Labelled Modal Sequents

Guido Governatori[1] and Antonino Rotolo[2]

[1] School of Computing and Information Technology, Griffith University, Nathan, QLD 4111, Australia, G.Governatori@cit.gu.edu.au
[2] CIRSFID, University of Bologna
Via Galliera 3, I-40121 Bologna, Italy, rotolo@cirfid.unibo.it

Abstract. In this paper we present a new labelled sequent calculus for modal logic. The proof method works with a more "liberal" modal language which allows inferential steps where different formulas refer to different labels without moving to a particular world and there computing if the consequence holds. World-paths can be composed, decomposed and manipulated through unification algorithms and formulas in different worlds can be compared even if they are sub-formulas which do not depend directly on the main connective. Accordingly, such a sequent system can provide a general definition of modal consequence relation. Finally, we briefly sketch a proof of the soundness and completeness results.

1 Introduction

Gabbay [12] proposes a new methodology called *Labelled Deductive Systems* (*LDS*) to deal, uniformly, with logical systems. This approach, where formulas are indexed with labels to bring meta-level features in the object language, is very flexible: it enables us to work not only on the logical part, but also on the labels (using an appropriate algebra), and both.

Gentzen systems are often used to define calculi as well as consequence relations. Nevertheless, such systems do not work so well when intensional operators are involved. In order to generalise them to modal logic, the most direct course is to try and devise rules for $\Box$ of the same kind as those governing the classical operators; in other words to force the classical pattern on the modal operator. Moreover, there is no single interpretation of modality: each of them requires its own consequence relation. This leads to the fact that modal sequents are far to be uniform (see [15] for an overview of such systems). Labelled sequents seem to offer an higher degree of uniformity, at least for classes of logics. Unfortunately almost all recent works proposing labels in sequent systems suffer from the same illness: they use labels properties (semantics) to reduce modal consequence to classical one. So they fail to provide a general system for defining real notions of modal deducibility.

In the spirit of *LDS* we present a general framework for modal sequent calculi ($\mathscr{LMS}$) suitable for providing a notion of modal consequence. Basically, it enjoys the following features:

– There is only a rule for modality, and the rules for the boolean connectives are generalized to the modal case.
– All the modal inferences are kept in the labels, no external constraints of modal axioms are needed.

We use KEM label language (see [1, 16]) that simulates accessibility relation, and an algorithm to determine the conditions under which two labels can be compared. If so, we can apply inference rules on the related formulas. We have two kinds of atomic labels *constants* — $w_1, w_2, \dots$ — and *variables* — $W_1, W_2, \dots$ — that might be combined into *path* labels. Roughly a constant corresponds to $\Diamond$ and a variable to $\Box$. A path is a label with the following form (i, i'), where i is an atomic label and i' is either a path or a constant, in the same way an atomic label corresponds to a single modality a path corresponds to a string of modalities. It is worth noting that labels may be split so that the parts can be considered separately. Another interesting feature of the present approach is that boolean and modal combinations of labelled formulas are permitted; so, if A, B are well formed formulas, and i, j, k are labels, then $A : i$, $B : j$ and $(A : i \rightarrow B : j) : k$ are expressions of our language. This fact forces us to reconsider the classical rules for propositional connectives. For example we have the following instance of *modal modus tollens*: $(A : i \rightarrow B : j) : k$ and $\neg\Box B : l$ imply $\neg A : m$, under the appropriate conditions on the labels. In particular l and k should be comparable and j corresponds to $\Box$.[1]

2 Labelled Modal Language

As we have already alluded to, we allow boolean and modal combination of labelled formulas, so we first introduce the appropriate label formalism and then we extend the language of modal logic to the labelled case.

2.1 Label Formalism

KEM has two basic kinds of atomic labels: variables and constants. The label scheme arises from such a basic alphabet, so that a "world" label is either a world-symbol or a "structured" sequence of world-symbols that we call a "world-path". Constant and variable world-symbols denote worlds and sets of worlds respectively (in a Kripke model), while a world-path conveys information about access between the worlds in it. KEM labels are built in a modular way and so they can be easily composed and decomposed. Furthermore, we shall use auxiliary "dummy" labels, that allow world-paths to be split into proper subpaths.

Definition 1. *Let $\Phi_A = \{w_0, w_0', \dots\}$ be a not empty set of auxiliary or actual world symbols; let $\Phi_C = \{w_1, w_2, \dots\}$ be a not empty set of constant world symbols (or simply constants); let $\Phi_V = \{W_1, W_2, \dots\}$ be a not empty set of*

[1] See section 4.2 for the actual definition of the modal modus tollens.

variable world symbols (or simply variables). The set $\Im$ *of label is then defined as follows*

$$\Im = \bigcup_{1 \leq i} \Im_i \text{ where } \Im_i :$$
$$\Im_1 = \Phi_A;$$
$$\Im_{n+1} = (\Phi_C \cup \Phi_V) \times \Im_n, \ (n \geq 1).$$

According to the above definition a label is either (i) an element of the set Φ_A, or (ii) a path term (k', k) where (iia) $k' \in \Phi_C \cup \Phi_V$ and (iib) $k \in \Phi_C$ or $k = (i', i)$ where (i', i) is a label. From now on we shall use $i, j, k, \ldots$ to denote arbitrary labels.

For any label $i = (k', k)$ we shall call k' the *head* of i, k the *body* of i, and denote them by $h(i)$ and $b(i)$ respectively. Notice that these notions are recursive (they correspond to projection functions): if $b(i)$ denotes the body of i, then $b(b(i))$ will denote the body of $b(i)$, $b(b(b(i)))$ will denote the body of $b(b(i))$; and so on. We call each of $b(i)$, $b(b(i))$, etc., a *segment* of i. Let $s(i)$ denote any segment of i (obviously, by definition every segment $s(i)$ of a label i is a label); then $h(s(i))$ will denote the head of $s(i)$. We shall call a label i *restricted* if $h(i) \in \Phi_C$, otherwise *unrestricted*.

