Labelled Tableaux for Multi-Modal Logics*

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Introduction

In this paper we present a tableau-like proof system for multi-modal logics based on D'Agostino and Mondadori's classical refutation system KE [DM94]. The proposed system, that we call KEM, works for the logics $S5A$ and $S5P_{(n)}$ which have been devised by Mayer and van der Hoek [MvH92] for formalizing the notions of actuality and preference. We shall also show how KEM works with the normal modal logics $K45, D45$, and $S5$ which are frequently used as bases for epistemic operators – knowledge, belief (see, for example [Hoe93, Wan90]), and we shall briefly sketch how to combine knowledge and belief in a multi-agent setting through KEM modularity.

1 Preliminaries

All the systems of Modal Logic we shall be concerned with are couched in a standard modal language consisting of: propositional variables; the usual logical constants and operators: $\neg, \wedge, \vee, \rightarrow, \equiv, \Box, \Diamond$ for negation, conjunction, disjunction, conditionality, biconditionality, necessity, and possibility respectively; the modal-like operators: $\Box_i, \Diamond_i$ for i-necessity and i-possibility, respectively. In what follows we shall use different names for different modal-like operators. Formulas are defined in the usual way. We shall use the letters $A, B, C, \ldots$ to denote arbitrary formulas. A system of Modal Logic will be denoted by L.

We define an *extended Kripke model* for a logic L (briefly an *L-model*) to be a structure $< W, \Sigma_1, \ldots \Sigma_m, R_1 \ldots R_n, \upsilon >$ where W is a non-empty set (the set of "possible worlds"), $\Sigma_i \subseteq W, (1 \leq i \leq m)$, $R_i, (1 \leq i \leq n)$ is a binary "accessibility" relation on W, and υ is a mapping from $S \times W$ to $\{T, F\}$ where S is the set of all the formulas of our language. The notion of L-model appropriate for the logic L can be obtained by restricting R_i to satisfy the conditions associated with L.

As usual [Smu68b] by a *signed formula (S-formula)* we shall mean an expression of the form SA where A is a formula and $S \in \{T, F\}$. Thus TA if $\upsilon(A, x) = T$

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and FA if $\upsilon(A, x) = F$ for some L-model $< W, \Sigma_1, \dots \Sigma_m, R_1, \dots, R_n, \upsilon >$ and $x \in W$. We shall denote by X, Y, Z arbitrary signed formulas.

By the *conjugate* X^C of a signed formula X we shall mean the result of changing S to its opposite. Moreover we assume that the S-formulas listed in the left column of the following table have as their conjugates both the S-formulas listed in the other columns.

X	X^C	
$T\Box_i A$	$F\Box_i A$	$T\Diamond_i \neg A$
$F\Diamond_i A$	$T\Diamond_i A$	$F\Box_i \neg A$
$F\Box_i A$	$T\Box_i A$	$F\Diamond_i \neg A$
$T\Diamond_i A$	$F\Diamond_i A$	$T\Box_i \neg A$

Where, for example, $T\Box_i A$ has both $F\Box_i A$ and $T\Diamond_i \neg A$ as its conjugates.

Two S-formulas X, Z such that $Z = X^C$, will be called *complementary*. For ease of exposition we shall use Smullyan-Fitting's "α, β, ν, π" unifying notation, that classifies S-formulas with respect to their modality, in the generalized form "$\alpha, \beta, \nu_i, \pi_i$". We begin by giving a concise exposition of the logics we shall be concerned with (for more details see [MvH92]).

1.1 The Logic $S5A$

The modal logic $S5A$ is obtained by enlarging the basic modal language with an actuality operator $\triangle$ indicating that a formula is actually true, i.e. holds in the actual world. The set of $S5A$-formulas consists of (i) all the $S5$-formulas and (ii) all the formulas of the form $\triangle A$. In addition to the customary $S5$ axioms we have the following axioms:

1. $\triangle(A \wedge B) \equiv (\triangle A \wedge \triangle B)$;
2. $\triangle \neg A \equiv \neg \triangle A$;
3. $\Box A \rightarrow \triangle A$;
4. $\triangle A \rightarrow \Box \triangle A$.

The semantic for $S5A$ is given in terms of a "mixed" $S5$-$D45$ Kripke model ($S5A$-model) $< W, R, \mathcal{A}, \upsilon >$ where W is a non empty set of "worlds", R is an equivalence relation on W, $\mathcal{A}$ is a constant function on W, so that

$$\mathcal{A} \subseteq R, \exists! a \in W : \forall w \in W, w\mathcal{A}a;$$

υ is as usual with the following additional clause:

$$\upsilon(\triangle A, w) = T \iff \forall a \in W : w\mathcal{A}a, \upsilon(A, a) = T.$$

$\mathcal{A}$ turns out to be serial, transitive and euclidean.

1.2 The Logic $S5P_{(n)}$

To obtain the logic $S5P_{(n)}$ we introduce in the basic modal language n modal operators $P_1, \ldots, P_n$ indicating that a formula holds in a set of "preferred" worlds. The set of $S5$-formulas is enlarged to include all the formulas of the form $P_i A, (1 \leq i \leq n)$. In addition to the customary $S5$ axioms we have the following axioms:

1. $\Box P_i A \equiv P_i A$;
2. $\neg P_i \bot \to (P_i P_j A \equiv P_j A)$;
3. $\neg P_i \bot \to (P_i \Box A \equiv \Box A)$;
4. $\Box A \to P_i A (1 \leq i \leq n)$.

The semantic for $S5P_{(n)}$ is given in terms of a "mixed" $S5$-$K45$ Kripke-model ($S5P_{(n)}$-model) $< W, \Sigma_1, \ldots, \Sigma_n, R, R_1, \ldots, R_n, \upsilon >$ where $\Sigma_i \subset W$, are subsets (possibly empty) of preferred worlds; $R_i = \Sigma \times \Sigma_i \subset R$ are transitive and euclidean relations on Σ_i; and R is an equivalence relation on W; υ is as usual with the following additional clause:

$$\upsilon(P_i A, w) = T \iff \forall x \in \Sigma_i : w R_i x, \upsilon(A, x) = T.$$

2 The System KEM

In this section we describe the computational framework KEM. Like resolution and tableau systems, KEM is a formalization of the search for countermodels and it can be adapted to all settings which have a Kripke-model based semantics.

