The Linear Zero-Forcing Crosstalk Canceler is Near-optimal in DSL Channels

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***Abstract*— Crosstalk is a serious problem in next-generation DSL systems such as VDSL. Several non-linear crosstalk cancelers and pre-compensators have been proposed to address this. Unfortunately they all suffer from high complexity, DFE error propagation and/or require modification of CPE. In this paper we propose the use of a simple linear *zero-forcing crosstalk canceler* in upstream transmission and a simple linear *diagonalizing crosstalk precoder* in downstream transmission.**

Certain properties of DSL channels ensure that these simple linear designs lead to near-optimal performance. We formulate a bound on the performance of these schemes and show that in 99% of upstream DSL channels the linear zero-forcing canceler achieves 97% of the theoretical channel capacity. Similarly in 99% of downstream DSL channels the linear diagonalizing precoder achieves 91% of the theoretical channel capacity.

I. Introduction

Next generation DSL systems such as VDSL aim at providing extremely high data-rates, up to 52 Mbps in the downstream. Such high data rates are supported by operating over short loop lengths and transmitting in frequencies up to 12 MHz. Unfortunately, the use of such high frequency ranges causes significant electromagnetic coupling between neighbouring twisted pairs within a binder group. This coupling creates interference, referred to as *crosstalk*, between the systems operating within a binder. Over short loop lengths crosstalk is typically 10-15 dB larger than the background noise and is *the* dominant source of performance degradation.

In upstream communications the receiving modems are co-located at the *central office* (CO) or at an *optical network unit* (ONU) located at the end of the street. This allows joint reception of the signals transmitted on the different lines, thereby enabling *crosstalk cancellation*.

Numerous techniques have been proposed for crosstalk cancellation, see e.g. [1]. Whilst these schemes lead to large performance gains they are unfortunately non-linear which results in high run-time complexities. Furthermore the use of decision feedback can cause problems with error propagation.

Existing crosstalk cancelers are typically based on techniques borrowed from the wireless field. For example the so-called *vectored receiver* proposed in [1] has an identical structure to the wireless *vertical BLAST* receiver. DSL channels have special properties not present in wireless channels which can be exploited to simplify crosstalk canceler design.

In this paper we present a simple linear *zero forcing* (ZF) crosstalk canceler. This structure has a low complexity and suffers no problems with error propagation. As we will show, certain properties of the DSL channel ensure that this simple, linear structure achieves near-optimal performance. We develop a bound which ensures that in 99% of DSL channels, the linear ZF canceler achieves at least 97% of the capacity.

This work was carried out in the frame of IUAP P5/22, *Dynamical Systems and Control: Computation, Identification and Modelling* and P5/11, *Mobile multimedia communication systems and networks*; the Concerted Research Action GOA-MEFISTO-666, *Mathematical Engineering for Information and Communication Systems Technology*; IWT SOLIDT Project, *Solutions for xDSL Interoperability, Deployment and New Technologies*; FWO Project G.0196.02, *Design of efficient communication techniques for wireless time-dispersive multi-user MIMO systems* and was sponsored by Alcatel-Bell.

II. System Model

Assuming that the modems are synchronized and *discrete multi-tone* (DMT) modulation is employed we can model transmission independently on each tone

$$\mathbf{y}_k = \mathbf{H}_k \mathbf{x}_k + \mathbf{z}_k \tag{1}$$

The vector $\mathbf{x}_k \triangleq \left[x_k^1, \cdots, x_k^N\right]$ contains transmitted signals on tone k. There are N lines in the binder and x_k^n is the signal transmitted onto line n at tone k. $\mathbf{y}_k$ and $\mathbf{z}_k$ have similar structures. $\mathbf{y}_k$ is the vector of received signals on tone k. $\mathbf{z}_k$ is the vector of additive noise on tone k and contains thermal noise, alien crosstalk, RFI etc. The tone index is k and lies in the range $1 \dots K$. We assume that the noise is spatially white such that $\mathcal{E}\left\{\mathbf{z}_k \mathbf{z}_k^H\right\} = \sigma_k^2 \mathbf{I}_N$. $\mathbf{H}_k$ is the $N \times N$ channel transfer matrix on tone k. $h_k^{n,m} \triangleq [\mathbf{H}_k]_{n,m}$ is the channel from TX m to RX n on tone k. The diagonal elements of $\mathbf{H}_k$ contain the direct-channels whilst the off-diagonal elements contain the crosstalk channels. We denote the transmit PSD of user n on tone k as $s_k^n \triangleq \mathcal{E}\left\{|x_k^n|^2\right\}$.

