Simplified Power Allocation and TX/RX Structure for MIMO-DSL

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***Abstract*— In this paper we investigate power allocation and crosstalk cancellation for MIMO-DSL (a.k.a. bonded-DSL) systems. Exploiting a property of the DSL channel, namely column-wise diagonal dominance, allows us to simplify the power allocation and crosstalk cancellation problems. This leads to a significant reduction in initialisation and run-time complexity whilst maintaining near-optimal performance. We develop a bound which relates the deviation of the simplified power allocation to the degree of column-wise diagonal dominance. Reliable transmission at near-capacity data-rates is also demonstrated through simulation.**

I. Introduction

Next generation DSL systems such as VDSL aim at providing extremely high data-rates, up to 52 Mbps in the downstream to the mass-consumer market. In VDSL such high data-rates are supported by operating over short loop lengths and transmitting in frequencies up to 12 MHz. Unfortunately, the use of such high frequency ranges causes significant electromagnetic coupling between neighbouring twisted-pairs within a binder group. This coupling creates interference, referred to as crosstalk, between the systems operating within a binder. Over short loop lengths crosstalk is typically 10-15 dB larger than the background noise and is *the* dominant source of performance degradation.

In MIMO-DSL (a.k.a. bonded-DSL), several lines are multiplexed to provide a service with even higher data-rate and reach than conventional DSL. MIMO-DSL is seen as a competitor to low end fiber-to-the-business services as well as cheaper alternative to fiber for Central Office (CO) to Remote Terminal (RT) links. An advantage of MIMO-DSL over conventional DSL is that both the transmitters (TX) and receivers (RX) of a service are co-located which allows transmission and reception to be co-ordinated on a signal level. This facilitates both Far-end Crosstalk (FEXT) and Near-end Crosstalk (NEXT) cancellation. FEXT cancellation has been shown to yield significant performance gains, sometimes doubling or even tripling achievable rates[1]. In addition, NEXT cancellation allows full duplex transmission to be employed which also effectively doubles the achievable data-rate. Due to the cyclic structure of Discrete Multi-Tone (DMT) transmission blocks NEXT cancellation can be done independently on each tone. This allows it to be implemented with reasonable complexity[2].

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Using such high frequencies in transmission also causes the DSL channel to exhibit severe frequency selectivity. Exploiting this property through adaptive power allocation schemes such as Shannon's classic waterfilling algorithm results in significant performance gains.

In this paper we investigate the power allocation problem and optimal TX/RX structures for the MIMO-DSL channel. Exploiting certain properties of the DSL channel allows us to significantly reduce the power allocation and TX/RX complexity whilst still operating close to capacity.

II. Channel Model

Through synchronized transmission and the cyclic structure of DMT blocks we model transmission independently on each tone[1]

$$\mathbf{y}_k = \mathbf{H}_k \mathbf{x}_k + \mathbf{z}_k$$

$\mathbf{x}_k \triangleq \left[x_k^1, \cdots, x_k^N\right]$ is the vector of transmitted signals on tone k. There are N lines in the bonded system and x_k^n is the signal transmitted onto line n at tone k. $\mathbf{y}_k$ and $\mathbf{z}_k$ have similar structures. $\mathbf{y}_k$ is the vector of received signals on tone k. $\mathbf{z}_k$ is the vector of additive noise on tone k and contains thermal noise, alien crosstalk, RFI etc. We assume $\mathcal{E}\left\{\mathbf{z}_k \mathbf{z}_k^H\right\} = \sigma_k^2 \mathbf{I}_N$. This is without loss of generality since a noise-whitening pre-filter can be applied at the RXs.

$\mathbf{H}_k$ is the $N \times N$ channel transfer matrix on tone k. $h_k^{n,m} \triangleq [\mathbf{H}_k]_{n,m}$ is the channel from TX m to RX n on tone k. The diagonal elements of $\mathbf{H}_k$ contain the direct-channels whilst the off-diagonal elements contain the crosstalk channels. We define the transmit correlation matrix on tone k $\mathbf{S}_k \triangleq \mathcal{E}\left\{\mathbf{x}_k \mathbf{x}_k^H\right\}$ and its elements $s_k^{n,m} \triangleq [\mathbf{S}_k]_{n,m}$

Wireline channels with co-located RXs exhibit column-wise diagonal dominance[1]. This ensures

$$h_k^{m,m} \gg h_k^{n,m}, \forall n \neq m \tag{1}$$

We measure the degree of diagonal dominance using

$$\alpha \triangleq \max_{n,m\, n\neq m} \arctan \frac{|h_k^{n,m}|}{|h_k^{m,m}|} \tag{2}$$

where small α corresponds to a strongly column-wise diagonal dominant channel.