For any label i, we define the length of i, $\ell(i)$, as the number of world-symbols in i, i.e., $\ell(i) = n \Leftrightarrow i \in \Im_n$. $s^n(i)$ will denote the segment of i of length n, i.e., $s^n(i) = s(i)$ such that $\ell(s(i)) = n$. We shall use $h^n(i)$ as an abbreviation for $h(s^n(i))$.

For any label $i, \ell(i) > n$, we define the *countersegment-n* of i, as follows:

$$c^n(i) = h(i) \times (\cdots \times (h^k(i) \times (\cdots \times (h^{n+1}(i), j)))) \ (n < \ell(i))$$

where j is an auxiliary label. In other words the countersegment-n of a label i is the label obtained from i by replacing $s^n(i)$ with an auxiliary world symbol.

There is a strict relationship between labels and possible world semantics. The intuitive reading of a constant is a possible world in a Kripke models, while a variable denotes a set of worlds. A path label moreover encodes the information about the accessibility relation. Indeed a label such as $(W_1, (w_1, w_0))$ represents the set of worlds accessible from the world denoted by w_1, which itself is accessible from the actual world w_0. An auxiliary world symbol stands for an actual world.

In general a label corresponds to the model generated from a formula with respect to the actual world: the actual world of the label. However, sometimes, we want to change our point of view, so we move our actual world inside a path, and to consider the truncated model. This effect is achieved by the notions of segment and countersegment. We split a label into two parts: the segment is the path which leads us to the current actual world from the previous one; the countersegment then is the truncated model.

2.2 Labelled Well-formed Formulas

The standard modal language L is extended by attaching to each well-formed formula of L (wff) a KEM label. So, the notion of *label formula* is defined as follows:

Definition 2. – *if A is a wff and i is a label, then $A : i$ is a* labelled formula *(lwff for short);*
- *if $A : i$ is a lwff and j is a label, then $A : i : j$ is a lwff;*
- *if $A : i$ and $B : j$ are lwff's, $\#$ is a binary connective, and k is a label, then $(A : i\#B : j) : k$ is a lwff.*
- *if $A : i$ is a lwff and j is a label, then $\Box(A : i) : j$, $\Diamond(A : i) : j$ and $\neg(A : i) : j$ are lwff's.*

Formulas without labels will be considered labelled with the auxiliary label w_0; so A will be regarded as $A : w_0$.

According to Smullyan-Fitting [11] unifying notation that classifies formulas we shall say that

Definition 3. *Two wffs A, and B of type ν and π are* complementary *iff A_0 and B_0 are* complementary.

In the previous section we have seen that the labels can be decomposed. Here we show how labels can be composed. Given a lwff $A : i : j$ we can compose i and j in a label k which satisfies the following conditions:

$$i = c^{\ell(j)}(k) \qquad\qquad j = s^{\ell(j)}(k) \tag{1}$$

2.3 From Labels to Modalities

In this section we shall examine the relationships between labels and modalities.

Our rules are designed in such a way that each modal step depends on the properties of the labels involved, which are defined to simulate the syntactical structure of modal formulas.

Why should we use then labels instead of modalities? The algebra of labels is extremely flexible and allows easy manipulations of them. However, sometimes, it may be useful to deal also with modalities mixed with labels, at least we want to translate the final steps of proofs in a plain modal language. Another example where we use mixed expressions is the generalization of classical principles such as *modus ponens*, *modus tollens* to the modal case, where a part of the inference pattern is expressed in label notation and the other uses modalities. To this end we need a function which translates labels into modalities.

We shall use $\mathfrak{m}, \mathfrak{n}, \mathfrak{p}, \mathfrak{q}, \ldots$ for strings of positive modalities; let $\mathfrak{M}$ be the set of positive modalities, by definition of modality the empty string of positive modalities is a modality, we use $\sharp$ to denote it.

Definition 4. *Let ϕ^+ be a map from $\Im$ to $\mathfrak{M}$ thus defined:*

$$\phi^+(i) = \begin{cases} \sharp & i \in \Im_1 \\ \Box\phi(b(i)) & \ell(i) > 1 \text{ and } h(i) \in \Phi_V \\ \Diamond\phi(b(i)) & \ell(i) > 1 \text{ and } h(i) \in \Phi_C \end{cases} \tag{2}$$

Let ϕ^- be a map from $\Im$ to $\mathfrak{M}$ thus defined:

$$\phi^-(i) = \begin{cases} \sharp & i \in \Im_1 \\ \Diamond\phi(b(i)) & \ell(i) > 1 \text{ and } h(i) \in \Phi_V \\ \Box\phi(b(i)) & \ell(i) > 1 \text{ and } h(i) \in \Phi_C \end{cases} \tag{3}$$

3 Unifications

The key feature of our approach is that in the course of proof labels are manipulated in a way closely related to the semantics of modal operators and "matched" using a specialized unification algorithm. That two labels i and k are unifiable means, intuitively, that the sets of worlds they "denote" have a non-null intersection. The basic element of the unification is the substitution function which maps each variable in label to a label, and each constant to itself. Formally

$$\sigma : \Phi_V \mapsto \Phi_A \cup \Phi_C \cup \Phi_V$$
$$\mathbf{1}_{\Phi_C \cup \Phi_A}$$

Applying the substitution recursively in a label we obtain the substitution of a label

$$\sigma(i) = \begin{cases} \sigma(i) & \ell(i) = 1 \\ (\sigma(h(i)), \sigma(b(i))) & \text{otherwise} \end{cases} \tag{4}$$

For two labels i and j, and a substitution σ, if σ is a unifier of i and j then we shall say that i, j are *σ-unifiable*. We shall use $(i, j)\sigma$ to denote both that i and j are σ-unifiable and the result of their unification. In particular

$$\forall i, j, k \in \Im, (i, j)\sigma = k \text{ iff } \exists\sigma \text{ such that both } \sigma(i) = \sigma(j) \text{ and } \sigma(i) = k \tag{5}$$

On this basis we may define several specialised, logic-dependent notions of σ-unification characterizing various modal logics. First of all, we have to define unifications (axiom unifications) corresponding to modal axioms. Then in the same way a modal logic is obtained by combining several axioms we define, using axiom unifications, combined unifications, that, when applied recursively, produce logic unifications.

