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CONTRIBUTION

TITLE: Proof of the Optimality of Optimal Spectrum Management

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ABSTRACT

This document contains a proof of the optimality of the *Optimal Spectrum Management* (OSM) algorithm.

It is presented for information only.

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Proof of the Optimality of Optimal Spectrum Management

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ABSTRACT

This document contains a proof of the optimality of the *Optimal Spectrum Management* (OSM) algorithm.

1. Optimal Spectrum Management

The *optimal spectrum management* (OSM) algorithm was presented in [1]. It provides a numerically tractable way of finding the optimal spectra for a network of DSL modems. This contribution provides a proof of the optimality of OSM.

We start by defining the spectrum management problem. Our goal is to maximize the weighted rate-sum of the modems within a binder.

$$S_1^*, \ldots, S_M^* = \arg \max_{S_1, \ldots, S_M} \sum_i w_i R_i(S_1, \ldots, S_M) \qquad (1)$$

$$\text{s.t. } \sum_n S_i(n) \cdot \Delta f \le P_{i,\max} \;\; \forall i \qquad (2)$$

Varying the weights w_i allows us to operate at different points on the rate region. The optimisation is subject to a total power constraint $P_{i,\max}$ on each line.

2. Equivalence to the Lagrangian

In OSM we find the optimal solution to (1) through the Lagrangian Dual problem which can be solved with a much lower complexity. The Lagrangian Dual to (1) is

$$S_1^D, \ldots, S_M^D = \arg \max_{S_1, \ldots, S_M} \sum_i w_i R_i(S_1, \ldots, S_M) + \lambda_i (P_{i,\max} - \sum_n S_i(n) \cdot \Delta f) \qquad (3)$$

For the Lagrangian Dual problem (3) to be equivalent to (1) the Lagrangian multipliers λ_i must be chosen to satisfy the *complimentary slackness* conditions for all i. The complimentary slackness conditions are:

$$\sum_n S_i^D(n) \cdot \Delta f = P_{i,\max} \text{ and } \lambda_i \ge 0$$
$$\text{or} \qquad (4)$$
$$\sum_n S_i^D(n) \cdot \Delta f \le P_{i,\max} \text{ and } \lambda_i = 0$$

Theorem: Solving the optimal spectrum management problem (1) and the Lagrangian Dual (3) are equivalent.

Proof: Consider a PSD tuple $S_1^D, \ldots, S_M^D$ that is optimal in (3) and satisfies the complimentary slackness conditions (4). We consider two cases for a particular i.

Case 1: $\sum_n S_i(n)\cdot\Delta f \le P_{i,\max}$ and $\lambda_i = 0$

Since $\lambda_i = 0$ the optimal solution to the dual problem (3) satisfies

$$S_1^D,\ldots,S_M^D = \arg\max_{S_1,\ldots,S_M} \sum_i w_i R_i(S_1,\ldots,S_M) \qquad (5)$$

Since $\sum_n S_i^D(n)\cdot\Delta f \le P_{i,\max}$ the PSD obeys the total power constraint. Now (5) implies that the PSDs maximize the weighted rate-sum, hence they are optimal in terms of (1) hence the dual problem is equivalent to the original problem.

Case 2: $\sum_n S_i(n)\cdot\Delta f = P_{i,\max}$ and $\lambda_i > 0$

Since $\lambda_i P_{i,\max}$ is a constant term independent of S_i we can remove it from (3) without changing the optimal solution

$$S_1^D,\ldots,S_M^D = \arg\max_{S_1,\ldots,S_M} \sum_i w_i R_i(S_1,\ldots,S_M) - \lambda_i \sum_n S_i(n)\cdot\Delta f \qquad (6)$$

Imagine that some other PSD tuple $\tilde{S}_1,\ldots,\tilde{S}_M$ is more optimal in terms of (1) than the solution to the dual problem $S_1^D,\ldots,S_M^D$. This implies that

$$\sum_i w_i R_i(\tilde{S}_1,\ldots,\tilde{S}_M) > \sum_i w_i R_i(S_1^D,\ldots,S_M^D)$$

Now for $\tilde{S}_1,\ldots,\tilde{S}_M$ to be a valid solution it must be true that $\sum_n \tilde{S}_i(n)\cdot\Delta f \le P_{i,\max}$. Now since $\sum_n S_i(n)\cdot\Delta f = P_{i,\max}$ this implies

$$\sum_n \tilde{S}_i(n)\cdot\Delta f \le \sum_n S_i^D(n)\cdot\Delta f$$

and since λ_i is positive

$$\sum_i w_i R_i(\tilde{S}_1,\ldots,\tilde{S}_M) - \lambda_i \sum_n \tilde{S}_i(n)\cdot\Delta f > \sum_i w_i R_i(S_1^D,\ldots,S_M^D) - \lambda_i \sum_n S_i^D(n)\cdot\Delta f$$

But this is contradicted by the optimality of $S_1^D,\ldots,S_M^D$ in (3). Therefore it is not possible that a PSD tuple $\tilde{S}_1,\ldots,\tilde{S}_M$ exists that gives a better solution to (1) than $S_1^D,\ldots,S_M^D$. This implies that $S_1^D,\ldots,S_M^D$ is optimal in terms of (1) and that the dual problem (3) and the original problem (1) are equivalent. In practice as long as one of the complimentary slackness conditions in (4) is satisfied for each modem then (1) and (3) are equivalent. ■