In *upstream* (US) transmission the receiver modems are co-located. As a result $\mathbf{H}_k$ is *column-wise diagonally dominant* (CWDD). This means that on each column of $\mathbf{H}_k$, the diagonal element has the largest magnitude

$$|h_k^{n,n}| \gg |h_k^{m,n}|, \; \forall m \neq n \tag{2}$$

The physical reason for this is that the crosstalk signal must propagate through the full length of the disturber's line, as depicted in Fig. 1. This together with the attenuation which results from shielding between twisted pairs ensures CWDD of $\mathbf{H}_k$. We can measure the degree of CWDD with α_k

$$|h_k^{m,n}| \leq \alpha_k \, |h_k^{n,n}|, \; \forall m \neq n \tag{3}$$

Note that receivers must be co-located for crosstalk cancellation to be possible since it relies on joint detection. CWDD has been verified through extensive measurement campaigns of real binders. In 99% of lines α_k is bounded

$$\alpha_k \leq K_{\text{fext}} f_k \sqrt{l} \tag{4}$$

where $K_{\text{fext}} = -22.5$ dB, l is the line length in kilometers, and f_k is the frequency on tone k in MHz[2]. On typical lines α_k is less than -11.3 dB. We will show later on that CWDD ensures that the channel matrix is well conditioned. This ensures that ZF crosstalk cancelers do not cause noise enhancement.

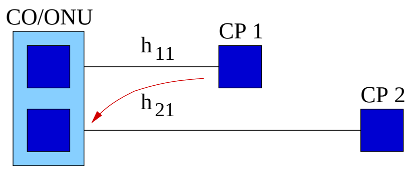

Fig. 1. Column-wise Diagonal Dominance $|h_{11}| \gg |h_{21}|$

III. Theoretical Capacity

We will start by considering the theoretically achievable channel capacity. A word on notation: we use $|x|$ to denote the absolute value of x, whilst $\det(\mathbf{X})$ denotes the determinant of the matrix $\mathbf{X}$.

Theorem 1: The theoretically achievable capacity for user n on tone k can be upper bounded

$$c_{k,\text{opt}}^n \leq \log_2 \left(1 + \sigma_k^{-2} s_k^n \left|h_k^{n,n}\right|^2 \Gamma^{-1} [1 + (N-1)\alpha_k^2]\right) \quad (5)$$

where Γ is the SNR-gap to capacity and is a function of the target BER, noise margin and coding gain.

Proof: Let us start by considering the so-called *single-user bound* which is the capacity achieved when only one user (customer premises modem) transmits and all receivers (CO modems) are used to detect that user. The single-user bound can be achieved by detecting a user last in a *successive interference cancellation* structure[1]. Using the single-user bound the maximum achievable capacity of user n on tone k is

$$c_{k,\text{opt}}^n = \log_2 \left(1 + \sigma_k^{-2} s_k^n \Gamma^{-1} \left\|\mathbf{h}_k^n\right\|_2^2\right)$$

where $\mathbf{h}_k^n \triangleq [\mathbf{H}_k]_{\text{col } n}$. Now using (3) we can bound

$$\left\|\mathbf{h}_k^n\right\|_2^2 \leq \left|h_k^{n,n}\right|^2 \left[1 + (N-1)\alpha_k^2\right]$$

which leads to (5). ∎

Examining (5) we can see that due to CWDD very little increase can be made in the *signal power* by using multiple *receivers* (RX) in the detection of user n. This is the case since the channel from *transmitter* (TX) n to RX m is so much weaker than the direct channel from TX n to RX n. There is no equivalent to *space diversity* in wireline channels.

This does not mean that using co-ordinated reception is pointless however. Instead the benefit comes primarily from the ability to do crosstalk cancellation. That is, co-ordinated reception does not increase signal power in DSL channels, but rather decreases *interference power*.

IV. Data-rate with the Linear ZF canceler (Upstream)

The linear ZF canceler forms an estimate of the transmitted vector

$$\widehat{\mathbf{x}}_k = \mathbf{H}_k^{-1} \mathbf{y}_k$$

This completely inverts the transmission channel, removing interference completely. Consider the *singular value decomposition* (SVD) of $\mathbf{H}_k$

$$\mathbf{H}_k \stackrel{\text{svd}}{=} \mathbf{U}_k \Lambda_k \mathbf{V}_k^H$$

The CWDD of $\mathbf{H}_k$ ensures that its columns are approximately orthogonal. As a result we can closely approximate $\mathbf{V}_k \simeq \mathbf{I}_N$ which implies that

$$\mathbf{H}_k^{-1} \simeq \Lambda_k^{-1} \mathbf{U}_k^H$$

Since $\mathbf{U}_k$ is orthonormal it will not cause noise enhancement. Furthermore Λ_k^{-1} is diagonal so it scales the noise and signal powers equally. As a result, due to the CWDD of $\mathbf{H}_k$, filtering the received signal with the matrix $\mathbf{H}_k^{-1}$ does not cause noise enhancement. This allows the linear ZF canceler to achieve near-optimal performance in DSL channels. This observation is made more rigorous in the following theorem.