The key feature of KEM, besides its being based on a combination of tableau and natural deduction inference rules which allows for a suitably restricted use of the cut rule, is that it automatically generates models and checks them for putative contradictions using a label scheme to bookkeep "world" paths. Briefly and informally, in the KEM-based approach S-formulas are labelled by worlds. A "world" label is a constant or a variable "world" symbol or a "structured" sequence of world-symbols we shall call a "world-path". Intuitively, constant and variable world-symbols can be viewed as denoting worlds and sets of worlds respectively, while a world-path conveys information about access between the worlds in it.

An S-formula SA with an associated label i (a *labelled signed formula*, or *LS*-formula, as we shall call it) means, intuitively, that A is true (false) at the (last) world (on the path) i. In the course of proof search, labels are manipulated in a way closely related to the semantics of modal operators and "matched" using a (specialized, logic-dependent) unification algorithm. That two structured labels i and k are unifiable means, intuitively, that they virtually represent the same path, i.e. any world which you could get to by the path i could be reached by the path k and vice versa. S-formulas whose labels are unifiable turn out to be true (false) at the same world(s) relative to the accessibility restrictions that hold in the class of L-models. In particular, two LS-formulas X, i X^C, k, whose labels are unifiable, will stand for formulas which are contradictory "in the same world".

Remark. The idea of using a label scheme to bookkeep "world" paths in modal theorem proving goes back at least to [Fi66]. Similar, or related, ideas are found in [Fit72, Fit83, Wri85] and, more recently, in [Cat91, JR89, Tap87, Wal90] and also in the "translation" tradition of [AE92, Ohl91], and in Gabbay's Discipline of Labelled Deductive Systems [Gab91] (see also [DG94] tableau extension with labels).

KEM combines two kinds of rules: rules for processing the propositional part (which are the same for all modal logics), and rules for manipulating labels according to the appropriate accessibility restriction. The key features of KEM are outlined as follows. For a more comprehensive presentation of KEM as applied to a wide variety of normal modal logics see [AG94, ACG94a].

2.1 Label Formalism

Let $\Phi^i_C = \{w^i_1, w^i_2, \ldots\}$ be non empty sets of constant world symbols, and let $\Phi^i_V = \{W^i_1, W^i_2, \ldots\}$ be non empty sets of variable world symbols, where $1 \leq i \leq n$ and n is the number of the different $\Box_i$ in the logic we are considering; for $i = 0$ ($\Box_0 = \Box$) we shall use w_j, W_j. (According to the conditions which hold is $S5A$ we have $\Phi^1_C = \Phi^1_V = \{a\}$). We define $\Phi_C = \bigcup_{0 \leq i \leq n} \Phi^i_C$ and $\Phi_V = \bigcup_{0 \leq i \leq n} \Phi^i_V$. Thus the set $\Im$ of labels is defined as follows:

$$\Im = \bigcup_{1 \leq i} \Im_i \text{ where } \Im_i \text{ is:}$$
$$\Im_1 = \Phi_C \cup \Phi_V;$$
$$\Im_2 = \Im_1 \times \Phi_C;$$
$$\Im_{n+1} = \Im_1 \times \Im_n.$$

That is a world-label is either (i) an element of the set Φ_C, or (ii) an element of the set Φ_V, or (iii) a path term (k', k) where (iiia) $k' \in \Phi_C \cup \Phi_V$ and (iiib) $k \in \Phi_C$ or $k = (m', m)$ where (m', m) is a label. $w_j, (W_j)$ is also used to denote a given world (a world) for which we do not have enough information to specify what is its i (i.e. we do not know what kind of label it is). From now on we shall use $i, j, k, \ldots$ to denote arbitrary labels. According to the above intuitive explanation, we may think of a label $i \in \Phi_C$ as denoting a world (a given one), and a label $i \in \Phi_V$ as denoting a set of worlds (any world) in some L-model. A label $i = (k', k)$ may be viewed as representing a path from k to a (set of) world(s) k' accessible from k.

Example 1. The label (W_1, w_1) represents a path which takes us to the set W_1 of worlds accessible from w_1; $(w_2, (W_1, w_1)))$ represents a path which takes us to a world w_2 accessible via any world accessible from w_1, (i.e., accessible from the subpath (W_1, w_1)) and so on. The label (w^i_2, w_1) represents a path which takes us to the world w^i_2 accessible through R_i from the world w_1.

For any label $i = (k', k)$ we call k' the head of i, k the body of i, and denote them by $h(i)$ and $b(i)$ respectively. Notice that these notions are recursive: if $b(i)$

denotes the body of i, then $b(b(i))$ will denote the body of $b(i)$, $b(b(b(i)))$ will denote the body of $b(b(i))$; and so on. For example, if i is $(w_4, (W_3, (w_3, (W_2, w_1))))$, then $b(i) = (W_3, (w_3, (W_2, w_1)))$, $b(b(i)) = (w_3, (W_2, w_1))$, $b(b(b(i))) = (W_2, w_1)$, $b(b(b(b(i)))) = w_1$. We call each of $b(i)$, $b(b(i))$, etc., a segment of i. Let $s(i)$ denote any segment of i (obviously, by definition every segment $s(i)$ of a label i is a label); then $h(s(i))$ will denote the head of $s(i)$.

For any label i, we define the length of i, $l(i)$, as the number of world-symbols in i. The segment of i whose length is n is denoted by $s^n(i)$.