3. Per-tone Decomposition

The Dual problem is equivalent to (6) since only a constant term has been removed. Furthermore, (6) can be rewritten as a summation of terms across tones

$$S_1^D,\ldots,S_M^D = \arg\max_{S_1,\ldots,S_M} \sum_n \left(\sum_i w_i b_i(n, S_1(n),\ldots,S_M(n)) - \lambda_i S_i(n)\cdot\Delta f\right) \quad (7)$$

Moving from (6) to (7) is possible since *the bitloading on each tone is only a function of the PSDs on that tone*. So the optimisation can be decoupled across tones. **Inherent to this is the assumption that sidelobes can be safely neglected**. In VDSL (using transmitter and receiver windowing) this is the case and OSM is the truly optimal DSM algorithm. In ADSL the sidelobes are more substantial however we still find that OSM yields as significant performance gain over other DSM techniques like *iterative waterfilling*[1].

Since the optimisation has been decoupled, we can find the optimal PSD tuples on a per-tone basis.

$$S_1^D(n),\ldots,S_M^D(n) = \arg\max_{S_1(n),\ldots,S_M(n)} \sum_i w_i b_i(n, S_1(n),\ldots,S_M(n)) - \lambda_i S_i(n)\cdot\Delta f \quad (8)$$

This greatly decreases complexity and allows the optimal solution to be found in a numerically tractable way. This is the true innovation behind the OSM algorithm.

The optimisation in (8) is solved independently on each tone. The only coupling occurs through the Lagrangian multipliers λ_i which are adjusted in an outer loop of the algorithm.

To solve (8) one can use an exhaustive search and this is the most straight-forward solution for small M. As M becomes larger it is more efficient to apply specialised non-convex optimisation techniques e.g. the Nelder-Mead method which is available in Matlab through the function *fminsearch*.

4. Convergence of OSM and the Complimentary Slackness conditions

So far we have shown that solving the optimisation problem independently on each tone (8) is equivalent to the original optimisation problem (1). Recall that this is only the case if the complimentary slackness conditions (4) are satisfied. In [2] we prove that OSM always converges and that at convergence the complimentary slackness conditions are satisfied. Thus OSM always yields the optimal solution. This proof is rather complex and so is not included here. The interested reader is referred to [2].

5. Summary

This contribution is **for information only**. We presented a proof of the optimality of the *Optimal Spectrum Management* (OSM) algorithm. We showed that OSM yields the optimal PSDs for the spectrum management problem, allowing any point in the theoretically optimal rate region to be achieved.

OSM is based on the inherent assumption that sidelobe effects are negligible. This is the case in VDSL where transmitter and receiver windowing minimises sidelobes. In ADSL sidelobe effects are more pronounced however OSM still yields significant gains over *iterative waterfilling*[1].

References

[1] R. Cendrillon, W. Yu, M. Moonen, J. Verlinden, T. Bostoen, "Optimal Spectrum Management," T1E1.4 Contribution 2003-459, May, 2004, Charlotte, North-Carolina

[2] R. Cendrillon, W. Yu, M. Moonen, J. Verlinden, T. Bostoen, "Optimal Multiuser Spectrum Management for Digital Subscriber Lines," *submitted to IEEE Transactions on Communications, accepted for IEEE Intl. Conf. on Communications (ICC) 2004*.
Available at http://www.esat.kuleuven.ac.be/~rcedrill/research/publications.html

[3] W. Yu, G. Ginis, J. Cioffi, "Distributed Multiuser Power Control for Digital Subscriber Lines," in *IEEE Journal on Selected Areas in Communications*, vol. 20, no. 5, pp. 1105 – 1115, June 2002.

Algorithm A.1.3.2: Optimal Spectrum Management

initialise $w_1,\ldots, w_{M\text{-}1},\ \lambda_1,\ldots,\lambda_M$

while $R_1 \neq R_1^{\text{target}}$

$\vdots$

while $R_{M-1} \neq R_{M-1}^{\text{target}}$

while ($\sum_n S_1(n)\cdot\Delta f \neq P_{1,\max}$) and ($\lambda_1 > 0$)

complimentary slackness conditions

$\vdots$

while ($\sum_n S_M(n)\cdot\Delta f \neq P_{M,\max}$) and ($\lambda_M > 0$)

$$w_M = 1 - \sum_{i=1}^{M-1} w_i$$

for each tone *n*:

find PSD tuple ($S_1(n),\ldots, S_M(n)$) which maximises

$$L(n) = \sum_i w_i \cdot b_i\big(n, S_1(n),\ldots,S_M(n)\big) - \lambda_i S_i(n)\cdot\Delta f$$

if $\sum_n S_M(n)\cdot\Delta f > P_{M,\max}$ increase λ_M, else decrease λ_M

end

$\vdots$

if $\sum_n S_1(n)\cdot\Delta f > P_{1,\max}$ increase λ_1, else decrease λ_1

end

if $R_{M-1} < R_{M-1}^{\text{target}}$ increase $w_{M\text{-}1}$, else decrease w_{M-1}

end

$\vdots$

if $R_1 < R_1^{\text{target}}$ increase w_1, else decrease w_1

end