Theorem 2: The data-rate achieved by the linear ZF crosstalk canceler can be lower bounded

$$c_{k,\text{zf}}^n \geq \log_2 \left(1 + \sigma_k^{-2} s_k^n \left|h_k^{n,n}\right|^2 \Gamma^{-1} f^{-1}(N, \alpha_k)\right) \quad (6)$$

where

$$f(N, \alpha_k) \triangleq \left(\frac{A_{\max}^{(N-1)}}{A_{\min}^{(N)}}\right)^2 + (N-1)\left(\frac{B_{\max}^{(N-1)}}{A_{\min}^{(N)}}\right)^2 \quad (7)$$

and

$$\begin{bmatrix} A_{\max}^{(N)} \\ B_{\max}^{(N)} \end{bmatrix} \triangleq \left(\prod_{i=1}^{N} \begin{bmatrix} 1 & (i-1)\alpha \\ \alpha & (i-1)\alpha \end{bmatrix}\right) \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad (8)$$

$$A_{\min}^{(N)} \triangleq 1 - \sum_{i=1}^{N} \alpha(i-1) B_{\max}^{(i-1)} \quad (9)$$

Proof: See Appendix I. ∎

In practice we will use this bound to show, in the section VI, that in 99% of DSL channels the linear ZF canceler achieves 97% of the theoretical channel capacity.

V. Data-rate with the Linear Diagonal precoder (Downstream)

In *downstream* (DS) communications the receiving modems reside within different *customer premises* (CP) so crosstalk cancellation is not possible. However since the transmitting modems are co-located at the CO it is possible to do transmission in a joint fashion. This allows some pre-distortion to be introduced into the signals on the different lines before transmission. This pre-distortion is designed to destructively interfere with the crosstalk introduced in the binder, a technique known as *crosstalk ensation*. Several non-linear techniques have been proposed for crosstalk precoding. The technique in [1] is based on the *Tomlinson-Harashima* precoder. Whilst this leads to large performance gains it requires modification of *customer premises equipment* (CPE). This is difficult due to the millions of CPEs already deployed, all owned and operated by different customers. For this reason precoders which only require modification of CO equipment are preferable.

In [3] we described a linear *diagonalizing precoder* (DP) which does not require modification of CPE. The precoder operates by pre-filtering the true symbols on each tone with a matrix $\mathbf{P}_k$ prior to transmission such that

$$\mathbf{x}_k = \mathbf{P}_k \widetilde{\mathbf{x}}_k$$

Here $\widetilde{\mathbf{x}}_k$ denotes the vector of true symbols on tone k, whilst $\mathbf{x}_k$ denotes the ensated symbols that are actually transmitted. The precoding matrix is defined

$$\mathbf{P}_k \triangleq \frac{1}{\beta_k} \mathbf{H}_k^{-1} \mathrm{diag}\{h_k^{1,1}, \ldots, h_k^{N,N}\} \tag{10}$$

where

$$\beta_k \triangleq \max_n \left\| \left[\mathbf{H}_k^{-1} \mathrm{diag}\{h_k^{1,1}, \ldots, h_k^{N,N}\} \right]_{\mathrm{row}\, n} \right\|_2 \tag{11}$$

In DS transmission the DSL channel is *row-wise diagonally dominant* (RWDD) and satisfies

$$|h_k^{n,m}| \leq \alpha_k \, |h_k^{n,n}| \,, \; \forall m \neq n \tag{12}$$

Interestingly the DP also achieves near-optimal performance and obeys the same bound as the linear ZF canceler.

Theorem 3: The data-rate achieved by the linear diagonalizing crosstalk precoder can be lower bounded

$$c_{k,\mathrm{dp}}^n \geq \log_2 \left(1 + \sigma_k^{-2} s_k^n \, |h_k^{n,n}|^2 \, \Gamma^{-1} f^{-1}(N, \alpha_k) \right) \tag{13}$$

Proof: See Appendix II. ∎

VI. Performance

In this section we evaluate the performance of the linear ZF canceler through simulation of a binder of 8 VDSL lines. 4 of the lines have lengths of 600 m. whilst the other 4 have lengths of L m. The performance is shown for a range of line lengths L.