In this paper we exploit this property to simplify both the power allocation problem and TX/RX structure. This leads to a significant reduction in both initialization and run-time complexity for the DSL modem. In related work [1] this property was used to demonstrate the near-optimality of the Zero Forcing Decision Feedback Equalizer (ZF-DFE) for crosstalk cancellation in DSL. Here we show that the decision feedback operation is not required and a simple linear design is sufficient to achieve a bitrate close to capacity. In addition, we give a bound on the deviation of the simplified power allocation (6) from the truly optimal solution (3).

The techniques are also applicable to MIMO-CDMA where considering the processing gain and mitigating the fast-fading effect ensures that the interference gain is typically 15-20 dB smaller than the main path gain.

III. Optimal Power Allocation for MIMO

A. Power Constraint for All Transmitters

Typical power allocation algorithms focus on maximizing data-rate with a constraint P on the total power of all TXs

$$\max_{\{\mathbf{S}_k\}_{k=1\dots K}} C \quad \text{s.t.} \quad s_k^{n,n} \geq 0,\ \forall n,k$$
$$\sum_n \sum_k s_k^{n,n} \leq P$$

where $C \triangleq \sum_k \log \left|\mathbf{I} + \sigma_k^{-2}\mathbf{H}_k\mathbf{S}_k\mathbf{H}_k^H\right|$. This constraint is well suited to wireless applications and is motivated by considering the limitations on the analog front-end (AFE) which drives the multi-element antenna. Using the SVD $\mathbf{H}_k \stackrel{\text{svd}}{=} \mathbf{U}_k\Lambda_k\mathbf{V}_k^H$. The resulting optimal power allocation consists of pre-coding with $\mathbf{V}_k$ and waterfilling against the singular values of the channel

$$\mathbf{S}_k = \mathbf{V}_k \left[\frac{1}{\lambda}\mathbf{I}_N - \sigma_k^2\Lambda_k^{-2}\right]^+ \mathbf{V}_k^H$$

where $[x]^+ \triangleq \max(0,x)$. $\frac{1}{\lambda}$ is the waterfilling level and is chosen such that the TX power constraint is met with equality.

B. Power Constraint per Transmitter

In DSL the power constraint P actually applies to each TX

$$\max_{\{\mathbf{S}_k\}_{k=1\dots K}} C \quad \text{s.t.} \quad s_k^{n,n} \geq 0,\ \forall n,k$$
$$\sum_k s_k^{n,n} \leq P,\ \forall n$$

This arises out of a limitation on the AFE of each modem and regulatory constraints. Formulating a Lagrangian and examining the K.K.T. conditions yields[3]

$$s_{k,\text{opt}}^{n,m} = \begin{cases} \left[\frac{1}{\lambda_n} - \sigma_k^2 t_k^{n,n}\right]^+ & n = m \\ -\sigma_k^2 t_k^{n,m} & n \neq m \end{cases} \tag{3}$$

where $\mathbf{T}_k \triangleq \left(\mathbf{H}_k^H\mathbf{H}_k\right)^{-1}$ and $t_k^{n,m} \triangleq [\mathbf{T}_k]_{n,m}$. The optimal TX correlation matrix is denoted $\mathbf{S}_{k,\text{opt}}$ where $s_{k,\text{opt}}^{n,m} = [\mathbf{S}_{k,\text{opt}}]_{n,m}$. Using (3) the off-diagonal elements of $\mathbf{S}_{k,\text{opt}}$ are known in closed form. The diagonal elements can be found using a conventional waterfilling algorithm against the channel $[(t_1^{n,n})^{-1/2} \cdots (t_K^{n,n})^{-1/2}]$. $\frac{1}{\lambda_n}$ is the waterfilling level for TX n and is chosen such that its total power constraint is met with equality.

Spectral masks can be included in the constraints to protect legacy systems like ISDN and HDSL. Their application is quite straightforward[4].