The general form of an axiom unification σ^A is:

$$(i, j)\sigma^A \iff (f_A(i), g_A(j))\sigma \text{ and } C^A$$

where f_A and g_A are given logic-dependent functions from labels to labels and C^A is a set of constraints.

A combined unification $\sigma^{A_1 \cdots A_n}$ is generally defined as the combination of the axiom unifications for the axioms characterizing the logic

$$(i,j)\sigma^{A_1 \cdots A_n} \iff \begin{cases} (i,j)\sigma^{A_1} & C^{A_1} \\ \vdots & \vdots \\ (i,j)\sigma^{A_n} & C^{A_n} \end{cases}$$

Applying recursively the above $\sigma^{A_1 \cdots A_n}$ unification we obtain the logic unification σ_L.

$$(i,j)\sigma_L = \begin{cases} (i,j)\sigma^{A_1 \cdots A_n} \\ (c^n(i), c^m(j))\sigma^{A_1 \cdots A_n} \end{cases}$$

where $w_0 = (s^n(i), s^m(j))\sigma_L$.

We have presented elsewhere unifications for a wide class of modal logics (see [16, 1, 13]). By way of example, here we show how to build the logic unification for $S4$. First of all, we have to define the axiom unification for 4 that mimics transitivity:

$$(i,k)\sigma^4 = \begin{cases} c^{\ell(i)}(k) & \ell(k) > \ell(i), h(i) \in \Phi_V \text{ and} \\ & w_0 = (i, s^{\ell(i)}(k))\sigma \\ c^{\ell(k)}(i) & \ell(i) > \ell(k), h(k) \in \Phi_V \text{ and} \\ & w_0 = (s^{\ell(k)}(i), k)\sigma \end{cases} \qquad (\sigma^4)$$

Take for example the labels $i = (W_3, (w_2, w_1))$ and $k = (w_5, (w_4, (w_3, (W_2, w_1))))$. Here $s^{\ell(i)}(k) = (w_3, (W_2, w_1))$. Then i and k σ^4-unify to $(w_5, (w_4, (w_3, (w_2, w_1))))$ since $(i, s^{\ell(i)}(k))\sigma = ((W_3, (w_2, w_1)), (w_3, (W_2, w_1)))\sigma$. This intuitively means that all the worlds accessible from a sub-path $s^{\ell(i)}(k)$ of k are accessible from any path i which leads to the same world(s) denoted by $s^{\ell(i)}(k)$.

Well, given the following combined unification:

$$(i,k)\sigma^{DT4} = \begin{cases} (i,k)\sigma^D & \ell(i) = \ell(k) \\ (i,k)\sigma^T & \ell(i) < \ell(k), h(i) \in \Phi_C \\ (i,k)\sigma^4 & \ell(i) < \ell(k), h(i) \in \Phi_V \end{cases} \qquad (\sigma^{DT4})$$

such that

$$(i,k)\sigma^D = (i,k)\sigma \qquad (\sigma^D)$$

$$(i,k)\sigma^T = \begin{cases} (s^{\ell(k)}(i), k)\sigma & \ell(i) > \ell(k), \text{ and} \\ & \forall n \geq \ell(k), (h^n(i), h(k))\sigma = (h(i), h(k))\sigma \\ (i, s^{\ell(i)}(k))\sigma & \ell(k) > \ell(i), \text{ and} \\ & \forall n \geq \ell(i), (h(i), h^n(k))\sigma = (h(i), h(k))\sigma \end{cases} \qquad (\sigma^T)$$

the logic unification for $S4$ is recursively defined as follows:

$$(i,k)\sigma^{S4} = \begin{cases} (c^n(i), c^m(k))\sigma^{DT4} \\ (i,k)\sigma^{DT4} \end{cases} \qquad (\sigma_{S4})$$

As usual the meaning of an unification is that the denotations of the terms have non-null intersection. However, in some cases, the information encoded in the labels is not enough to determine whether two labels unify, and we need information from other labels. For example let us assume a non serial modal logic and the labels $i = (W_1, w_0)$, $j = (W_2, w_0)$, and $k = (w_1, w_0)$. According to the meaning of the labels, both i and j denote the set of world accessible from the actual world w_0, while k denotes a world accessible from it. Since our logic is not serial the set of world accessible from w_0 may be empty; however, this is not the case since the non-emptiness of such a set is granted by k. This is the reason for the next unification.[2]

Definition 5. *Let $\mathcal{L}$ be a set of labels. Then $(i, j)\sigma_L^{\mathcal{L}}$ iff*

1. $(i, j)\sigma_L$ *or*
2. $\exists k \in \mathcal{L}, \exists n, m \in \mathbb{N}$ *such that*
   - $(s^n(i), k)\sigma_L^{\mathcal{L}} = (s^m(j), k)\sigma_L^{\mathcal{L}}$ *and*
   - $(c^n(i), c^m(j))\sigma_L^{\mathcal{L}}$ *where* $w_0 = (s^n(i), k)\sigma_L^{\mathcal{L}}$

Traditionally formulas in sequents are evaluated as true if they occur in the antecedent, otherwise as false. When we move formulas from one side to the other we have to change their signs, but their contents are left unchanged. Since we use labelled formulas we have to move formulas as well as their labels. In section 2.3 we defined two translation functions from labels to modalities: each one is the opposite (dual) of the other. As we shall see, the first translation function is applied to labels occurring in the antecedent and the latter for labels in the consequent. So when a label moves from the antecedent to the consequent (or the other way around) it changes its sign; where the sign of a label is defined as follows:

Definition 6. *For any label i the* specular image *of i, denoted by $\bar{\imath}$ is defined as follows:*

$$\bar{\imath} = \begin{cases} i & i \in \Phi_A \\ \bar{\imath} & i \in \Phi_C \cup \Phi_V \\ (\overline{h(i)}, \overline{b(i)}) & \textit{otherwise} \end{cases}$$

Furthermore the specular image of a label i satisfies the following properties:

1. *If i is a label so is $\bar{\imath}$, similarly for $i \in \Phi_C$ and $i \in \Phi_V$;*
2. $\bar{\bar{\imath}} = i$;

[2] It is not the aim of this paper to give a computational characterization of label unifications. However, we believe that KEM label unification provides a good starting point. In fact it is not hard to show that the basic unification always terminates and is decidable in linear time, and so are the logic unifications for some of the standard modal logics. On the other hand, we argue that any other (string/label) unification would be suitable for the present approach if (1) it corresponds to the intended semantics, and (2) it modular in the sense that it is possible to split labels in several parts and compute independently their unifications.

3. $\phi^+(\bar{\imath}) = \phi^-(i)$,
$\phi^-(\bar{\imath}) = \phi^+(i)$;

We further assume that a constant and its specular image unify, and the result of their unification is the specular image if the result occurs in the consequent of a sequent, otherwise it is the label itself.

For specular images we can prove

Lemma 1. *1. For all* $i, j \in \Im$, $(\bar{\imath}, j)\sigma_L = \overline{(i, \bar{\jmath})\sigma_L}$;
2. for all $i, j \in \Im$, $(i, j)\sigma_L^{\mathcal{L}}$ *iff* $(\bar{\imath}, j)\sigma_L^{\mathcal{L}}$.

Proof. By case inspection.

4 Modal Sequents

A drawback of standard modal sequents is that they define consequence relations for set of modal formulas, but they do not provide a true notion of modal consequence. Moreover such a consequence is defined for an (the) actual world. Labels give a first partial relief to this problem insofar as they define a modal consequence with respect to an (the) actual world. However this solution is not general enough. Semantically we can jump from a world to another and set the latter as the current actual world, establish a modal consequence relation with respect to the world, using it to draw inferences, and then we can carry the information thus obtained to another world or back to the original actual world. Composing and decomposing labels corresponds to this mechanism, and KEM labels are very well-suited to this task (see [13, 14]). However, this is just the first step in order to define a general modal consequence relation: what we need is to introduce connectives/operators wherever in the formula, not only as main ones. In the next section we show how to achieve this result.

4.1 Inference Rules

The heart of $\mathscr{LMS}$ is constituted by the following sequent rules which are designed to work both as inference rules (to make deductions from both the declarative and the labelled part of wff formulas), and as ways of manipulating labels during proofs.

Axiom

$$A \vdash A$$

Negation

$$\frac{\Gamma, A : i \vdash \Delta}{\Gamma \vdash \neg A : \bar{\imath}, \Delta} \vdash \neg \qquad \frac{\Gamma \vdash A : i, \Delta}{\Gamma, \neg A : \bar{\imath} \vdash \Delta} \neg \vdash$$

Conjunction

$$\frac{\Gamma, A : i, B : j \vdash \Delta}{\Gamma, (A : c^n(i) \wedge B : c^n(j)) : s^n(i) \vdash \Delta} \wedge \vdash$$

where $s^n(i)$ is a segment shared by i and j.

$$\frac{\Gamma \vdash A : i, \Delta \qquad\qquad \Gamma' \vdash B : j, \Delta'}{\Gamma, \Gamma' \vdash (A : c^n(i) \wedge B : c^m(j)) : (s^n(i), s^m(j))\sigma_L^{\mathcal{L}}, \Delta, \Delta'} \vdash \wedge$$

Cut

$$\frac{\Gamma \vdash A : i, \Delta \qquad \Gamma', B : j \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}$$

if $(s^n(i), s^m(j))\sigma_L^{\mathcal{L}}$ and $\phi^-(c^n(i))A = \phi^+(c^m(j))B$. Moreover all the constants occurring in $c^n(i)$ and $c^m(j)$ do not occur in $\Gamma, \Gamma', \Delta, \Delta'$.

Contraction

$$\frac{\Gamma, A : i, B : j \vdash \Delta}{\Gamma, A : i \vdash \Delta} \qquad\qquad \frac{\Gamma \vdash A : i, B : j, \Delta}{\Gamma \vdash A : i, \Delta}$$

if $\ell(i) > \ell(j)$ and $\exists n$ such that $\phi(c^n(i))A = \phi(c^n(j))B$ and $s^n(i) = s^n(j)$.

Weakening

$$\frac{\Gamma \vdash A : i, \Delta}{\Gamma \vdash A : i, B : j, \Delta} \qquad\qquad \frac{\Gamma, A : i \vdash \Delta}{\Gamma, A : i, B : j \vdash \Delta}$$

Given a set of lwff Γ we shall use $\Gamma^{\mathcal{L}}$ to denote the set of labels occurring in Γ.

Modal rule

$$\frac{\Gamma, A_1 : j_1, \ldots, A_n : j_n \vdash B : j_0, \Delta}{\Gamma, A_1 : j_1 : i_1, \ldots, A_n : j_n : i_n \vdash B : j_0 : k_0, \Delta} RM$$

if

1. $j_0 : k_0 = (\hat{p}, \hat{q})\sigma_L^{\mathcal{L}}$, $0 \leq p, q \leq n$; or
2. $\forall \hat{p}, \hat{q}((\hat{p}, j_0 : k_0)\sigma_L^{\mathcal{L}}, (\hat{q}, j_0 : k_0)\sigma_L^{\mathcal{L}})\sigma_L^{\mathcal{L}}$

where $\mathcal{L} = \bigcup_{1 \leq p \leq n} j_p : i_p \cup \Gamma^{\mathcal{L}} \cup \Delta^{\mathcal{L}}$ and $\hat{p} = j_p : i_p$ or $\hat{p} = j_0 : k_0$.