The lines have diameters of 0.5mm. Each modem has a coding gain of 3 dB, a noise margin of 6 dB and a target error probability of 10^{-7} or less which results in $\Gamma = 12.9$ dB. The modems use 4096 tones, the 998 FDD bandplan and transmit at -60 dBm/Hz. We use ETSI noise model A and the semi-empirical transfer functions of [2].

Shown in Fig. 2 are the data-rates achieved on the L m. lines. As can be seen the linear ZF canceller has near-optimal performance, operating quite close to capacity. We also include the lower bound (6) on the performance of the linear ZF canceler. As can be seen the bound is quite tight and close to the theoretical capacity.

The important thing to note is that the bound depends only on the direct channel gain and the background noise power. Good models for both of these characteristics exist based on extensive measurement campaigns. Crosstalk channels on the other hand are much more poorly understood and actual channels can deviate significantly from the few empirical models that exist. This can make provisioning of services difficult.

Using the bound (6) allows us to overcome this problem. The bound tells us that the crosstalk channel gain is not important as long as CWDD is observed. CWDD is a well understood and modeled phenomenon. As a result (6) allows provisioning to be done in a reliable and accurate fashion.

A note of explanation may be necessary at this point. It may seem that CWDD allows us to easily predict (or at least bound) the crosstalk power that a RX experiences. However this is *not* the case. The crosstalk power that a RX experiences depends on the magnitude of elements along a *row* (not *column*) of $\mathbf{H}_k$. This in turn depends on configuration of the other lines within the binder which can vary dramatically from case to case. So knowledge of the full configuration of a binder would be necessary to predict the performance of a single line. CWDD on the other hand applies to *all* DSL lines when RXs are co-located. No knowledge of the actual binder configuration is necessary. Using (6) the performance of a line can be estimated using only information about the line itself.

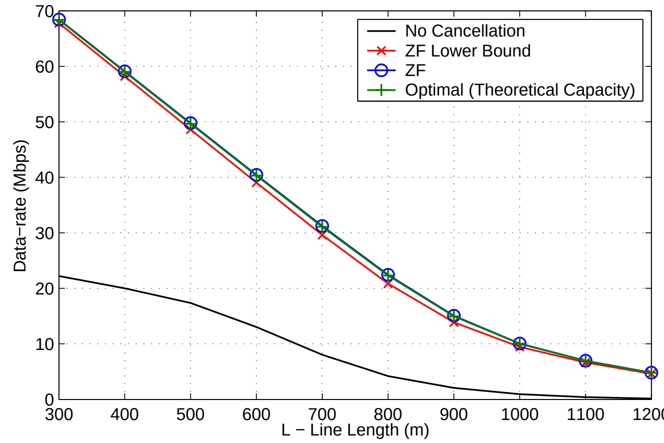

Fig. 2. Upstream Data-rate Achieved with ZF Canceler and Lower Bound

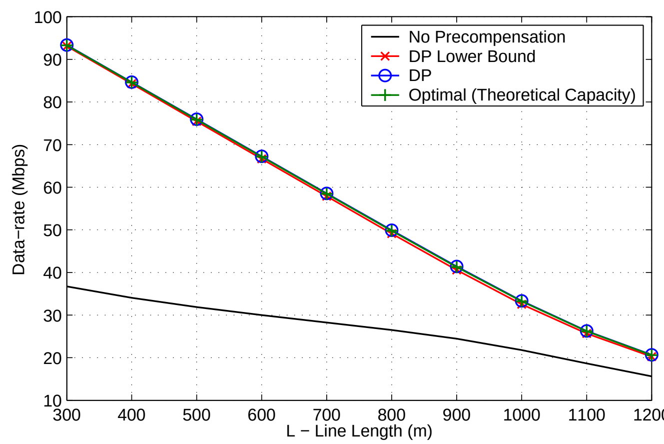

Fig. 3. Downstream Data-rate Achieved with DP and Lower Bound

The value for α_k from (4) is based on worst 1% case models. Hence for 99% of DSL lines α_k will be smaller. Examining the lower bound allows us to conclude that for 99% of DSL lines, the linear ZF canceler achieves 97% of the theoretical channel capacity. So whilst here we only gave simulation results for one binder configuration, the bound ensures us that in all other cases the linear ZF canceller will also give near-optimal performance.