We call a label i restricted if $h(i) \in \Phi_C$, otherwise we call it unrestricted. We shall say that a label k is *i-preferred* iff $k \in \Im^i$ where $\Im^i = \{k \in \Im : h(k) \text{ is either } w^i_m \text{ or } W^i_m, 1 \leq i \leq n\}$, and that a label k is *i-ground* $(1 \leq i \leq n)$ iff

1. $\forall s(k) : h(s(k)) \notin \Phi^i_V$, and
2. if $\exists s^m(k) : h(s^m(k)) \in \Phi^i_V$ then $\exists s^j(k), j < m : h(s^j(k)) \in \Phi^i_C$.

2.2 High Unifications

We define a substitution in the usual way as a function

$$\begin{aligned} \sigma : \Phi^0_V &\longrightarrow \Im^- \\ : \Phi^i_V &\longrightarrow \Im^i, (1 \leq i \leq n). \end{aligned}$$

where $\Im^- = \Im - \Phi_V$. For two labels i, k and a substitution σ we shall use $(i,k)\sigma$ to denote both that i and k are unifiable (briefly, are σ-unifiable) and the result of their unification. On this basis we define several logic-dependent notions of σ-unification [ACG94a, ACG95, AG94]. The notion of two labels i, k being σ^L-unifiable, for the logics we are considering is as follows:

$$\begin{aligned}
(i,k)\sigma^* &= (i,k)\sigma \iff \text{ either} \\
&\quad \text{at least one of } i \text{ and } k \text{ is restricted, or} \\
&\quad i,k \in \Phi^0_V \text{ for every } s(i), s(k), l(s(i)) = l(s(k)), (s(i), s(k))\sigma^*; \\
(i,k)\sigma^K &= (i,k)\sigma \iff \\
&\quad \text{at least one of } i \text{ and } k \text{ is restricted, and} \\
&\quad \text{for every } s(i),\ s(k),\ l(s(i)) = l(s(k)),\ (s(i), s(k))\sigma^K; \\
(i,k)\sigma^D &= (i,k)\sigma; \\
(i,k)\sigma^{K45} &= (h(i), h(k))\sigma^K \times (s^1(i), s^1(k))\sigma^K \iff \\
&\quad l(i), l(k) > 1 \text{ and } (s^2(i), s^2(k))\sigma^K; \\
(i,k)\sigma^{D45} &= (h(i), h(k))\sigma \times (s^1(i), s^1(k))\sigma \iff \\
&\quad l(i), l(k) > 1; \\
(i,k)\sigma^{S5} &= (h(i), h(k))\sigma; \\
(i,k)\sigma^{S5A} &= (h(i), h(k))\sigma; \\
(i,k)\sigma^{S5P_{(n)}} &= (h(i), h(k))\sigma^* \text{ if} \\
&\quad i,k \text{ are } i\text{-ground}, 1 \leq i \leq n, \text{ or} \\
&\quad \exists s(i), s(k) : h(s(i)), h(s(k)) \in \Phi^i, \text{ and } (h(s(i)), h(s(k))\sigma^{S5P_{(n)}}.
\end{aligned}$$

Example 2. The notions of σ^K- and σ^D-unification are related in an obvious way to the idealization condition. Thus, $(w_2, (W_1, w_1))$, $(W_3, (W_2, w_1))$ are σ^D-unifiable but not σ^K-unifiable (since the segments $(W_1, w_1), (W_2, w_1)$ are not σ^K-unifiable by the above definition). The reason is that in the "non idealisable" logic K the "denotations" of W_1 and W_2 may be empty (i.e., there can be no worlds accessible from w_1), which obviously makes their unification impossible, while in the "idealisable" logic D they are not empty, which makes them unifiable "on" any constant. Similar intuitive motivations hold for the other σ^L-unifications.

2.3 Low Unifications

We are now able to define what it means for two labels i, k to be σ_L-unifiable for $L = K45, D45, S5, S5A, S5P_{(n)}$:

$$\begin{aligned}
(i,k)\sigma_{K45} &= \begin{cases} (i,k)\sigma^K & l(i), l(k) \leq 2 \\ (i,k)\sigma^{K45} & \text{otherwise} \end{cases} \\
(i,k)\sigma_{D45} &= \begin{cases} (i,k)\sigma^D & l(i), l(k) \leq 2 \\ (i,k)\sigma^{D45} & \text{otherwise} \end{cases} \\
(i,k)\sigma_{S5} &= (i,k)\sigma^{S5} \\
(i,k)\sigma_{S5A} &= (i,k)\sigma^{S5A} \\
(i,k)\sigma_{S5P_{(n)}} &= (i,k)\sigma^{S5P_{(n)}}
\end{aligned}$$

Remark. It is worth noting that the notion of σ^L or "high" unification is meant to mirror a single constraint on R, while the notion of σ_L or "low" unification (which includes the former) is used to simulate the full accessibility restrictions which hold in the various L-models. In general both high and low unifications are necessary for multi-modal logics, where we have several modalities acting differently, and each modality has its own high unification, and the various high unifications are combined into the low unification which models such logics.

Remark. The modal proof system proposed by Jackson and Reichgelt [JR89] is the most closely related to ours. The index formalism is almost identical, but the unification algorithm used to resolve complementary formulas in the various modal logics does not work for the non-idealisable K logics. This is due to the fact that their unification scheme is not recursive inside the world path, and the accessibility relation is external and it is not built-in into the unification as in our system.