The first condition allows us to introduce two unifiable labels in the antecedent and their unification in the consequent, while the second permits the introduction in the consequent of a label that unify uniformly with the relevant labels introduced in the antecedent. Thus, according to the basic unifications the following rules are respectively instances of RM with the first condition

$$\frac{A : w_0, B : w_0 \vdash A \wedge B : w_0}{A : (W_2, (w_1, w_0)), B : (w_2, (W_1, w_0)) \vdash A \wedge B : (w_2, (w_1, w_0))}$$

and with the second condition

$$\frac{A : w_0, B : w_0 \vdash A \wedge B : w_0}{A : (W_2, (w_1, w_0)), B : (w_2, (W_1, w_0)) \vdash A \wedge B : (W_4, (W_3, w_0))}$$

Notice that in both rules the labels unify according to the basic unification defined in (5).

4.2 Derived Rules

Introduction of disjunction and implication The rules

$$\frac{\Gamma, A : i \vdash \Delta \qquad\qquad \Gamma', B : j \vdash \Delta'}{\Gamma, \Gamma', (A : c^n(i) \vee B : c^m(j)) : (s^n(i), s^m(j))\sigma^{\mathcal{L}}_{L} \vdash \Delta, \Delta'} \vdash \vee$$

$$\frac{\Gamma \vdash A : i, B : j, \Delta}{\Gamma \vdash (A : c^n(i) \vee B : c^n(j)) : s^n(i), \Delta} \vee \vdash$$

where $s^n(i)$ is a segment shared by i and j; and

$$\frac{\Gamma, A : i \vdash B : j, \Delta}{\Gamma \vdash (A : c^n(\bar{\imath}) \rightarrow B : c^n(j)) : s^n(j), \Delta} \vdash\rightarrow$$

where $s^n(j)$ is a segment shared by i and j

$$\frac{\Gamma \vdash A : i, \Delta \qquad\qquad \Gamma', B : j \vdash \Delta'}{\Gamma, \Gamma', (A : c^n(\bar{\imath}) \rightarrow B : c^m(j)) : (s^n(\bar{\imath}), s^m(j))\sigma^{\mathcal{L}}_{L} \vdash \Delta, \Delta'} \rightarrow\vdash$$

are derived rules. Here we prove only $\vee \vdash$, the others are proofs are similar.

$$\cfrac{\cfrac{\cfrac{\cfrac{\Gamma, A : i \vdash \Delta}{\Gamma \vdash \neg A : \bar{\imath}, \Delta} \vdash \neg \qquad\qquad \cfrac{\Gamma', B : j \vdash \Delta'}{\Gamma' \vdash \neg B : \bar{\jmath}, \Delta'} \vdash \neg}{\Gamma, \Gamma' \vdash (\neg A : c^n(\bar{\imath}) \wedge \neg B : c^m(\bar{\jmath})) : (s^n(\bar{\imath}), s^m(\bar{\jmath}))\sigma^{\mathcal{L}}_{L}, \Delta, \Delta'} \vdash \wedge}{\Gamma, \Gamma', \neg(\neg A : c^n(\bar{\bar{\imath}}) \wedge \neg B : c^m(\bar{\bar{\jmath}})) : \overline{(s^n(\bar{\imath}), s^m(\bar{\jmath}))\sigma^{\mathcal{L}}_{L}} \vdash \Delta, \Delta'} \neg \vdash}{\Gamma, \Gamma', (A : c^n(i) \vee B : c^m(j)) : (s^n(i), s^m(j))\sigma^{\mathcal{L}}_{L} \vdash \Delta, \Delta'} \textit{RM} \text{ and cut}$$

Another bunch of derived rules is the set of the "semantic" version of the α-rules. For example the following rule

$$\frac{\Gamma, A : i, B : j \vdash \Delta}{\Gamma, (A : c^n(i) \wedge B : c^m(j)) : (s^n(i), s^m(j))\sigma^{\mathcal{L}}_{L} \vdash \Delta} \wedge_{\sigma^{\mathcal{L}}_{L}} \vdash$$

can be derived by using cut and modal rule. Similarly for the other α-rules, i.e., $\vdash\rightarrow_{\sigma^{\mathcal{L}}_{L}}$ and $\vdash \vee_{\sigma^{\mathcal{L}}_{L}}$.

Modus Ponens The modus ponens

$$\frac{\vdash A \to B \quad \vdash A}{\vdash B}$$

is just an instance of the generalized modal version

$$\frac{\Gamma, \vdash (A : i \to B : j) : k, \Delta \qquad \Gamma' \vdash C : l, \Delta'}{\Gamma, \Gamma' \vdash B : j : c^n(k) : (s^n(k), s^m(l))\sigma^{\mathfrak{L}}_L, \Delta, \Delta'}\ MP$$

where $\phi^+(i : c^n(k))A = \phi^+(c^m(l))C$ e $(s^n(k), s^m(l))\sigma^{\mathfrak{L}}_L$

On the contrary modus tollens can be derived without limitation only in its propositional version, whereas modal version requires some complex conditions. This is due to the directionality of modalities and negation.

Modus tollens

$$\frac{\vdash (A : i \to B : j) : k, \quad \vdash C : l}{\vdash \neg A : i : \bar{k}}$$

if $\exists \hat{k} \in \Im, \exists m, p \in \mathbb{N}$ such that $(k, \hat{k})\sigma^{\mathfrak{L}}_L = k'$ and $(s^m(j : \hat{k}), s^p(l))\sigma^{\mathfrak{L}}_L$, where $\phi^+(c^m(j : \bar{k}))B$, and $\phi^+(c^p(l))C$ are complementary.