We ran similar simulations for DS transmission and examined the performance of the linear DP. Shown in Fig. 3 are the data-rates achieved on the L m. lines. As can be seen the DP achieves near-optimal performance. We also include the lower bound (13) on the performance of the DP. Examining the lower bound assures us that for 99% of DSL lines, the linear DP achieves 91% of the theoretical channel capacity.

VII. Conclusions

Crosstalk is a serious problem in next-generation DSL systems such as VDSL. Several non-linear crosstalk cancelers and pre-compensators have been proposed to address this. Unfortunately they suffer from high complexity, DFE error propagation and/or require modification of CPE.

In this paper we proposed the use of a simple linear ZF crosstalk canceler in the US and a simple linear DP in the DS.

Due to the CWDD (RWDD) of US (DS) channels these simple linear designs leads to near-optimal performance in DSL.

It was shown that in 99% of US DSL channels the linear ZF canceler achieves 97% of the theoretical channel capacity. Similarly in 99% of DS DSL channels the linear DP achieves 91% of the theoretical channel capacity.

APPENDIX I
SUB-OPTIMALITY BOUND FOR LINEAR ZF CANCELLER

We will make use of the following theorem in our proof.

Theorem 4: Consider any $N \times N$ matrix $\mathbf{A}^{(N)} \triangleq [a_{n,m}]$ which satisfies $a_{n,n} = 1,\ \forall n$ and

$$|a_{n,m}| \leq \alpha,\ \forall n \neq m \tag{14}$$

and any $N \times N$ matrix $\mathbf{B}^{(N)} \triangleq [b_{n,m}]$ which satisfies $b_{n,n} = 1,\ \forall n < N$ and

$$|b_{N,N}| \leq \alpha \tag{15}$$

$$|b_{n,m}| \leq \alpha,\ \forall n \neq m \tag{16}$$

The magnitude of the determinants of $\mathbf{A}^{(N)}$ and $\mathbf{B}^{(N)}$ can be bounded as follows

$$\left|\det(\mathbf{A}^{(N)})\right| \leq A_{\max}^{(N)} \tag{17}$$

$$\left|\det(\mathbf{A}^{(N)})\right| \geq A_{\min}^{(N)} \tag{18}$$

$$\left|\det(\mathbf{B}^{(N)})\right| \leq B_{\max}^{(N)} \tag{19}$$

where $A_{\max}^{(N)}$, $B_{\max}^{(N)}$ and $A_{\min}^{(N)}$ are defined in (8) and (9).

Proof: We use proof by induction. We start by assuming that the bounds (17), (18) and (19) hold for any $N \times N$ matrices of the form $\mathbf{A}^{(N)}$ and $\mathbf{B}^{(N)}$ for some specific value of N. Now consider any $(N+1) \times (N+1)$ matrix of the form $\mathbf{A}^{(N+1)}$

$$\mathbf{A}^{(N+1)} = \begin{bmatrix} & \mathbf{A}^{(N)} & & \begin{matrix} a_{1,N+1} \\ \vdots \\ a_{N,N+1} \end{matrix} \\ a_{N+1,1} & \cdots & a_{N+1,N} & 1 \end{bmatrix}$$

We're interested in finding bounds for the determinant of $\mathbf{A}^{(N+1)}$. If we expand the determinant along the last row of $\mathbf{A}^{(N+1)}$ it can be seen that

$$\begin{aligned} &\left|\det(\mathbf{A}^{(N+1)})\right| \\ &= \Big|\det(\mathbf{A}^{(N)}) \\ &+ \textstyle\sum_{m=1}^{N} (-1)^{N+1-m} a_{N+1,m} \det\left(\left[\begin{array}{cc} \overline{\mathbf{A}}_m^{(N)} & \mathbf{a}_{N+1} \end{array}\right]\right)\Big| \\ &\leq \left|\det(\mathbf{A}^{(N)})\right| + \textstyle\sum_{m=1}^{N} \alpha \left|\det\left(\left[\begin{array}{cc} \overline{\mathbf{A}}_m^{(N)} & \mathbf{a}_{N+1} \end{array}\right]\right)\right| \end{aligned} \tag{20}$$

where $\overline{\mathbf{A}}_m^{(N)}$ is the sub-matrix formed by removing column m from $\mathbf{A}^{(N)}$ and $\mathbf{a}_{N+1} \triangleq [a_{1,N+1} \ldots a_{N,N+1}]^T$. We've exploited the fact that row permutation does not affect the magnitude of a determinant, and we use (14) in the third line. Define the permutation matrix