2.4 Inference Rules

The rules of KEM will be defined for LS-formulas. Two LS-formulas X, i, Z, k such that $Z = X^C$ and $(i,k)\sigma_L$ will be called σ_L-*complementary*. The following inference rules hold for all the logics we are considering (i, k, and m stand for arbitrary labels).

$$\frac{\alpha, i}{\alpha_1, i} \qquad\qquad \frac{\alpha, i}{\alpha_2, i} \tag{α}$$

$$\frac{\begin{array}{c}\beta, i\\ \beta_1^C, k\end{array}}{\beta_2, (i,k)\sigma_L}[(i,k)\sigma_L] \qquad\qquad \frac{\begin{array}{c}\beta, i\\ \beta_1^C, k\end{array}}{\beta_2, (i,k)\sigma_L}[(i,k)\sigma_L] \tag{β}$$

$$\frac{\nu_i, i}{\nu_0, (m,i)}[m \in \Phi_V^i \text{ and new}] \tag{ν_i}$$

$$\frac{\pi_i, i}{\pi_0, (m,i)}[m \in \Phi_C^i \text{ and new}] \tag{π_i}$$

$$\frac{}{X, i \qquad X^C, i}[i \text{ restricted}] \tag{PB}$$

$$\frac{\begin{array}{c}X, i\\ X^C, k\end{array}}{\times(i,k)\sigma_L}[(i,k)\sigma_L] \tag{PNC}$$

Here the α-rules are just the usual linear branch-expansion rules of the tableau method, while the β-rules correspond to such common natural inference patterns as *modus ponens, modus tollens*, etc.

The rules for the modal operators bear a not unexpected resemblance to the familiar quantifier rules of the tableau method. "m new" in the proviso for the ν_i- and π_i-rule obviously means: m must not have occurred in any label yet used, which obviously does not hold for $S5A$ when the actuality operator is involved.

Notice that in all inferences via an α-rule the label of the premise carries over unchanged to the conclusion, and in all inferences via a β-rule the labels of the premises must be σ_L-unifiable, so that the conclusion inherits their unification.

PB (the "Principle of Bivalence") represents the (LS-version of the) semantic counterpart of the cut rule of the sequent calculus (intuitive meaning: a formula A is either true or false in any given world).

PNC (the "Principle of Non-Contradiction") corresponds to the familiar branch-closure rule of the tableau method, saying that from a contradiction of the form (occurrence of a pair of σ_L-complementary LS-formulas) $X, i,\ X^C, k$ on a branch we may infer the closure of the branch. The $(i,k)\sigma_L$ in the "conclusion" of PNC means that the contradiction holds "in the same world".

3 Soundness and Completeness

We shall show that the KEM versions of the logics L we have been considering are equivalent to their respective axiomatic formulations. In order to do this, we have to prove (i) that the characteristic axioms and the inference rules of the axiomatic L are derivable in KEM, and (ii) that the rules of KEM are derived rules in the axiomatic L. To prove (ii) we show that the rules of KEM hold in a model for the respective L.

Let $\mathcal{F} =< W, \Sigma_1, \dots \Sigma_m, R_1, \dots, R_n >$ be an extended Kripke frame and let $\mathcal{M} =< W, R_1, \dots, R_n, \upsilon >$ be an extended Kripke model with the usual conditions on their elements; R_i is defined as $\Gamma R_i \Gamma' \Leftrightarrow \{A : \Box_i A \in \Gamma\} \subseteq \Gamma'$, where Γ denotes an element of the non empty set W; and υ is as before.

We now define a translation function g from labels to the model's frame as follows: $g : \Im \rightarrow \mathcal{F}$ so that:

(a) If $i \in \Phi_C$ then $g(i) = \exists \Gamma \in W$;
(b) If $i \in \Phi_V$ then $g(i) = \forall \Gamma \in W$;
(c) If $i = a$, then $g(i) = \Gamma^*$ (which is intended to denote the "actual world");
(d) If $i \in \Phi^i_C$, then $g(i) = \exists \Sigma^m_i \in \Sigma_i$;
(e) If $i \in \Phi^i_V$, then $g(i) = \forall \Sigma^m_i \in \Sigma_i$;
(f) If $l(i) = n > 1$ then we denote by i^m the $h(j)$ such that $l(j) = m,\ m \leq n$, and j is a segment of i; thus

$$g(i) = Qg^1 Qg^2(g^1 Bg^2 \# Qg^3(g^2 Bg^3 \# \cdots \# Qg^n(g^{n-1} Bg^n) \cdots))$$

where g^m denotes the element associated by g to the segment of i of length m; Qg^m denotes $\forall \Gamma^m, \exists \Gamma^m, \forall \Sigma^m_i, \exists \Sigma^m_i, \emptyset$ respectively if its i^m is W, w, W^i, w^i, a; $\#$ is $\supset$ if the associate Qg is $\forall$, otherwise it is $\wedge$; $g^k Bg^m$ is $g^k R_i g^m$ if i^m is W^i, or w^i, $g^k \mathcal{A} g^m$ if i^m is a, othewise it is $g^k R g^m$.

Let f be the translation function from LS-formulas to the model defined as:

$$f(SA, i) = g(i), \upsilon(A, g(h(i))) = S.$$

Lemma 1. *For any $i, k \in \Im$, $L = K45, D45, S5, S5A, S5P_{(n)}$, if X, i and $(i, k)\sigma^L$, then X, k.*

We only give the proofs for $S5, S5A, S5P_{(n)}$; for the other logics see [AG94]

Proof $L = S5$. Let us suppose that X, i, X^C, k and $(i, k)\sigma^{S5}$, but $(i, k)\sigma^{S5} \iff (h(i), h(k))\sigma^{S5}$. From the definition of f we have

$$f(X, i) = g(i), \upsilon(A, g(h(i))) = S,$$

and

$$f(X^C, k) = g(k), \upsilon(A, g(h(k))) = S^C;$$

from the equivalence relation of the model we get

$$f(X, i) = Qg(h(i)), \upsilon(A, g(h(i))) = S,$$

and

$$f(X^C, k) = Qg(h(k)), \upsilon(A, g(h(k))) = S^C.$$

We now analyse what kind of labels $h(i), h(k)$ are.

1. $h(i), h(k)$ are two constants;
2. $h(i), h(k)$ are a constant and a variable;

3. $h(i), h(k)$ are two variables.

Case 1. Two constants unify iff they are the same constant, and so $h(i) = h(k)$ and $g(h(i)) = g(h(k))$, thus obtaining a contradiction.