IV. Simplified Power Allocation for MIMO-DSL

Using column-wise diagonal dominance (1) we write the novel bounds

$$\frac{1}{|h_k^{n,n}|^2}[1 - f_3(\alpha)] \leq t_k^{n,n} \leq \frac{1}{|h_k^{n,n}|^2}[1 + f_2(\alpha)] \tag{4}$$

$$|t_k^{n,m}| \leq \frac{1}{|h_k^{n,n}|}\frac{1}{|h_k^{m,m}|}f_4(\alpha),\ \forall n \neq m \tag{5}$$

where $f_2(\alpha)$, $f_3(\alpha)$ and $f_4(\alpha)$ go rapidly to 0 as $\alpha \to 0$.

Proof: See Appendix. ■

In DSL typically $\alpha \leq 0.01$ (see e.g. the crosstalk channel measurements in [5]). With $\alpha = 0.01$ and $N = 8$: $0.9993\,|h_k^{n,n}|^{-2} \leq t_k^{n,n} \leq 1.0099\,|h_k^{n,n}|^{-2}$ and $|t_k^{n,m}| \leq 0.0522\,|h_k^{n,n}|^{-1}\,|h_k^{m,m}|^{-1}$, $\forall n \neq m$.

We can now approximate the optimal transmit correlation matrix with

$$s_{k,\text{simp}}^{n,m} = \begin{cases} \left[\frac{1}{\lambda_n} - \frac{\sigma_k^2}{|h_k^{n,n}|^2}\right]^+ & n = m \\ 0 & n \neq m \end{cases} \tag{6}$$

The approximate (simplified) TX correlation matrix is denoted $\mathbf{S}_{k,\text{simp}}$ where $s_{k,\text{simp}}^{n,m} = [\mathbf{S}_{k,\text{simp}}]_{n,m}$. Note that (6) allows us to allocate power with reduced complexity. Each transmitter simply waterfills against their direct channel as if crosstalk were not present. This is possible since column-wise diagonal dominance assures us of a good basis for crosstalk cancellation at the RX. As a result we can avoid the matrix inversion necessary in the calculation of $\mathbf{T}_k$.

Using (4) and (5) we can bound the error between the truly optimal power allocation and our simplified one

$$\left|s_{k,\text{opt}}^{n,m} - s_{k,\text{simp}}^{n,m}\right| \leq \begin{cases} \frac{\sigma_k^2}{|h_k^{n,n}|^2}\max(f_2(\alpha), f_3(\alpha)) & n = m \\ \frac{1}{|h_k^{n,n}||h_k^{m,m}|}f_4(\alpha) & n \neq m \end{cases}$$

where the bound goes rapidly to 0 as $\alpha \to 0$. This bound assures us that any capacity loss arising from the simplification in (6) will be minimal. This is later confirmed through simulation in Sec. VII.

Since the off-diagonal elements of $\mathbf{S}_{k,\text{simp}}$ are zero, TX-side co-ordination is not required to operate at channel capacity provided that a maximum-likelihood receiver is used. Furthermore as we will show in Sec. VI, in MIMO channels which exhibit column-wise diagonal dominance (1), reliable transmission at data-rates close to capacity can be achieved with a simple linear RX structure and no TX co-ordination.

V. Optimal Tx/Rx Structure for MIMO

In a MIMO system with TX and RX co-ordination, transmission at channel capacity for a given $\mathbf{S}_k$ can be easily achieved using simple linear pre and post-processing and a standard decision device (slicer)[6]. Using the Cholesky decomposition $\mathbf{S}_{k,\text{opt}} \stackrel{\text{chol}}{=} \mathbf{B}_k^H\mathbf{B}_k$. Define $\overline{\mathbf{H}}_k \triangleq \mathbf{H}_k\mathbf{B}_k^H$ and using the SVD $\overline{\mathbf{H}}_k \stackrel{svd}{=} \overline{\mathbf{U}}_k\overline{\Lambda}_k\overline{\mathbf{V}}_k^H$ where $\overline{\Lambda}_k \triangleq \text{diag}\{\overline{\rho}_1, \dots, \overline{\rho}_N\}$. We assume that $\overline{\mathbf{H}}_k$ is full rank which is virtually always the case in practice. The QAM-encoder generates the set of symbols $\overline{\mathbf{x}}_k$ with normalized correlation matrix $\mathcal{E}\left\{\overline{\mathbf{x}}_k\overline{\mathbf{x}}_k^H\right\} = \mathbf{I}_N$. These are pre-filtered to generate the transmitted signals