$$\Pi_m \triangleq [\mathbf{e}_1 \cdots \mathbf{e}_{m-1}\ \mathbf{e}_{m+1} \ldots \mathbf{e}_N\ \mathbf{e}_m]$$

where $\mathbf{e}_m$ is defined as the mth column of the $N \times N$ identity matrix. Note that $\Pi_m^T [\overline{\mathbf{A}}_m^{(N)}\ \mathbf{a}_{N+1}]$ is of the form $\mathbf{B}^{(N)}$ hence from (19)

$$\left|\det\left(\Pi_m^T \left[\begin{array}{cc} \overline{\mathbf{A}}_m^{(N)} & \mathbf{a}_{N+1} \end{array}\right]\right)\right| \leq B_{\max}^{(N)} \tag{21}$$

Using the fact that row permutations have no effect on the magnitude of a determinant, together with (17) and (20) yields

$$\left|\det(\mathbf{A}^{(N+1)})\right| \leq A_{\max}^{(N)} + \alpha N B_{\max}^{(N)}$$

hence

$$A_{\max}^{(N+1)} = A_{\max}^{(N)} + \alpha N B_{\max}^{(N)} \tag{22}$$

We now turn our attention to finding bounds for any $(N+1) \times (N+1)$ matrix of the form $\mathbf{B}^{(N+1)}$. Consider

$$\mathbf{B}^{(N+1)} = \begin{bmatrix} & \mathbf{C}^{(N)} & & \begin{matrix} b_{1,N+1} \\ \vdots \\ b_{N,N+1} \end{matrix} \\ b_{N+1,1} & \cdots & b_{N+1,N} & b_{N+1,N+1} \end{bmatrix}$$

where the $N \times N$ matrix $\mathbf{C}^{(N)} \triangleq [c_{n,m}]$, $c_{n,n} = 1,\ \forall n$ and $|c_{n,m}| \leq \alpha,\ \forall n \neq m$. Expanding the determinant along the last row of $\mathbf{B}^{(N+1)}$ yields

$$\begin{aligned} &\left|\det(\mathbf{B}^{(N+1)})\right| \\ &= \Big|b_{N+1,N+1} \det(\mathbf{C}^{(N)}) \\ &+ \textstyle\sum_{m=1}^{N} (-1)^{N+1-m} b_{N+1,m} \det\left(\left[\begin{array}{cc} \overline{\mathbf{C}}_m^{(N)} & \mathbf{b}_{N+1} \end{array}\right]\right)\Big| \end{aligned} \tag{23}$$

where $\overline{\mathbf{C}}_m^{(N)}$ is the sub-matrix formed by removing column m from $\mathbf{C}^{(N)}$ and $\mathbf{b}_{N+1} \triangleq [b_{1,N+1} \ldots b_{N,N+1}]^T$. Note that $\mathbf{C}^{(N)}$ is of the form $\mathbf{A}^{(N)}$ hence from (17)

$$\left|\det(\mathbf{C}^{(N)})\right| \leq A_{\max}^{(N)}$$

In a similar fashion to (21) it can be shown that

$$\left|\det\left(\Pi_m^T \left[\begin{array}{cc} \overline{\mathbf{C}}_m^{(N)} & \mathbf{b}_{N+1} \end{array}\right]\right)\right| \leq B_{\max}^{(N)}$$

Using the fact that row permutations have no effect on the magnitude of the determinant, together with (15), (16) and (23) yields

$$\left|\det(\mathbf{B}^{(N+1)})\right| \leq \alpha A_{\max}^{(N)} + \alpha N B_{\max}^{(N)}$$

hence

$$B_{\max}^{(N+1)} = \alpha A_{\max}^{(N)} + \alpha N B_{\max}^{(N)} \tag{24}$$

Combining (22) and (24) in matrix form yields

$$\begin{bmatrix} A_{\max}^{(N+1)} \\ B_{\max}^{(N+1)} \end{bmatrix} = \begin{bmatrix} 1 & \alpha N \\ \alpha & \alpha N \end{bmatrix} \begin{bmatrix} A_{\max}^{(N)} \\ B_{\max}^{(N)} \end{bmatrix} \tag{25}$$

