ (i,j)\sigma^{\mathcal{C}A_n} & C^{A_n} \end{cases}$$

Applying recursively the above $\sigma^{\mathcal{C}A_1\cdots A_n}$ unification we obtain the logic unification σ_{Σ}.

$$(i,j)\sigma_{\Sigma} = \begin{cases} (i,j)\sigma^{\mathcal{C}A_1\cdots A_n} \\ (c^n(i), c^m(j))\sigma^{\mathcal{C}A_1\cdots A_n} \end{cases}$$

where $w_0 = (s^n(i), s^m(j))\sigma_{\Sigma}$.

We shall denote the constants occurring in labels obtained as the result of an unification with $*$, and we shall denote the set of such constants by Φ_C^*.

It is worth noting that the variables can be mapped on more than a label in the course of a proof; imposing restriction on the quantity of labels a variable can be mapped to in the course of a proof we are able to characterize the classes of modal logic at hand. More precisely

Monotonic Logic

$$\rho^M : \Phi_V \mapsto \Im_{\text{branch}} \qquad \text{injective}$$
$$\mathbf{1}_{\Phi_C^*}$$

The condition for monotonic logics states that a variable can be mapped to a unique label in a branch of a KEM-proof, while constants are mapped on themselves only if they are the result of a unification. It is worth noting that it is now possible to map a variable on different labels if they occur in distinct branches.

Regular Logic

$$\rho^R : \Phi_V \mapsto \Im$$
$$\mathbf{1}_{\Phi_C^*}$$

For regular logics the restriction on variables is released, while that on constants still obtains.

Normal Logic

$$\rho^K : \Phi_V \mapsto \Im$$
$$\mathbf{1}_{\Phi_C}$$

The substitution for normal logics is obtained from that for regular by dropping the restriction on constants.

6 Inference Rules

In displaying the rules of KEM we shall use Smullyan-Fitting α, β, ν, π unifying notation [6]. If X is an LS-formula, X^C denotes the *conjugate* of X, i.e., the result of changing the sign of X to its opposite; two LS-formulas X, i and X^C, k such that $(i, k)\sigma_\Sigma$ will be called σ_Σ*-complementary*.

$$\frac{\alpha, i}{\alpha_n, i}[n = 1, 2] \tag{α}$$

The α rules are just the familiar linear branch-expansion rules of the tableau method.

$$\frac{\begin{array}{c}\beta, i\\ \beta_n^C, j\end{array}}{\beta_{3-n}(i, j)\sigma_\Sigma}[(i, j)\sigma_\Sigma,\ n = 1, 2] \tag{β}$$

The β are nothing else than natural inference patterns such as Modus Ponens, Modus Tollens and Disjunctive syllogism generalized to the modal case. In order to apply such rules it is required that the labels of the premises unify and the label of the conclusion is the result of their unification.

$$\frac{\nu, i}{\nu_0, (W_n, i)}[W_n \text{ new}] \tag{ν}$$

$$\frac{\pi, i}{\pi_0, (w_n, i)}[w_n \text{ new}] \tag{π}$$

ν and π rules allow us to expand labels according to the intended semantics, where, with new we means that the label does not occur previously in the tree. It is worth noting that the proviso W_n new is not necessary for normal logics, but this is not the case for non-normal ones; this is due to the fact that the meaning of W_n wrt to normal modal logic is the set of worlds accessible from i, while for non-normal modal logic it denotes a set of neighbors of i, and a world may have several sets of neighbors.

$$\frac{}{X, i \qquad X^C, i}[i \text{ restricted}] \tag{PB}$$

PB (the "Principle of Bivalence") represents the (LS-version of the) semantic counterpart of the cut rule of the sequent calculus (intuitive meaning: a formula A is either true or false in any given world).

$$\frac{\begin{array}{c} X, i \\ X^C, j \end{array}}{\times}[(i, j)\sigma_\Sigma] \tag{PNC}$$

PNC (the "Principle of Non-Contradiction") corresponds to the familiar branch-closure rule of the tableau method, saying that from a contradiction of the form (occurrence of a pair of σ_Σ-complementary LS-formulas) X, i and X^C, j on a branch we may infer the closure of the branch. The $(i, j)\sigma_\Sigma$ in the "conclusion" of PNC means that the contradiction holds "in the same world".

As usual with refutation methods, a proof of a formula A of Σ consists of attempting to construct a countermodel for A by assuming that A is false in some arbitrary model for Σ. Every successful proof discovers a contradiction in the putative countermodel. In what follows by a KEM-*tree* we shall mean a tree generated by the inference rules of KEM. A branch τ of a KEM-tree will be said to be σ_Σ-*closed* if it ends with an application of PNC. A KEM-tree $\mathcal{T}$ will be said to be σ_Σ-closed if all its branches are σ_Σ-closed. Finally, by a Σ-proof of a formula A we shall mean σ_Σ-closed KEM-tree starting with FA, i, where i is a constant world-symbol. We shall use $\vdash_{\mathrm{KEM}(\Sigma)} A$ to denote that there is a Σ-proof of A.