Case 2. This case implies

$$f(X, i) = \forall g(h(i)), \upsilon(A, g(h(i))) = S,$$

and

$$f(X^C, k) = \exists g(h(k)), \upsilon(A, g(h(k))) = S^C$$

which lead to a contradiction.

Case 3. We get a contradiction because a variable unifies with any label, and W is not empty.

Proof $L = S5A$. The proof is the same as for $S5$ apart from the case where $h(i)$ and $h(k)$ are both a.

Let us then suppose that the lemma does not hold. This means that in the model we have:

$$f(X, i) = g(i), \upsilon(A, g(h(i))) = S,$$

and

$$f(X^C, k) = g(k), \upsilon(A, g(h(k))) = S^C.$$

Since $S5A$ is like $S5$ and $\mathcal{A} \subseteq R$ we have only to analyse the case in which both labels end with a; but this implies

$$f(X, i) = g(i), \upsilon(A, \Gamma^*) = S,$$

and

$$f(X^C, k) = g(k), \upsilon(A, \Gamma^*) = S^C$$

thus obtaining a contradiction.

Proof $L = S5P_{(n)}$. We analyse only the cases which are different from $S5$. i, k unify if they are not two variables of the same type, and for each variable of type i there is a constant of the same type.

$$\begin{aligned} f(X, i) = \cdots g^1 B g^2 \# \cdots \# & \\ Qg(h(i))(g(b(h(i))Bg(h(i)))), \upsilon(A, g(h(i))) = S, & \end{aligned} \tag{1}$$

and

$$\begin{aligned} f(X^C, k) = \cdots g^1 Bg(s^2(k)) \# & \\ \cdots \# Qg(h(k))(g(b(h(k))Bg(h(k)))), \upsilon(A, g(h(k))) = S^C. & \end{aligned} \tag{2}$$

This is done to ensure that the corresponding Σ_i is not empty. Since each $R_i \subset R$ (1) and (2) imply, respectively

$$Qg(h(i)), \upsilon(A, g(h(i))) = S,$$

and

$$Qg(h(k)), \upsilon(A, g(h(k))) = S^C$$

from which by analysing as before what kind of labels $h(i), h(k)$ are, we get a contradiction.

Theorem 2. $\vdash_L A \Leftrightarrow \vdash_{KEM(L)} A$ *for* $L = K45, D45, S5, S5A, S5P_{(n)}$.

Proof $\Rightarrow$. The *modus ponens* and the characteristic axioms for L are provable in KEM (see [AG93] for the proof of the axioms and [DM94] for a proof that *modus ponens* is a derived rule in KE).

We shall give a KEM-proof of the rule of necessitation. Let us assume that $\vdash_{KEM(L)} A$. Then the following is the KEM-proof of $\Box A$.

$$
\begin{array}{ll}
1.\ F\Box A & w_1 \\
2.\ FA & (w_2, w_1) \\
3.\ \times & (w_2, w_1)
\end{array}
$$

The closure follows from the fact that A is provable in KEM, i.e. there is a closed KEM-tree for FA and that KE and KEM enjoy the "subproof" property (i.e. the transitivity property of proof, see [DM94]).

As regards the axioms of $S5A$ and $S5P_{(n)}$ we shall show example proofs of $\triangle A \to \Box\triangle A$ and $\neg P_i\bot \to (P_iP_jA \equiv P_jA)$ (for the other axioms see [ACG94b]).
$\vdash_{KES5A} \triangle A \to \Box\triangle A$

$$
\begin{array}{lr}
1.\ F\triangle A \to \Box\triangle A & w_1 \\
2.\ T\triangle A & w_1 \\
3.\ F\Box\triangle A & w_1 \\
4.\ TA & (a, w_1) \\
5.\ F\triangle A & (w_2, w_1) \\
6.\ FA & (a, (w_2, w_1)) \\
7.\ \times & a
\end{array}
$$

$\vdash_{KES5P_{(n)}} \neg P_i\bot \to (P_iP_jA \equiv P_jA)$

$$
\begin{array}{lr}
1.\ F\neg P_i\bot \to (P_iP_jA \equiv P_jA) & w_1 \\
2.\ T\neg P_i\bot & w_1 \\
3.\ FP_iP_jA \equiv P_jA & w_1 \\
4.\ FP_i\bot & w_1 \\
5.\ F\bot & (w_2^i, w_1)
\end{array}
$$

$$
\begin{array}{lr|lr}
6.\ TP_iP_jA & w_1 & 7.\ FP_iP_jA & w_1 \\
8.\ FP_jA & w_1 & 13.\ TP_jA & w_1 \\
9.\ TP_jA & (W_1^i, w_1) & 14.\ FP_jA & (w_4^i, w_1) \\
10.\ TA & (W_2^j, (W_1^i, w_1)) & 15.\ FA & (w_5^j, (w_4^i, w_1)) \\
11.\ FA & (w_3^j, w_1) & 16.\ TA & (W_3^j, w_1) \\
12.\ \times & w_3^j & 17.\ \times & w_5^j
\end{array}
$$

Proof $\Leftarrow$. The α-rules and PB are obviously derived rules in L. For the β-rules and PNC: by hypothesis $(i,k)\sigma_L$ and hence, by the above lemma and the definitions of the σ_L-unifications, the formulas involved have the same value in $i\,(k)$ and $(i,k)\sigma_L$; after that these rules become rules of KE, and thus they are derived rules in L.

For the ν_i-rules let us suppose that ν_i is not a derived rule of L; from which it follows $\upsilon(\nu_i, g(i)) = S$ and $g(i)R_j\Gamma, \upsilon(\nu_0, \Gamma) = S^C$, but the former implies $\forall(\exists)\Gamma \in W, g(i)R_j\Gamma, \upsilon(\nu_0, \Gamma) = S$, thus obtaining a contradiction.

The proof for the π_i-rules is similar.