Now observe that $\mathbf{A}^{(1)} = 1$ and $\mathbf{B}^{(1)} = a_{11} \leq \alpha$ so (17) and (19) hold for $N = 1$. Hence through induction we see that (17) and (19) hold for all N. This concludes the proof for the upper bounds (17) and (19). We now turn our attention to the lower bound (18). Assume that α is small enough such that

$$\left|\det(\mathbf{A}^{(N)})\right| \geq \left|\sum_{m=1}^{N} (-1)^{N+1-m} a_{N+1,m} \det([\overline{\mathbf{A}}_m^{(N)}\ \mathbf{a}_{N+1}])\right|$$

This is a necessary condition for our lower bound to hold. This can be easily checked by seeing whether the resulting bound $A_{\min}^{(N+1)}$ is positive. Using (20), (14) and (21) we have

$$\left|\det(\mathbf{A}^{(N+1)})\right| \geq \left|\det(\mathbf{A}^{(N)})\right| - \alpha N B_{\max}^{(N)}$$

which together with (18) yields

$$A_{\min}^{(N+1)} = A_{\min}^{(N)} - \alpha N B_{\max}^{(N)} \tag{26}$$

Now observe that $\mathbf{A}^{(1)} = 1$ so $A_{\min}^{(1)} = 1$ and (19) holds for $N = 1$. Hence through induction we see that (26) holds for all N. This concludes the proof for the lower bound (19). ■
The following theorem will also prove useful.

Theorem 5: Consider any $N \times N$ matrix $\mathbf{G}$ of the form $\mathbf{A}^{(N)}$. The magnitude of the elements of $\mathbf{G}^{-1}$ can be bounded

$$\left|\left[\mathbf{G}^{-1}\right]_{n,m}\right| \leq \begin{cases} A_{\max}^{(N-1)}/A_{\min}^{(N)} & n = m \\ B_{\max}^{(N-1)}/A_{\min}^{(N)} & n \neq m \end{cases} \tag{27}$$

where $A_{\max}^{(N)}$, $B_{\max}^{(N)}$ and $A_{\min}^{(N)}$ are defined in (8) and (9).

Proof: By definition of the matrix inverse

$$\left|\left[\mathbf{G}^{-1}\right]_{n,m}\right| = \left|\det(\overline{\mathbf{G}}^{m,n})\right| / \left|\det(\mathbf{G})\right| \tag{28}$$

where $\overline{\mathbf{G}}^{m,n}$ is the sub-matrix formed by removing row m and column n from $\mathbf{G}$. Now $\mathbf{G}$ is of the form $\mathbf{A}^{(N)}$ so from theorem 4

$$\left|\det(\mathbf{G})\right| \geq A_{\min}^{(N)} \tag{29}$$

If $m = n$ then $\overline{\mathbf{G}}^{m,n}$ is of the form $\mathbf{A}^{(N-1)}$ and from theorem 4

$$\left|\det(\overline{\mathbf{G}}^{m,m})\right| \leq A_{\max}^{(N-1)}, \forall m \tag{30}$$

If $m \neq n$ then $\Pi_n^T \overline{\mathbf{G}}^{m,n} \Pi_m$ is of the form $\mathbf{B}^{(N-1)}$ and from theorem 4

$$\left|\det(\overline{\mathbf{G}}^{m,n})\right| = \left|\det(\Pi_n^T \overline{\mathbf{G}}^{m,n} \Pi_m)\right| \leq B_{\max}^{(N-1)}, \forall m \neq n \tag{31}$$

Combining (28), (29), (30) and (31) yields (27) which concludes our proof. ■
Let us now return our attention to the proof of theorem 2. Define the matrix $\mathbf{G}_k \triangleq [g_k^{n,m}]$ where $g_k^{n,m} \triangleq h_k^{n,m}/h_k^{m,m}$. Since the US channel is CWDD (3) ensures us that $\mathbf{G}_k$ is of the form $\mathbf{A}^{(N)}$. Now

$$\mathbf{H}_k = \mathbf{G}_k \mathrm{diag}\{h_k^{1,1}, \ldots, h_k^{N,N}\} \tag{32}$$

After application of the linear ZF canceler, the soft estimate of the transmitted symbol is

$$\begin{aligned} \widehat{x}_k^n &= \left[\mathbf{H}_k^{-1}\right]_{\text{row } n} \mathbf{y}_k \\ &= x_k^n + \left[\mathbf{H}_k^{-1}\right]_{\text{row } n} \mathbf{z}_k \end{aligned}$$

where we use (1) in the second line. Hence the post-cancellation signal power is s_k^n, post cancellation interference power is zero, and the post cancellation noise power is

$$\begin{aligned} \widetilde{\sigma}_{k,n}^2 &= \mathcal{E}\left\{\left|\left[\mathbf{H}_k^{-1}\right]_{\text{row } n} \mathbf{z}_k\right|^2\right\} \\ &= \left\|\left[\mathbf{H}_k^{-1}\right]_{\text{row } n}\right\|_2^2 \sigma_k^2 \end{aligned} \tag{33}$$

where we use the fact that the noise is spatially white in the second line. So the data-rate achieved by the linear ZF canceler is

$$c_{k,\text{zf}}^n = \log_2(1 + s_k^n \widetilde{\sigma}_{k,n}^{-2} \Gamma^{-1}) \tag{34}$$