Introduction of modalities The rules for introducing modalities

$$\frac{\Gamma, A : i \vdash \Delta}{\Gamma, \phi^+(c^n(i))A : s^n(i) \vdash \Delta}\phi\vdash \qquad \frac{\Gamma \vdash A : i, \Delta}{\Gamma \vdash \phi^-(c^n(i))A : s^n(i), \Delta}\vdash\phi$$

where the constants occurring in $\phi^+(c^n(i))$ and $\phi^-(c^n(i))$ do not occur elsewhere, are derived rules.

$$\frac{\dfrac{(1)\phi^-(c^n(i))A \vdash \phi^-(c^n(i))A}{(2)\phi^-(c^n(i))A : j \vdash \phi^-(c^n(i))A : s^n(i)} \quad (3)\Gamma \vdash A : i, \Delta}{(4)\Gamma \vdash \phi^-(c^n(i))A : s^n(i), \Delta}$$

The relevant step is step 2, which has obtained from 1 by an application of the modal rule. Notice that we introduce on the antecedent a label j that unifies with $s^n(i)$. At this point we can apply the cut rule to obtain the desired result.

Elimination of modalities The rules for eliminating the modalities

$$\frac{\Gamma \vdash \mathfrak{m}A : i, \Delta}{\Gamma \vdash (\mathfrak{n}A : j) : k : i, \Delta} \qquad \frac{\Gamma, \mathfrak{m}A : i \vdash \Delta}{(\Gamma, \mathfrak{n}A : j) : k : i \vdash \Delta}$$

where $\mathfrak{m}A = \phi(k)\mathfrak{n}\phi(j)A$ and the constants occurring in j and k do not occur elsewhere, are derived rules;

$$\cfrac{\Gamma \vdash \mathfrak{m}A : i \quad \cfrac{\cfrac{\cfrac{\cfrac{A \vdash A}{A : j^* \vdash A : j}\text{ Modal Rule}}{\mathfrak{n}A : j^* \vdash \mathfrak{n}A : j}\text{ Introduction of }\mathfrak{n}}{(\mathfrak{n}A : j^*) : k^* \vdash (\mathfrak{n}A : j) : k}\text{ Modal Rule}}{(\mathfrak{n}A : j^*) : k^* : i^* \vdash (\mathfrak{n}A : j) : k : i}\text{ Modal Rule}}{\Gamma \vdash (\mathfrak{n}A : j) : k : i}\text{ cut}$$

It is worth noting that these rules allow a general treatment of modalities. In particular, given a formula such as $\Diamond\Box\Diamond A : i$ in the consequent of a sequent, we are able to translate each modality even if it is not the main (most external) operator of the formula. Suppose we want to translate just $\Box$. In this case the elimination of this operator produces the following results

$$\Diamond((\Diamond A) : (w_2, w_0)) : i$$

Necessitation The necessitation rule

$$\frac{\vdash A}{\vdash \Box A}$$

is also a derived rule. In fact it can be derived as follows:

$$\dfrac{\dfrac{\vdash A : i}{\vdash A : i : (w_1, w_0)}\text{ Modal Rule}}{\vdash \Box(A : i)}\vdash\Box$$

Notice that we have applied the modal rule only with respect to the most external label (w_1, w_0), for which $((w_1, w_0), (w_1, w_0))\sigma_L^{\mathcal{L}} = (w_1, w_0)$ holds; namely it satisfies the first condition for the applicability of the modal rule. We show now that the application of the modal rule only with respect to the most external label is safe.

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdash A : i}{\vdash \phi^-(i)A}\vdash\phi^-(i)}{\vdash \phi^-(i)A : (w_1, w_0)}\text{ Modal Rule}}{\vdash \Box\phi^-(i)A}\vdash\Box}{\vdash \Box(A : i)}\text{ Elimination of }\phi^-(i)$$

5 Examples

In the previous section we have shown how $\mathscr{LMS}$ works by proving some derived rules. In addition, here we provide a couple of proofs which use the unification for $S4$ presented in section 3.

$$\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{A : (w_1, (W_1, w_0)) \vdash A : (W_2, (w_2, (w_1, (W_1, w_0))))}\text{ Modal Rule}}{A : (w_1, (W_1, w_0)) \vdash \Box\Diamond A : (w_1, (W_1, w_0))}\vdash\phi}{\vdash (A \to \Box\Diamond A) : (w_1, (W_1, w_0))}\vdash\to}{\vdash \Diamond\Box(A \to \Box\Diamond A)}\vdash\phi$$

We apply the Modal Rule to the axiom $A \vdash A$ introducing the labels $i = (w_1, (W_1, w_0))$ and $j = (W_2, (w_2, (w_1, (W_1, w_0))))$. The labels i and j $\sigma_{S4}^{\mathcal{L}}$-unify in so far as $c^2(i) = (w_1, w_0')$ and $c^4(j) = (W_2, w_0')\ \sigma^D$, once w_0' has been identified with the $\sigma_{S4}^{\mathcal{L}}$-unification of $s^2(i) = (W_1, w_0)$ and $s^4(j) = (w_2, (w_1, (W_1, w_0)))$; it is immediate to see that $s^2(i)$ and $s^4(j)$ $\sigma^{\mathcal{L}}_{S4}$-unify, since they σ^4- and therefore σ^{DT4}-unify.

We translate part of the label of the formula occurring on the right in a modality as follows: $\phi^-((W_2,(w_3,w'_0))A) = \Box\Diamond A$.