$$\mathbf{x}_k = \mathbf{P}_k\overline{\mathbf{x}}_k$$

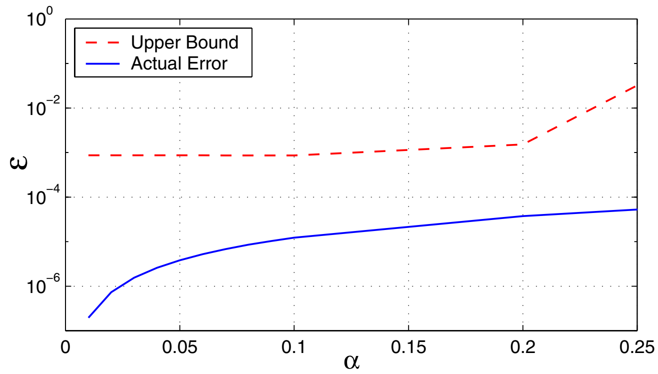

Fig. 1. Relative Error in $\mathbf{S}_{k,\text{simp}}$ vs. α

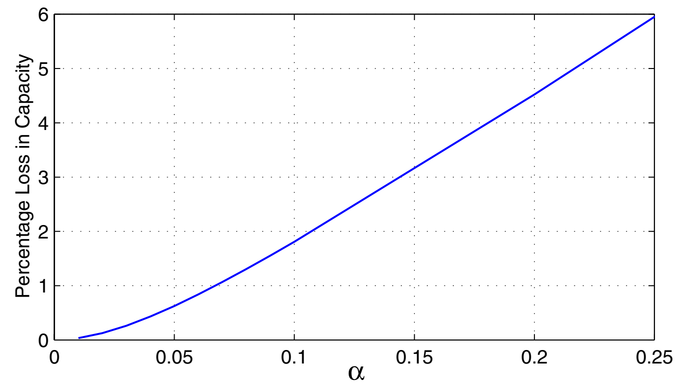

Fig. 2. Percentage Loss in Capacity vs. α

where the pre-filtering matrix $\mathbf{P}_k \triangleq \mathbf{B}_k^H \overline{\mathbf{V}}_k$. Application of $\mathbf{P}_k$ ensures that $\mathcal{E}\left\{\mathbf{x}_k \mathbf{x}_k^H\right\} = \mathbf{S}_{k,\text{opt}}$ which is indeed the optimal transmit power and correlation for tone k.

At the RX we apply the filter $\mathbf{W}_k \triangleq \overline{\Lambda}_k^{-1} \overline{\mathbf{U}}_k^H$ to estimate the transmitted symbols

$$\begin{aligned}\widehat{\mathbf{x}}_k &= \mathbf{W}_k \mathbf{y}_k \\ &= \overline{\mathbf{x}}_k + \overline{\Lambda}_k^{-1} \overline{\mathbf{U}}_k^H \mathbf{z}_k\end{aligned}$$

The SNR on line n is thus $\sigma_k^{-2} \overline{\rho}_n^2$ and its capacity $c_n = \log_2\left(1 + \sigma_k^{-2} \overline{\rho}_n^2\right)$. It is straightforward to show

$$\sum_n c_n = \log_2 \left| \mathbf{I}_N + \sigma_k^{-2} \mathbf{H}_k \mathbf{S}_{k,\text{opt}} \mathbf{H}_k^H \right|$$

which is indeed the theoretical capacity of the channel. Hence reliable transmission at near channel capacity can be achieved with simple linear pre and post-processing and a standard slicer.

VI. Simplified Tx/Rx Structure for MIMO-DSL

Using the QR decomposition $\mathbf{H}_k \overset{\text{qr}}{=} \mathbf{Q}_k \mathbf{R}_k$. Through (8) (see appendix) it is possible to make the approximation $\mathbf{R}_k \simeq \text{diag}\left\{\mathbf{R}_k\right\}$. Furthermore $\mathbf{S}_{k,\text{simp}}$ is diagonal under simplified power allocation (6) hence

$$\mathbf{B}_k = \text{diag}\{\sqrt{s_{k,\text{simp}}^{1,1}}, \ldots, \sqrt{s_{k,\text{simp}}^{N,N}}\}$$