4 Proof Search

As usual with refutation methods, a proof of a formula A of L consists of attempting to construct a countermodel for A by assuming that A is false in some arbitrary L-model. Every successful proof discovers a contradiction in the putative countermodel. In this section we describe an algorithm which does this job (and that has been implemented in Prolog, see [ACG94a, ACG94b]).

In what follows by a KEM*-tree* we shall mean a tree generated by the inference rules of KEM.

A branch τ of a KEM-tree will be said to be σ_L*-closed* if it ends with an application of PNC. A KEM-tree $\mathcal{T}$ will be said to be σ_L*-closed* if all its branches are σ_L-closed. Finally, by an L-proof of a formula A we shall mean a σ_L-closed KEM-tree starting with FA, i.

Given a branch τ of a KEM-tree, we shall call an LS-formula X, i E*-analysed in* τ if either (i) X is of type α and both α_1, i and α_2, i occur in τ; or (ii) X is of type β and one of the following conditions is satisfied: (a) if β_1^C, k occurs in τ and $(i,k)\sigma_L$, then also $\beta_2, (i,k)\sigma_L$ occurs in τ, (b) if β_2^C, k occurs in τ and $(i,k)\sigma_L$, then also $\beta_1, (i,k)\sigma_L$ occurs in τ; or (iii) X is of type ν_i and $\nu_0, (m,i)$ occurs in τ for some $m \in \Phi_V$ not previously occurring in τ, or (iv) X is of type π_i and $\pi_0, (m,i)$ occurs in τ for some $m \in \Phi_C$ not previously occurring in τ.

We shall call a branch τ of a KEM-tree E*-completed* if every LS-formula in it is E-analysed and it contains no complementary formulas which are not σ_L-complementary. We shall say a branch τ of a KEM-tree *completed* if it is E-completed and all the LS-formulas of type β in it either are analysed or cannot be analysed. We shall call a KEM-tree *completed* if every branch is completed.

The following procedure starts from the 1-branch, 1-node tree consisting of FA, i and applies the rules of KEM until the resulting KEM-tree is either closed or completed. At each stage of the proof search (i) we choose an open non completed branch τ. If τ is not E-completed, then (ii) we apply the 1-premise rules until τ becomes E-completed. If the resulting branch τ' is neither closed nor completed, then (iii) we apply the 2-premise rules until τ becomes E-completed. If the resulting branch τ' is neither closed nor completed, then (iv) we choose an LS-formula of type β which is not yet analysed in the branch and apply PB so that the resulting LS-formulas are β_1, i' and β_1^C, i' (or, equivalently β_2, i' and β_2^C, i'), where $i = i'$ if i is restricted, otherwise i' is obtained from i by instantiating $h(i)$ to a constant not occurring in i; (v) if the branch is not E-completed nor closed, because of complementary formulas which are not σ_L-complementary, then we have to see whether a restricted label unifying with both the labels of the complementary formulas occurs previously in the branch; if such a label exists or can be built using already existing labels and the unification rules, then the branch is closed, (vi) we repeat the procedure in each branch generated by PB.

Remark. As is well known [Smu68a], what destroys analyticity is losing the (weak) subformula property [Fit90], and not having a cut rule restricted to subformulas. Otherwise each tableau system is not analytic, since from the formula $\neg(A \rightarrow B)$ we obtain two branches containing respectively $\neg A$ and B, but, obviously, $\neg A$ is not an immediate (strong) subformula of $\neg(A \rightarrow B)$. Moreover a clever and ruled use of the cut could reduce sharply the complexity of the proof [Boo84, DM94]. Finally it can be used to check closure in a modal setting. In fact step (v) of the above procedure, called "Modal PB", prescribes that we apply PB to one of the complementary formulas whose label unifies with both the labels, thus closing the branches.

The above procedure is based on the procedure for canonical KEM-trees. A KEM-tree is called canonical iff all the applications of 1-premise rules come before the applications of 2-premise rules, which precede the applications of the 0-premise rule.

Two interesting properties of canonical KEM-trees are (i) that a canonical KEM-tree always terminates, since for each formula there are a finite number of subformulas and the number of labels which can occur in the KEM-tree for a formula A (of L) is limited by the number of modal operators belonging to A, and (ii) that for each closed KEM-tree a closed canonical KEM-tree exists.

Let ϕ be the function which deletes the modal operators from given formulas.

Lemma 3. $\not\vdash_{KE} \phi A \Rightarrow \not\vdash_{KEM(L)} A$

Proof. Obvious. For the details see [ACG95].

This lemma gives a first termination check for the canonical KEM-trees; in fact a KEM-tree finds out whether complementary formulas exist and it verifies (through the σ_L-unifications) whether the paths denoted by the labels of the complementary formulas lead to the same world, if so the branch is closed, but if there are no complementary formulas there are no σ_L-complementary formulas.

We shall define the complexity of an LS-formula as the number of logical symbols occurring in it.

Theorem 4. *A canonical KEM-tree always terminates*

Proof. We show that each step produces at most a finite number of new LS-formulas, where with *new* we mean that the label has not been previously used with the S-formula.

The procedure for canonical KEM-trees stops either when

1. there are no LS-formulas whose complexity is greater than 1, or
2. there are no β-formulas that cannot be analysed, or

3. there are no complementary formulas which are not σ_L-complementary and a label which unifies with both the labels of the given formulas does not exist.

We prove the theorem by induction on the length of proof.

At the step 0, the α-, ν_i-, and π_i-rules produce a new LS-formula of less complexity, and PB produces 2 branches where there is a new LS-formula of less complexity.

At the n-th step α-rule produces at most 2 new LS-formulas of less complexity; and both ν_i-, π_i-rules produce a new LS-formulas of less complexity; the β-rules produce at most m new LS-formulas of less complexity, where m is the number of LS-formulas which are the conjugate of an immediate subformula of a β-formula, and whose labels σ_L-unify with the label of the β-formula; by induction m is finite. PB produces 2 branches where there are at most k new LS-formulas of less complexity, where k is the number of restricted labels which σ_L-unify with the label of the formula on which PB is applied; k is finite by induction.