Let's examine $\widetilde{\sigma}_{k,n}^2$ more closely. From (32)

$$\begin{aligned} \left[\mathbf{H}_k^{-1}\right]_{n,m} &= \left[\mathrm{diag}\{h_k^{1,1}, \ldots, h_k^{N,N}\}^{-1} \mathbf{G}_k^{-1}\right]_{n,m} \\ &= \frac{1}{h_k^{n,n}} \left[\mathbf{G}_k^{-1}\right]_{n,m} \end{aligned} \tag{35}$$

Since $\mathbf{G}_k$ is of the form $\mathbf{A}^{(N)}$we can use theorem 5 to bound

$$\left|\left[\mathbf{H}_k^{-1}\right]_{n,m}\right| \leq \begin{cases} |h_k^{n,n}|^{-1} A_{\max}^{(N-1)}/A_{\min}^{(N)} & n = m \\ |h_k^{n,n}|^{-1} B_{\max}^{(N-1)}/A_{\min}^{(N)} & n \neq m \end{cases}$$

Hence

$$\left\|\left[\mathbf{H}_k^{-1}\right]_{n,m}\right\|_2^2 \leq |h_k^{n,n}|^{-2} f(N, \alpha_k)$$

where $f(N, \alpha_k)$ is defined as in (7). Together with (33) this yields

$$\widetilde{\sigma}_{k,n}^2 \leq \sigma_k^2 \, |h_k^{n,n}|^{-2} f(N, \alpha_k)$$

Combining this with (34) leads to (6) which concludes our proof.

Appendix II
Sub-optimality Bound for Diagonalizing Precoder

In this case the DS channel is RWDD, hence we make a slightly different definition of $\overline{\mathbf{G}}_k \triangleq [\overline{g}_k^{n,m}]$ with $\overline{g}_k^{n,m} \triangleq h_k^{n,m}/h_k^{n,n}$. Note that each element is now divided by the diagonal element on the corresponding row. This is in contrast to $\mathbf{G}_k$ in the previous appendix where the division was done by the diagonal element on the corresponding column.

Since the DS channel is RWDD (12) ensures us that $\overline{\mathbf{G}}_k$ is of the form $\mathbf{A}^{(N)}$. Now

$$\mathbf{H}_k = \mathrm{diag}\{h_k^{1,1}, \ldots, h_k^{N,N}\} \overline{\mathbf{G}}_k \tag{36}$$

From (10) we see that after application of the diagonalizing precoder the signal at the receiver is

$$\mathbf{y}_k = \beta_k^{-1} \mathrm{diag}\{h_k^{1,1} \ldots h_k^{N,N}\} \mathbf{x}_k + \mathbf{z}_k$$

Hence the received signal power for user n is $\beta_k^{-2} \, |h_k^{n,n}|^2 \, s_k^n$, the received interference is zero, and the received noise power is σ_k^2. So the data-rate achieved by the diagonalizing precoder is

$$c_{k,\text{zf}}^n = \log_2(1 + \beta_k^{-2} \, |h_k^{n,n}|^2 \, s_k^n \sigma_k^{-2} \Gamma^{-1}) \tag{37}$$

Let's examine β_k more closely. From (11)

$$\begin{aligned} \beta_k^2 &= \max_n \left\|\left[\mathbf{H}_k^{-1} \mathrm{diag}\{h_k^{1,1}, \ldots, h_k^{N,N}\}\right]_{\text{row } n}\right\|_2^2 \\ &= \max_n \left\|\left[\overline{\mathbf{G}}_k^{-1}\right]_{\text{row } n}\right\|_2^2 \end{aligned}$$

where we use (36) in the second line. Since $\overline{\mathbf{G}}_k$ is of the form $\mathbf{A}^{(N)}$ we can use theorem 5 to bound

$$\beta_k^2 \leq f(N, \alpha_k)$$

where $f(N, \alpha_k)$ is defined as in (7). Combining this with (37) leads to (13) which concludes our proof.

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