Thus

$$\begin{aligned}\overline{\mathbf{H}}_k &= \mathbf{H}_k \mathbf{B}_k^H \\ &\simeq \mathbf{Q}_k \text{diag}\left\{\mathbf{R}_k\right\} \mathbf{B}_k^H\end{aligned}$$

Since $\mathbf{Q}_k$ is orthogonal and $\mathbf{B}_k$ is diagonal we can approximate $\overline{\mathbf{U}}_k \simeq \mathbf{Q}_k$, $\overline{\Lambda}_k \simeq \text{diag}\left\{\mathbf{R}_k\right\} \mathbf{B}_k^H$ and $\overline{\mathbf{V}}_k \simeq \mathbf{I}_N$. Therefore the *pre-filtering matrix*

$$\mathbf{P}_k \simeq \mathbf{B}_k^H$$

Since $\mathbf{B}_k$ is diagonal pre-filtering only modifies the transmit powers of the different modems. Signal level co-ordination is not required between the TXs to eliminate crosstalk. At the RX

$$\begin{aligned}\mathbf{W}_k &\simeq \mathbf{B}_k^{-H} \text{diag}\left\{\mathbf{R}_k\right\}^{-1} \mathbf{Q}_k^H \\ &\simeq \mathbf{B}_k^{-H} \mathbf{R}_k^{-1} \mathbf{Q}_k^H \\ \mathbf{W}_k &\simeq \mathbf{B}_k^{-H} \mathbf{H}_k^{-1}\end{aligned}$$

which is simply a *linear ZF crosstalk canceller* with compensation for the power allocation. The important observation here is that the linear ZF design is a good approximation of the optimal (SVD based) TX/RX scheme; hence it allows us to transmit reliably at near-capacity data-rates.

An important observation here is that since TX side co-ordination is not required, all the results in this paper apply to systems with RX-only co-ordination as well. RX-only co-ordination is available in the upstream direction of conventional DSL systems where transmitting modems are located in different customer premises, but receiving modems are co-located at the CO or at a RT. Note that column-wise diagonal dominance of the channel matrix is a characteristic of DSL systems with co-located RXs. The TXs do *not* have to be co-located. Thus the results of this paper find much broader application than in bonded systems (ie. co-located TXs and RXs) alone. Furthermore, we see that conventional DSL systems with appropriate RX side co-ordination can achieve similar capacity to fully bonded systems where co-ordination is available between both TXs and RXs.

VII. Performance

As we saw in Sec. IV and VI, reliable transmission at near-capacity data-rates can be achieved using a simplified power allocation algorithm and TX/RX structure which results in decreased initialisation and run-time complexity.

We now evaluate the performance of these simplifications against the optimal power allocation and TX/RX structure of Sec. III and V.

We simulate a bonded system consisting of 8 VDSL lines. The simulation environment uses 4096 tones, ETSI alien noise model A, a coding gain of 3 dB, noise margin of 6 dB and total power constraint 11.5 dBmW on each modem. The target error probability is $< 10^{-7}$ and all lines are 0.5mm (24-Gauge). Semi-empirical transfer functions and FDD bandplan 998 are used. For further details on the VDSL parameters and channel model see [7].

We begin by examining the effect of α on the accuracy of the simplified waterfilling approximation. The relative error in the simplified waterfilling approximation is defined

$$\epsilon \triangleq \left\|\mathbf{S}_{k,\text{opt}} - \mathbf{S}_{k,\text{simp}}\right\|_F / \left\|\mathbf{S}_{k,\text{opt}}\right\|_F$$

where $\|.\|_F$ denotes the Frobenius norm. We artificially set $h_k^{n,m} = h_k^{m,m} \tan\alpha$, $\forall n \neq m$ to investigate the effect of α on ϵ. The result is plotted in Fig. 1 for a bonded system with