If there are some complementary formulas which are not σ_L-complementary, modal PB controls whether a restricted label which σ_L-unifies with both the labels of the complementary formulas occurs in the tree. But the number of the restricted labels occurring in the tree is finite, since at most it is equal to the number of the LS-formulas occurring in the tree which is finite.

In what follows we shall use $\mathcal{T}$ and $\mathcal{C}$ to denote respectively a KEM-tree and a KEM-canonical tree. Moreover whe shall use $\vdash_{KEM} A$ to mean that there exists a closed KEM-tree for A^C and $\vdash_{KEMc} A$ to mean that there exists a closed KEM-canonical tree for A^C.

For two label i, k, we shall say that i extends k ($i \trianglerighteq k$) if an $s(i)$ so that $(s(i), k)\sigma_L$ exists.

Theorem 5. *A KEM-tree for a formula A is closed $\iff$ the canonical KEM-tree for A is closed.*

Proof $\Rightarrow$. The if part is obvious since a canonical KEM-tree is a KEM-tree, and the steps which are not essential could be freely deleted.

Proof $\Leftarrow$. We give the proof only for $S5A$ and $S5P_{(n)}$; for the other logics see [ACG95]

We have to show $\vdash_{KEM} A \Rightarrow \vdash_{KEMc} A$ or its equivalent $\nvdash_{KEMc} A \Rightarrow \nvdash_{KEM} A$.

Let us suppose that the theorem does not hold, which means $\nvdash_{KEMc} A$ but $\vdash_{KEM} A$. According to the previous lemma and the properties of KE (see [DM94]) we have that, in the proof of a formula A, PB occurs only on subformulas of the given formulas. And so each unnecessary occurence of PB can be freely deleted.

If A^C is of type α, ν_i or π_i then both $\mathcal{T}$ and $\mathcal{C}$ have the same development. The same happens with PB apart from complexity results. The only difference between a KEM-tree and a KEM-canonical tree relies on the application of the β-rules.

$$\begin{array}{ll}
\vdots & \\
\beta & i \qquad (3)\\
\beta_1^C & j \qquad (4)\\
\beta_2 & k, k = (i,j)\sigma_L \qquad (5)\\
\vdots & \\
T\phi(\beta_2) \rightarrow \psi(\beta_1^C) & l \qquad (6)\\
\psi(\beta_1^C) & m, m = (k,l)\sigma_L \qquad (7)\\
\vdots & \\
\beta_1^C & n, n \trianglerighteq m \qquad (8)\\
\beta_2 & o, o = (i,n)\sigma_L \qquad (9)\\
\vdots & \\
\beta_2^C & p \qquad (10)
\end{array}$$

From the hypothesis we get that β_2, o, and β_2^C, p are σ_L-complementary but β_2, k, and β_2^C, p are not. The correspondent canonical tree does not have β_2, o (9) since its rules do not allow us to use a β-rule on a β formula as a major premise more than in a "level", where, roughly, a level represents the set of formulas obtained from the 1-premise rules before using a β-rule. But it controls closure using PB with an existing restricted label on complementary formulas which are not σ_L-complementary. We are now going to show that the absurd hypothesis leads to a contradiction; so we have to prove that either $(k,p)\sigma_L$-unify or o, p do not σ_L-unify or a restricted label which unifies with both k and p, exists.

Proof L=S5A. If $h(i) \in \Phi_C$ then $k = m = o = h(i)$ but both $(o,p)\sigma_{S5A}$ and k, p do not σ_{S5A}-unify and so we have a contradiction. If $h(i) \in \Phi_V$ we have to see what kind of labe p is. If it is unrestricted, then it σ_{S5A}-unifies with each label, and also with k which implies a contradiction. Otherwise let us suppose that $h(p) \neq h(j)$; if so also in this case it σ_{S5A}-unifies with k which is not the case; but, since β_2^C, p does not depend on $\phi(\beta_2) \rightarrow \psi(\beta_1^C), l$ we can apply a β-rule on β, i and β_2^C, p itself, obtaining $\beta_1, h(p)$ which is one of the σ_{S5A}-complementary formulas of β_1^C, n because $h(n)$ must σ_{S5A}-unify with $h(i)$, and their result, o, σ_{S5A}-unifies with p. If n were restricted or different from a then $o = h(n) \in \Phi_C$, but $(o,p)\sigma_{S5A}$ and thus either $h(p) \in \Phi_V$, which is a contradiction, or $o = h(p) = a$ which means that $h(n) = a$ or n is unrestricted.

Proof $L = S5P_{(n)}$. In what follows we use the expression "the label i is i-ground" to mean either that i is i-ground or that the modal PB can generate an i-ground label unifying with it. Let us suppose also in this case that the theorem does not hold. We have to analyse what kind of labels are i, k and p. If $h(i) \in \Phi_C$, then $k = o = h(i)$, therefore $(k,p)\sigma_{S5P_{(n)}}$, from which we get a contradiction. We pass now to analyse the case in which i is unrestricted. Let us first suppose that also j is unrestricted; since they unify, they are i-ground $(1 \leq i \leq n)$, and the result of their unifications is any restricted label which appears in the tree; we shall call it w_i. If also p is unrestricted we have to see the head of n; if it is unrestricted, o must be a restricted label, w_j, different from w_i. But since $(o,p)\sigma_{S5P_{(n)}}$, even p is i-ground and therefore, in the canonical tree we must have applied a β-rule on β, i and β_2^C, p thus obtaining β_1, r, where r is a restricted label unifying with j; this implies that the resulting canonical tree is closed. If i, p are unrestricted, $h(j) = w_j$, and $h(n) = w_n$ then both are i-ground, otherwise they could not unify with j, and o respectively, and so we can repeat the reasoning of the previous case. If i, n are unrestricted, $h(j) = w_j$ and $h(p) = w_p$ therefore since p is a label already existing and we have β_2^C, p we obtain β_1, w_p which is a $\sigma_{S5P_{(n)}}$-complementary formula of β_1^C, n; but this implies that the canonical tree is closed.