At this point we introduce $\rightarrow$ on the right using i as the common shared label. To finish the proof we have to transform the label into a modality; namely: $\phi^-((w_1,(W_1,w_0))B) = \Diamond\Box B$.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{A:(W_1,w_0) \vdash A:(w_1,w_0)}\text{ Modal Rule}}{A:(W_1,w_0) \vdash \Box A}\vdash\phi \qquad C \vdash C}{A:(W_1,w_0), \Box A \rightarrow C \vdash C}\rightarrow\vdash}{A:(W_1,w_0):(w_2,w'_0), (\Box A \rightarrow C):(W_2,w'_0) \vdash C:(w_3,(w_2,w'_0))}\text{ Modal Rule}}{\Box A:(w_2,w'_0), \Box(\Box A \rightarrow C) \vdash \Box C:(w_2,w'_0)}\phi\vdash\phi}{\Box(\Box A \rightarrow C) \vdash (\Box A \rightarrow \Box C):(w_2,w'_0)}\vdash\rightarrow}{\Box(\Box A \rightarrow C) \vdash \Box(\Box A \rightarrow \Box C)}\vdash\phi}{\vdash \Box(\Box A \rightarrow C) \rightarrow \Box(\Box A \rightarrow \Box C)}\vdash\rightarrow
$$

The only step of the above proof deserving a clarification is the second application of the modal rule. Here we introduce three labels: $i = (W_1,w_0):(w_2,w'_0)$, which corresponds to $(W_1,(w_2,w'_0))$; $j = (W_2,w'_0)$; and $k = (w_3,(w_2,w_1))$. It is immediate to see that $(i,k)\sigma^{\mathcal{L}}_L = k$, $(j,k)\sigma^{\mathcal{L}}_L = k$, and $(k,k)\sigma^{\mathcal{L}}_L = k$; therefore the labels satisfy the second condition of the modal rule.

It is worth noting that we have to defer the translation of $A:(W_1,w_0)$, until the second application of the modal rule; otherwise it would not be possible to apply such a rule with the appropriate labels.

Notice also that it is possible to generalise the rules for introducing modalities: every label attached to a sub-formula can be transform in a modality, and not only the label of a formula. In this way, we can postpone the translation of labels until the last steps of proofs.

6 Soundness and Completeness

In this section we briefly sketch how to prove soundness and completeness results for $\mathscr{LMS}$. Basically, we have to show that (1) the rules and the axioms corresponding to a given Hilbert system L for modal logic are respectively derived rules and theorems in $\mathscr{LMS}$, and (2) the rules of $\mathscr{LMS}$ are sound with respect to the semantic conditions for L. In what follows we assume that the Hilbert system L is complete with respect to the appropriate Kripke models.

Theorem 1. *If* $\vdash_L A$ *then* $\vdash_{\mathscr{LMS}} A$.

Proof. In Section 4.2 we have already seen how to prove modus ponens and necessitation. Modal axioms are derivable as follows:

$$\dfrac{\dfrac{\dfrac{A \vdash A}{A : i \vdash A : j}\ \text{Modal rule}}{\vdash A : \bar{\imath} \to A : j}\ \vdash\to}{\vdash \phi^{-}(\bar{\imath})A \to \phi^{-}(j)A}\ \vdash \phi$$

where $(i,j)\sigma_L^{\mathcal{L}}$. This proof relies on the fact that each σ^A-unification corresponds to a generalization including necessitation and self recursion of the modal axiom A, and the various $\sigma_L^{\mathcal{L}}$ are built upon the σ^A of the axioms characterizing the logic L (see [16, 1]).

Let us first define some functions which map labels into elements of Kripke models. Given a model $\mathcal{M} = \langle W, R, v \rangle$, such functions translate labels into elements of $\mathcal{M}$ according to the structure of the labels.

Let g be a function from $\Im$ to $\wp(\mathcal{W})$ thus defined:

$$g(i) = \begin{cases} h(i) = \{h(i)\} & \text{if } h(i) \in \Phi_C \\ h(i) = \{w_i \in \mathcal{W} : g(b(i))Rw_i\} & \text{if } h(i) \in \Phi_V \end{cases}$$

The above function is not defined for composed labels, i.e., labels of the form $i : j$. However it can be extended to them by stipulating that $g(s^1(i)) = g(h(j))$; see [14] for a full account of the combination of labels.

Let r be a function from $\Im$ to R thus defined:

$$r(i) = \begin{cases} \emptyset & \text{if } l(i) = 1 \\ g(i^1)Rg(i^2), \ldots, g(i^{n-1})Rg(h(i)) & \text{if } l(i) = n > 1 \end{cases}$$

Let f be a function from lwff's to v thus defined:

$$f(A : i) =_{def} v(A, w_j) = T$$

for all $w_j \in g(i)$.

As second step, we need the following lemma.

Lemma 2. *For any $i, k \in \Im$ if $(i,k)\sigma_L$ then $g(i) \cap g(k) \neq \emptyset$.*

Proof. See [1, 16]

This lemma shows that if two labels unify, then the result of their σ^L-unification corresponds to an element of the appropriate model. In this way, we are able to build the Kripke model for the labels involved in a $\mathscr{LMS}$ proof, and so we can check every rule of $\mathscr{LMS}$ in a standard semantic setting:

Theorem 2. $\vdash_{\mathscr{LMS}(L)} A \Rightarrow \models_L A$.

Theorem 3. $\models_L A \iff \vdash_L A$.

From theorems 1, 2, and 3 we obtain:

Theorem 4. $\vdash_{\mathscr{LMS}} A \iff \models_L A$.

7 Discussion and Future Work

In this paper we have just presented a new sequent system for modal logic. The main interest in such a system is that it can provide a general definition of the notion of modal consequence relation. Accordingly, we propose a more "liberal" modal language which allows to draw inferences with respect to a given world w_i and then move to another world w_j where new inferences can be drawn taking into account the semantical conditions corresponding to the previous inferential steps. Furthermore, world-paths can be composed or decomposed so that new paths are manipulated through unification algorithms.