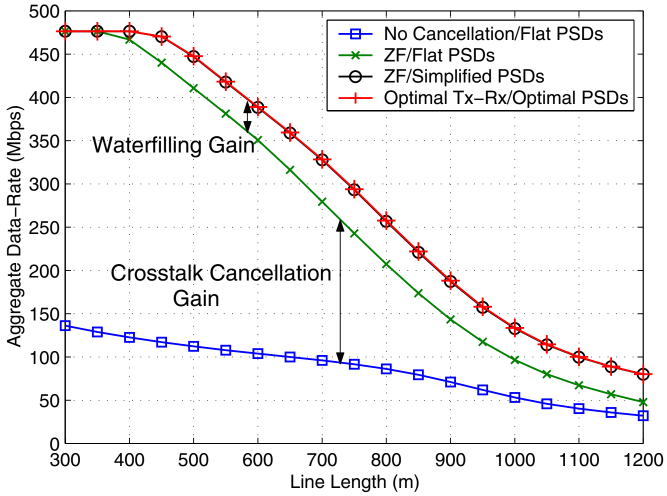

Fig. 3. Bonded-DSL Aggregate Data-Rate vs. Reach

900 m. lines. As can be seen ϵ remains quite low even at large values of α. In DSL typically $\alpha \leq 0.01$ and the error in $\mathbf{S}_{k,\text{simp}}$ is negligible. An upper bound on ϵ can be made using (4) and (5) and this is also shown in Fig. 1. The bound is quite loose, particularly for large α but is still useful for ensuring near-optimality in the ranges of α seen in DSL.

In Fig. 2 we plot the percentage loss in capacity which results from using the simplified waterfilling and RX structure again on a set of 900 m. bonded lines. Capacity loss is minimal for values of $\alpha < 0.05$. Interestingly enough at $\alpha = 1$ capacity loss is approximately 100%. The reason for this is that as $\alpha \to 1$ the channel becomes poorly conditioned. This results in a ZF design that causes severe noise enhancement and hence cannot support reliable transmission at any data-rate.

Fig. 3 shows the aggregate upstream rate for bonded systems of varying line length. Also shown for comparison are the rates achieved with no crosstalk cancellation + flat transmit spectra, and ZF crosstalk cancellation + flat transmit spectra to highlight the benefits of crosstalk cancellation and waterfilling respectively. The flat transmit spectra are at -60 dBmW/Hz as in conventional VDSL. The crosstalk channel model from [7] is used which determines α as a function of frequency and line length. This crosstalk channel model is based on worst-case scenarios and the value of α is actually less in 99% of lines. This implies that performance degradation using the simplified waterfilling and RX structure will be even less than that shown here for 99% of lines.

As predicted the simplified power allocation and RX structure yield near-optimal performance. Both crosstalk cancellation and waterfilling yield significant gains emphasizing their importance. Crosstalk cancellation gains are greatest on short lines where the DSL system is crosstalk-limited rather than noise-limited. Waterfilling gains are greatest on long lines where the channel is highly frequency selective.

VIII. Conclusions

In this paper we investigated power allocation and crosstalk cancellation in MIMO-DSL. Column-wise diagonal dominance is a property inherent in DSL channels with co-located RXs. We use this property to simplify both power allocation and TX/RX structure leading to a significant reduction in initialisation and run-time complexity for MIMO-DSL systems. Column-wise diagonal dominance is also present in MIMO-CDMA where these results are also applicable.

We derived a simplified power allocation algorithm where each transmitter waterfills against their direct channel as if interference were not present. It was shown that this simplified algorithm, combined with a linear ZF crosstalk canceller enables reliable transmission at near-capacity data-rates. This was done by providing an upper bound on the simplified algorithm's deviation from the truly optimal power allocation. Near-optimality was also verified through simulation.

In this work we compared the simplified algorithm to the optimal one in terms of TX correlation matrices. In future work it would be interesting to express the sub-optimality through an upper bound on capacity loss. Also, whilst $\mathcal{E}\left\{\mathbf{z}_k\mathbf{z}_k^H\right\} = \sigma_k^2\mathbf{I}_N$ is a reasonable assumption for DSL, it may not always hold in wireless channels (ie. MIMO-CDMA). In this case noise pre-whitening may affect the column-wise diagonal dominance of the whitened channel seen by the RX. This is the subject of current work.