5 Final Remarks

We would like to point out some of the advantages of KEM: unlike resolution or translation based methods it does not require the input formula to be preprocessed, nor to be transformed either in clausal form or in another formalism (it is worth noting that certain modal logics like $S4$ do not have a clausal form and other logics, like GL, cannot be expressed in a first-order formalism); all modal logics are treated in the same way and only constraints on the unifications, which are intended to represent the conditions on the various accessibility relations, distinguish one logic from another; the rules it uses are strictly related with the modal semantics; like axiomatic systems it is highly modular. To illustrate the last point we summarize how obtain the Kraus and Lehmann's logic for knowledge and belief $\mathcal{KB}$ [Hoe93]. Knowledge modalities act as $S5$ modalities whereas belief modalities act as $D45$ modalities, and it is well known that adding reflexivity to $D45$ leads to $S5$, therefore our basic high unification is σ^{D45}. A way to have reflexivity in KEM is to assume the following function (reduction) concerning label: $r(i) = b(i)$ if $h(i) \in \Phi_V^{K_i}$ otherwise $r(i) = (h(i), r(b(i)))$. Finally low unification works by applying recursively the σ^{D45} either to the labels or to the possible reductions of the labels. Such items will be investigated in a future work.

References

[ACG94a] A. Artosi, P. Cattabriga and G. Governatori. An Automated Approach to Deontic Reasoning. In J. Breuker (ed.), *Artificial Normative Reasoning*, Workshop ECAI 1994: 132–145.

[ACG94b] A. Artosi, P. Cattabriga and G. Governatori. *An Automated Modal Framework for Nonmonotonic Reasoning.* CIRFID Technical Report October 94.

[ACG95] A. Artosi, P. Cattabriga and G. Governatori. *KEM: A Modal Theorem Prover.* Forthcoming in Annali dell'Università di Ferrara, Sez. III, Fil., Discussion paper series.

[AG93] A. Artosi and G. Governatori. Labelled Modal Proofs. In A. Artosi (ed.),*Two Papers in Proofs Theory.* In *Annali dell'Università di Ferrara*, Sez. III,Filosofia, Discussion papers 33, 1–16. Ferrara.

[AG94] A. Artosi and G. Governatori. Labelled Model Modal Logic. In *Proceedings of the CADE-12 Workshop on Automated Model Building*, 1994: 11–17.

[AE92] Y. Auffray and P. Enjalbert. Modal Theorem Proving: An Equational Viewpoint. *Journal of Logic and Computation*, 2, 1992: 247–259

[Boo84] G. Boolos. Don't Eliminate Cut. *Journal of Philosophical Logic*, 13, 1984: 373–378.

[Cat91] L. Catach. Tableaux: A General Theorem Prover for Modal Logics. *Journal of Automated Reasoning*, 7, 1991: 489–510.

[DG94] M. D'Agostino and D.M. Gabbay. A Generalization of Analytic Deduction via Labelled Deductive Systems I: Basic Substructural Logics. *Journal of Automated Reasoning*, 13: 243–281.

[DM94] M. D'Agostino and M. Mondadori. The Taming of the Cut. *Journal of Logic and Computation*, 4, 1994: 285–319.

[Fi66] F.B. Fitch. Tree Proofs in Modal Logic (abstract). *Journal of Symbolic Logic*, 31, 1966: 152.

[Fit72] M. Fitting. Tableau Methods of Proof for Modal Logic. *Notre Dame Journal of Formal Logic*, 13, 1972: 237–247.

[Fit83] M. Fitting. *Proof Methods for Modal and Intuitionistic Logic*, D. Reidel Publishing Company, Dordrecht, 1983.

[Fit90] M. Fitting. *First-Order Logic and Automathed Theorem Prover.* Springer-Verlag, Berlin, 1990.

[Gab91] D.M. Gabbay. *Labelled Deductive System, Part I.* Technical Report CIS-Bericht-90-22, CIS-Universitaät München, February 1991.

[Hoe93] W. van der Hoek. Systems for Knowledge and Belief. *Journal of Logic and Computation*, 3, 1993: 173–196.

[JR89] P. Jackson and H. Reichgelt. A General Proof Method for Modal Predicate Logic. In P. Jackson, H. Reichgelt, and F. van Harmelen, *Logic-Based Knowledge Representation.* The MIT Press, Cambridge, Mass., 1989.

[MvH92] J.J.Ch. Mayer and W. van der Hoeck. A Modal Logic for Nonmonotonic Reasoning. In W. van der Hoeck, J.J.Ch. Mayer, Y. H. Tan and C. Witteveen (eds.), *Non-Monotonic Reasoning and Partial Semantics.* Ellis Horwood: N.Y., 1992: 37–77.

[Ohl91] H.J. Ohlbach. Semantics-Based Translation Methods for Modal Logics. *Journal of Logic and Computation*, 1, 1991: 691–746.

[Smu68a] R. Smullyan. Analityc Cut. *Journal of Symbolic Logic*, 33, 1968: 560–564.

[Smu68b] R. Smullyan. *First-Order Logic*, Springer-Verlag: Berlin, 1968.

[Tap87] B.L. Tapscott. A Simplified Natural Deduction Approach to Certain Modal Systems. *Notre Dame Journal of Formal Logic*, 28, 1987: 371–383.

[Wal90] L.A. Wallen. *Automated Deduction in Non-Classical Logics*, The MIT Press, Cambridge, Mass., 1990.

[Wan90] H. Wansing. A General Possible Words Framework for Reasoning about Knowledge and Belief. *Studia Logica*, 49, 1990: 523–539.

[Wri85] G. Wrightson. Non-classical Logic Theorem Proving, *Journal of Automated Reasoning*, 1, 1985: 35–37.