7 Soundness and Completeness

In order to prove soundness and completeness of KEM with respect to the classes of logics and models of Theorem 2, we have to show that the rules , RM, RR, and RK are derived rules in KEM. This can be easily achieved by drawing a KEM-proof for them. Here we just provide the proof for RM in M; the proofs for the remaining rules and logics are similar.

$$\begin{array}{ll}
1.\ F\Box A \to \Box B & w_1 \\
2.\ T\Box A & w_1 \\
3.\ F\Box B & w_1 \\
4.\ TA & (W_1, w_1) \\
5.\ FB & (w_2, w_1)
\end{array}$$

$$\begin{array}{ll|ll}
6.\ TA \to B & (w_2, w_1) & 7.\ FA \to B & (w_2, w_1) \\
8.\ TB & (w_2^*, w_1) & 10.\ \mathcal{B} & (w_2^*, w_1) \\
9.\ \times & (w_2^*, w_1) & 11.\ \times &
\end{array}$$

To show that there is a KEM-proof we have to provide a closed KEM-tree for $F\Box A \to \Box B$ given that a KEM-tree ($\mathcal{B}$) for $FA \leftrightarrow B$ closes. The steps 1–5 are immediate; at this point we apply PB wrt to $A \to B$, and label (w_2, w_1). In the left branch we can apply a β-rule on 4 and 6, thus obtaining 8 and closing the branch. In the right branch we can repeat the proof for $FA \to B$ with label (w_2^*, w_1), and so also this branch is closed.

D'Agostino and Mondadori [5] have proved that the Modus Ponens (if $\vdash A$, and $\vdash A \to B$, them $\vdash B$) is a derived rule of KE, the propositional modulo of KEM. Moreover they proved that KE is sound and complete with respect to classical propositional logic.

From the above considerations and Theorem 2 we can conclude

Theorem 3. $\models_\Sigma A \Rightarrow \vdash_{\mathrm{KEM}(\Sigma)} A$

To prove the second part of the correspondence theorem for KEM, we have to show that KEM rules and unifications are sound with respect to the appropriate model. To this end we define some functions mapping LS-formulas on elements of models, according to the structure of the labels.

Let g be a function from $\Im$ to $\mathscr{P}(W)$ such that:

$$g(i) = \begin{cases} h(i) = \{h(i)\} & h(i) \in \Phi_C \\ h(i) = \{w_i \in \mathcal{W} \ : \exists \mathcal{X}(\mathcal{X} \in N_{g^*(b(i))} \wedge w_i \in \mathcal{X})\} & h(i) \in \Phi_V \end{cases}$$

where g^* is a function over $g(i)$; if $h(i) \in \Phi_C$ and $\ell(i) > 1$:

$$g(i) = \{h(i)\} \subseteq \{w_i \in \mathcal{W} \ : \exists \mathcal{X}(\mathcal{X} \in N_{g^*(b(i))} \wedge w_i \in \mathcal{X})\}$$

Let r be a function from $\Im$ to $\bigcup N_w$ such that:

$$r(i) = \begin{cases} \emptyset & \ell(i) = 1 \\ g^*(i^1)N\ g(i^2), \dots, g^*(i^{n-1})N\ g(i^n) & \ell(i) > 1 \end{cases}$$

Finally let f be a function from the set of LS-formulas to v such that:

$$f(S\ A, i)\ =_{def}\ v(A, w_j) = S$$

for every $w_j \in g(i)$.

Let $\mathcal{F}$ be a set of LS-formulas and $\mathcal{L}$ be the set of labels occurring in $\mathcal{F}$; the function g_Σ, $\Sigma = \mathsf{M}, \mathsf{R}$, from $\mathcal{L}$ to $\mathscr{P}\mathscr{P}(\mathcal{W})$ produces an Σ-model starting from the LS-formulas in $\mathcal{F}$.

$$\begin{aligned} & g_\Sigma(\mathcal{L}) = \forall i \in \mathcal{L} \ \bigcup g(i), \text{ such that } \forall i, j, k \in \mathcal{L} \\ & \qquad \Sigma = \mathsf{M} : \text{if } g(i) \subseteq g(k) \text{ and } g(i) \in N_{g^*(j)} \text{ then } g(k) \in N_{g^*(j)} \\ & \qquad \Sigma = \mathsf{R} : \text{if } g(i) \in N_{g^*(j)} \text{ and } g(k) \in N_{g^*(j)} \text{ then } g(i) \cap g(k) \in N_{g^*(j)}. \end{aligned}$$

A KEM-tree with n branches is a collection of $\mathcal{F}_1, \dots, \mathcal{F}_n$ where $\bigcap \mathcal{F}_n \neq \emptyset$ since it contains at least the origin of the tree.

Lemma 1. *For any $i, k \in \Im$ if $(i, k)\sigma_L$ then $g(i) \cap g(k) \neq \emptyset$.*

Proof. The proof is similar to that given by [1, 10] for normal modal logics.

This lemma shows that if two labels unify, then the result of their σ_Σ-unification corresponds to an element of the appropriate model. In this way, we are able to build the neighborhood model for the labels involved in a KEM-proof, and so we can check every rule of KEM in a standard semantic setting:

Theorem 4. $\vdash_{\mathrm{KEM}(\Sigma)} A \Rightarrow \models_\Sigma A$.

From theorems 3 and 4 we obtain:

Theorem 5. $\vdash_{\mathrm{KEM}(\Sigma)} A \iff \models_\Sigma A$.

8 Conclusion

In this paper we have provided a uniform and modular automated proof system for non-normal modal logics. The system enjoys two orthogonal kinds of modularity: the first one with respect to the substitutions determining the classes of modal logic and the second one with respect to the unifications corresponding to the various modal axioms.

It is possible to claim that the system here presented is more efficient than the system proposed in [6]. Such a method takes a direct approach and uses prefixes to keep trace of the relation among possible worlds. However, it suffers from the drawback of duplicating formulas. It is easy to see that when the duplicate formulas behave disjunctively, the duplication implies an exponential increase of the complexity. On the other hand the complexity of KEM unification algorithm is linear (at least for the basic cases), so we can build examples such that the length of KEM proof is linear while Fitting's prefix tableaux has exponential proofs.

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