Appendix

Using the QR decomposition $\mathbf{H}_k \overset{\text{qr}}{=} \mathbf{Q}_k\mathbf{R}_k$. Define $r_k^{n,m} \triangleq [\mathbf{R}_k]_{n,m}$. From [1]

$$|h_k^{n,n}|\sqrt{1-4\alpha^2} \leq |r_k^{n,n}| \leq |h_k^{n,n}|\sqrt{1+(N-1)\tan^2\alpha} \quad (7)$$

Define $[\mathbf{A}]_{col\,m}$ as the mth column of the matrix $\mathbf{A}$. Now since $\|[\mathbf{R}]_{col\,m}\|_2 = \|[\mathbf{H}]_{col\,m}\|_2, \forall m$

$$\begin{aligned}
|r_k^{n,m}|^2 &= \|[\mathbf{H}]_{col\,m}\|_2^2 - |r_k^{m,m}|^2 - \sum_{i \neq n,m} \left|r_k^{i,m}\right|^2 \\
&\leq |h_k^{m,m}|^2 + \sum_{i \neq m} \left|h_k^{i,m}\right|^2 - |r_k^{m,m}|^2 \\
&\leq |h_k^{m,m}|^2 \left[1+(N-1)\tan^2\alpha\right] - |r_k^{m,m}|^2 \\
&\leq |r_k^{m,m}|^2 \left[\frac{1+(N-1)\tan^2\alpha}{1-4\alpha^2} - 1\right] \\
&= |r_k^{m,m}|^2 \left[\frac{4\alpha^2+(N-1)\tan^2\alpha}{1-4\alpha^2}\right], \forall n \neq m
\end{aligned}$$

The equations above all assume $n \neq m$. Hence

$$\frac{|r_k^{n,m}|}{|r_k^{m,m}|} \leq f_1(\alpha), \forall n \neq m \quad (8)$$

where

$$f_1(\alpha) \triangleq \sqrt{\frac{4\alpha^2+(N-1)\tan^2\alpha}{1-4\alpha^2}}$$

Note $\lim_{\alpha \to 0} f_1(\alpha) = 0$. Define $\mathbf{G}_k \triangleq \mathbf{R}_k^{-1}$. Hence $\mathbf{T}_k = \left(\mathbf{R}_k^H\mathbf{Q}_k^H\mathbf{Q}_k\mathbf{R}_k\right)^{-1} = \mathbf{G}_k\mathbf{G}_k^H$ and

$$t_k^{n,m} = \sum_i g_k^{n,i}\left(g_k^{m,i}\right)^* \quad (9)$$

Since $\mathbf{R}_k$ is upper-triangular

$$g_k^{n,m} = \begin{cases} 0 & m < n \\ \frac{1}{r_k^{m,m}} & m = n \\ -\frac{1}{r_k^{m,m}} \sum_{i=n}^{m-1} r_k^{i,m} g_k^{n,i} & m > n \end{cases}$$

Hence

$$
\begin{aligned}
|g_k^{n,m}| &\leq \sum_{i=n}^{m-1} \frac{\left|r_k^{i,m}\right|}{|r_k^{m,m}|} \left|g_k^{n,i}\right|, \forall m > n \\
&\leq f_1(\alpha) \sum_{i=n}^{m-1} \left|g_k^{n,i}\right|, \forall m > n
\end{aligned}
$$

For example

$$
\begin{aligned}
\left|g_k^{n,n+1}\right| &\leq \frac{1}{|r_k^{n,n}|} f_1(\alpha) \\
\left|g_k^{n,n+2}\right| &\leq \frac{1}{|r_k^{n,n}|} \left(f_1(\alpha) + f_1(\alpha)^2\right)
\end{aligned}
$$

In general

$$
\left|g_k^{n,n+i}\right| \leq \frac{1}{|r_k^{n,n}|} \sum_{l=1}^{i} \mathrm{C}_{l-1}^{i-1} f_1(\alpha)^l, \, \forall i > 0 \tag{10}
$$

where $\mathrm{C}_j^i \triangleq \binom{i}{j}$.

A. Diagonal Elements of $\mathbf{T}_k$

Using the derivations so far let us examine the diagonal elements of $\mathbf{T}_k$. From (9)

$$
t_k^{n,n} = \frac{1}{|r_k^{n,n}|^2} + \sum_{i=1}^{N-n} \left|g_k^{n,n+i}\right|^2 \tag{11}
$$

Now from (10)

$$
\sum_{i=1}^{N-n} \left|g_k^{n,n+i}\right|^2 \leq \frac{1}{|r_k^{n,n}|^2} \sum_{i=1}^{N-n} \left(\sum_{l=1}^{i} \mathrm{C}_{l-1}^{i-1} f_1(\alpha)^l \right)^2
$$