A possible objection to the above claim is that even $\mathscr{LMS}$ reduces modal inferences to inferences for classical connectives with respect to comparable labels (sets of worlds) involved in the process. As a matter of fact, this argument holds to some extent for all (labelled) sequent systems for modal logic. However, our system allows to do something more. $\mathscr{LMS}$ can handle modal operators wherever in a given formula. For example, under appropriate conditions sub-formulas which do not depend directly on the main connective can be involved during the proofs. A consequence of this fact is that we can provide new and more general definitions of substitution for modal equivalents and of distributivity of modal operators with respect to the boolean ones.

This is not a trivial result. On the contrary it seems to be a key feature for representing a general notion of modal deducibility. In fact, an $\mathscr{LMS}$ proof can be structured in such a way that different formulas can be compared in the same time even if they hold in distinct worlds and are not immediate sub-formulas. We believe this is a good starting point to devise a modal inference not just as a classical inference with respect to a particular world. On the other side, to achieve fully this goal it is necessary to prove soundness and completeness for Hilbert-style modal consequence and not only for theoremhood. This is a matter of future work.

Furthermore, we think that a more extensive investigation on $\mathscr{LMS}$ could point out some meta-theoric properties.

At first sight Došen's principle (cf. [26]) seems to be violated since the rules for logical operators involve more than the immediate sub-formulas where logical forms and structures (labels) are mixed. However, the format of the rules is just the modal generalization of standard ones, and the case of a "pure" operator is nothing else than a special case of the rule. Moreover the rules and the set of rules are unchanged in passing from a logic to another. All modifications are only over the unification procedure.

Secondly, thanks to the above features our method can easily cover other intensional logics such as conditional logics. In this case, since $>$ can be regarded as a necessity operator relative to its antecedent, KEM label language is extended by formula-indexed labels so that new specific unifications can be defined (see [2, 3]). Thus a similar definition of substitution for modal equivalents could offer an elegant and powerful method for composing proofs in a sequent setting and for establishing when two formulas are equivalent with respect to $>$.

A final point can be remarked as a matter for future works. It is well-known that for every tableaux proof for a formula A it is possible to build a corresponding (reverse) sequent proof for it. The label formalism we have presented was originally designed for a tableau-like system for modal logics called KEM (see [1, 16]), where the cut can be restricted to an analytic version; moreover KEM can be extended with the modal generalization of the rules we have proposed for $\mathscr{LMS}$, so it is a suitable tool for such a transformation. Finally our proof method enjoys an interesting property: since the order in which modal principles are applied in the proof is stored in the unifications, it is not hard to reconstruct a Hilbert style proof for A from the order of unifications (see [24, 19]). This is important because in this way we can produce constructive proofs without references to non-constructive (semantic) methods or to external resources such as labels or other devices.

8 Related Works

Although it is not the aim of the present work to compare $\mathscr{LMS}$ with other proof systems for modal logic, in this section we expose briefly the main differences. Several proposal have been put forth to find a general framework for modal deduction. As a consequence a plethora of formalisms and proof systems have been developed (for more exhaustive overviews and trends in this field see [15, 25]); however, a common feature can be identified: adding "structures" to deductive systems. These structures are meant to capture – in a proof-theoretic environment and in a more appropriate way – the intensional nature of modal logic. Mainly two strategies has been devised for this non easy task: explicit vs. implicit structures.

It is well-known that Gentzen sequent calculi are the archetype of proof systems with structures; so, methods adopting the first strategy add more structures to standard classical sequents; on the other hand labels or indices are used to represent semantic structures in the language and in the calculi.

We refer, in particular, to hypersequents [4], multidimensional sequents [10, 8, 20], and display modal logic [26, 18]. All these approaches define the notion of modal derivability in terms of a relativized classical consequence relation and are mainly concerned with the eliminability of cut. In our opinion they fail to provide a genuine and comprehensive modal consequence relation for the following two reasons: 1) the modal and boolean components are kept separate; 2) they do not perform modal operations across structures, i.e., modal operations are defined over the same main (more external) structure.

Here a more fine grained division is needed; so we classify labelled systems for modal logic in two classes: 1) semantic based translation methods, and 2) label propagation methods.

In the first class it is worth mentioning, among others, the works by Ohlbach [22], Russo [23], Basin, Matthews and Viganò [6]. The principal feature of these systems is that they translate the modal formula into first-order expression, or use a first-order condition as a parameter for inference rule involving modal operators.

It is clear that the first-order translation methods cannot be used in relation to logics whose semantics is not first-order based. For example they cannot deal with modal logics that can be represented in terms of neighbourhood models such as classical, regular, monotonic modal logics. However, in [17] KEM has been used for such classes of logics with minimal (modular) modifications on the substitution function. Another drawback is that translation methods can be used as effective decision methods only in case of definite subsets of first-order logic, which in general is semi-decidable. So, in general, they are not able to define actual inference rules for modalities.[3]

In the second class we mention the works by Fitting [11], Massacci [21], and Baldoni, Giordano and Martelli [5]. Basically, in this kind of system it is not possible to define uniform procedures independent from the order of application of the rules, therefore for each procedure pathological formulas can be found (here with pathological formulas we mean formulas requiring a great deal of useless information to the essential proof) (cf. [9]). In this category we include also the work by Beckert and Goré [7]. Here, similarly to what we have done, the label propagation is presented by means of variables, but, on the contrary, the rules governing the expansion of variables are given in term of first-order conditions; so, they overcome the drawback of uniform procedures but they suffer from the problem of the translation methods.

KEM label formalism, in general, is free from the above shortcomings. On the positive side KEM label formalism is strictly connected with Gabbay's fibring methodology; therefore, the same basic strategy can be used on various kinds of combinations of modal logics, given a KEM label algebra for the component logics (on this points see [14]).

Acknowledgments

We would like to thank Rajeev Goré and the anonymous referees of M4M and Tableaux 2000 for their useful comments and suggestions on a previous version of this paper. This research was partially supported by the Australia Research Council under Large Grant No. A49803544.

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[3] An inference rule is a relation R^n over the set of admissible expressions such that is decidable whether $f_1, \ldots, f_n \in R^n$.

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