Hence using (7)

$$
\begin{aligned}
t_k^{n,n} &\leq \frac{1}{|h_k^{n,n}|^2} \frac{1}{1-4\alpha^2} \left[1 + \sum_{i=1}^{N-1} \left(\sum_{l=1}^{i} \mathrm{C}_{l-1}^{i-1} f_1(\alpha)^l \right)^2 \right] \\
&\leq \frac{1}{|h_k^{n,n}|^2} \left[1 + f_2(\alpha)\right]
\end{aligned}
$$

where

$$
f_2(\alpha) \triangleq \frac{1}{1-4\alpha^2} \left[4\alpha^2 + \sum_{i=1}^{N-1} \left(\sum_{l=1}^{i} \mathrm{C}_{l-1}^{i-1} f_1(\alpha)^l \right)^2 \right]
$$

Note $\lim_{\alpha \to 0} f_2(\alpha) = 0$. Using (11) and (7) we define a lower bound

$$
\begin{aligned}
t_k^{n,n} &\geq \frac{1}{|r_k^{n,n}|^2} \\
&\geq \frac{1}{|h_k^{n,n}|^2} \frac{1}{1 + (N-1)\tan^2\alpha} \\
&\geq \frac{1}{|h_k^{n,n}|^2} \left[1 - f_3(\alpha)\right]
\end{aligned}
$$

where $f_3(\alpha) \triangleq \frac{(N-1)\tan^2\alpha}{1+(N-1)\tan^2\alpha}$. Note $\lim_{\alpha \to 0} f_3(\alpha) = 0$. Thus $t_k^{n,n}$ can be bounded

$$
\frac{1}{|h_k^{n,n}|^2} \left[1 - f_3(\alpha)\right] \leq t_k^{n,n} \leq \frac{1}{|h_k^{n,n}|^2} \left[1 + f_2(\alpha)\right]
$$

with the bounds becoming tight as $\alpha \to 0$.

B. Off-Diagonal Elements of $\mathbf{T}_k$

Let us examine the off-diagonal elements of $\mathbf{T}_k$. From (9)

$$
t_k^{n,m} = \sum_{i=\max(n,m)}^{N} g_k^{n,i} \left(g_k^{m,i}\right)^*
$$

Hence we can bound

$$
|t_k^{n,m}| \leq \sum_{i=\max(n,m)}^{N} \left|g_k^{n,i}\right| \left|g_k^{m,i}\right|
$$

We first focus on the case $n > m$. Here

$$
\begin{aligned}
|t_k^{n,m}| &\leq |g_k^{n,n}|\,|g_k^{m,n}| + \sum_{i=1}^{N-n} \left|g_k^{n,n+i}\right| \left|g_k^{m,n+i}\right| \\
&\leq \frac{1}{|r_k^{n,n}|} \frac{1}{|r_k^{m,m}|} \left[\sum_{l=1}^{n-m} \mathrm{C}_{l-1}^{n-m-1} f_1(\alpha)^l \right. \\
&\quad \left. + \sum_{i=1}^{N-n} \sum_{l=1}^{i} \mathrm{C}_{l-1}^{i-1} f_1(\alpha)^l \sum_{l'=1}^{i+n-m} \mathrm{C}_{l'-1}^{i+n-m-1} f_1(\alpha)^{l'} \right] \\
&\leq \frac{1}{|h_k^{n,n}|} \frac{1}{|h_k^{m,m}|} f_4(\alpha), \, \forall n > m
\end{aligned} \tag{12}
$$

where we use (10) to get from line 1 to 2. In line 4 we define

$$
\begin{aligned}
f_4(\alpha) &\triangleq \frac{1}{1-4\alpha^2} \left[\sum_{l=1}^{N-1} \mathrm{C}_{l-1}^{N-2} f_1(\alpha)^l \right. \\
&\quad \left. + \sum_{i=1}^{N-1} \sum_{l=1}^{i} \mathrm{C}_{l-1}^{i-1} f_1(\alpha)^l \sum_{l'=1}^{i+N-1} \mathrm{C}_{l'-1}^{i+N-2} f_1(\alpha)^{l'} \right]
\end{aligned}
$$

Note $\lim_{\alpha \to 0} f_4(\alpha) = 0$. The same bound can be shown to hold for $n < m$. Thus $t_k^{n,m}$ can be bounded

$$
|t_k^{n,m}| \leq \frac{1}{|h_k^{n,n}|} \frac{1}{|h_k^{m,m}|} f_4(\alpha), \, \forall n \neq m
$$

with the bound becoming zero as $\alpha \to 0$.

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