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Multiscale Modelling Of The Stress Singularity At A Mode I Crack Tip

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ABSTRACT: Direct modelling technique is used to study stresses at the tip of a mode I crack where domain of the material microstructure is explicitly discretised and represented by finite elements. Two cases of internal (at crack faces) and external uniform loading are considered. A multiscale asymptotic model accounting for both the external boundaries and the non-singular stresses at the crack tip are used to analyse the numerical results. It is shown that the parameters (coefficients) of the expansion of the stress concentration at the crack tip can be recovered for both cases of loading from the simulated stress distribution ahead of the crack tip. Applicability of this technique is validated by recovering the parameters for a wider range of heterogeneous material properties.

1 INTRODUCTION

Analysis of fracture phenomena controlled by complex material microstructure requires, at least as a first step, explicit numerical simulation of fracture propagation. Particularly attractive is a simulation method in which the microstructure is represented by numerical elements (e.g. finite element and finite difference methods), in which fractures are treated as sequences of soft elements with largely reduced or zero moduli. However, results of such simulations are difficult to interpret in terms of conventional theory of the LEFM. The applicability of this type of simulations as well as the LEFM interpretation of the results is considered in this paper for the case of mode I crack. The simulations are conducted by the finite difference code FLAC and the results of the analyses are compared with a multiscale asymptotic model developed to account for both external boundaries and the non-singular stresses at the crack tip.

The analysis involves three scales: (1) the scale of the modelling domain, which in this particular case is approximated by an infinite strip; at this scale the crack is seen as a point defect. (2) the scale of the crack at which the process zone is seen as a point; at this scale the singular term and two non-singular terms are determined. (3) the scale of the process zone; at this scale the interpretation of the simulation results and comparison with the analytical model are conducted.

Two cases of loading are considered, which produce different non-singular parts of the stress concentration – external uniform loading and uniform internal loading at crack faces. The macrostructural property of the material is modelled by randomly varying Young's moduli. Stresses are calculated numerically at different distances ahead of the crack tip along its axis and the coefficients of the expansion of the stress concentration at the crack tip are recovered and compared with an analytical asymptotic solution.

2 DIRECT FLAC SIMULATION OF STRESS FIELD IN MODE I CRACK PROPAGATION

Figure 1 shows a finite difference mesh of 200 x 100 elements is used in this plane stress analysis. Crack is represented as a straight slot-like opening of one finite element width. This was achieved by effective removal of elements by setting it to null elements where all the stress tensors and forces in these elements become zero.

Two types of loadings are considered: remote loading $\sigma_{yy}=p$, and tractions applied at the crack faces, $t_y=p$. Other stress components at the sample boundary and the crack faces are zero. Owing to the linearity, the particular calculations are conducted with p=1 MPa. The numerical model is assumed to be in equilibrium condition when the FLAC out-of-balance force is small compared to the total applied external force *viz* within an error of 0.03%.

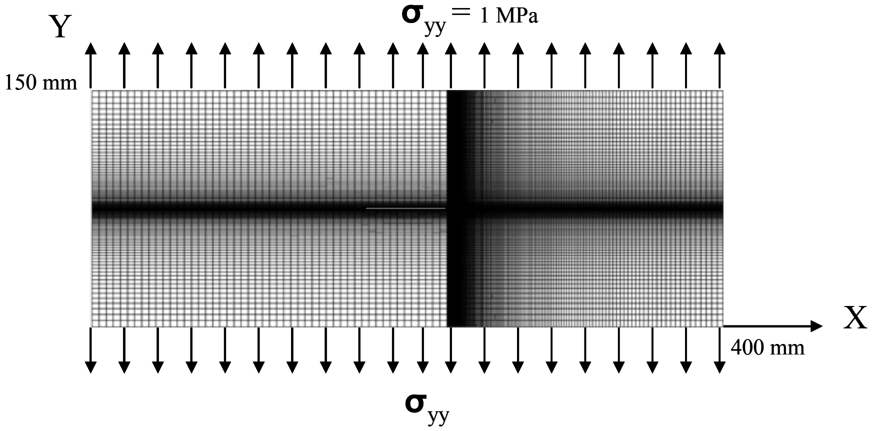

Figure 1. Typical finite difference mesh, loading and FLAC-element representation of crack used in the simulation.

This type of simulation involves a natural macroscopic crack propagation criterion – a criterion of local failure under the stress in the element at the crack tip. The rational behind such a criterion is that the FLAC elements represent both structural and crack elements of the material. Therefore, microscopic structural details of the crack propagation cannot be simulated by such codes.

In the numerical simulations the stress component σ_{yy} normal to the crack length was calculated for different types of loading at 16 points ahead of the crack tip (Figure 2) assuming that the material is homogeneous. In reality, rocks are heterogeneous on any scale, which inevitably influences fracture processes. This material heterogeneity effect was modelled by assigning uniformly distributed random variations of Young's modulus in each FLAC element. The ranges of variation were 10, 20 and 40% (±5, ±10 and ±20% of the mean value respectively). Figures 3 and 4 show the stresses calculated along the crack axis (the *x*-axis) for the case of loading at crack faces for different ranges of variation of Young's modulus. It is seen that the stress distribution is barely affected by these heterogeneity variations. This can be explained as follows. The lengths of the FLAC-elements at the crack tip represent the microstructure of the material in the numerical simulation. This length of the microstructural element is very small as compared to the crack length, i.e. $d \ll a$, where d is the length of the microstructural element. The value of the distance from the crack tip, r, varies from d to $14d$ which is still very small in comparison to the crack length. Therefore, between these two scales a mesoscale associated with the dimension H of the representative volume element can be introduced, i.e. $d \ll H \ll a$. The stress field will only slightly differ for the one obtained from the modelling performed in an effective continuum medium of the scale H which corresponds to averaging of stress and strain fields over the volume elements of size H. This effective medium is characterised by uniform effective moduli replacing heterogeneous moduli of the original material. That is why the influence of the modulus heterogeneity is negligible everywhere except in the immediate vicinity of the crack tip where the effective medium modelling is no longer applicable. Correspondingly, Figures 3 and 4 show that the stresses near the crack tips are fluctuating as compared to the case of uniform modulus.

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3 ASYMPTOTICS OF NON-SINGULAR PART OF THE STRESS FIELD

LEFM assumes that the process zone at the crack tip is negligibly small compared to the crack length or the concept of small-scale yielding [Rice, 1968]. This assumption allows reasonable approximation of stresses near the crack tip by considering only the singular part of stress concentration. However, when the process zone length is not very small compared to the crack length, the contribution of non-singular terms becomes important. Therefore, it is convenient to continue usage of classical elastic crack solution, as in LEFM, but as a next step take into account the non-singular part of stress concentration [Dyskin, 1997].

The expression for normal stress at and ahead of the tip of a crack axis loaded by tractions *–p(t)* applied to the crack faces has the following form [Muskhelishvili, 1953] with the origin of the *x*-axis being placed at the crack centre

$$\sigma_y(x) = \frac{1}{\pi\sqrt{x^2 - a^2}} \int_{-a}^{a} \frac{p(t)\sqrt{a^2 - t^2}}{x - t} dt \qquad (1)$$

where 2*a* is the crack length, *r* is the distance from the crack tip and *–p(t)* is the traction applied to the crack faces.

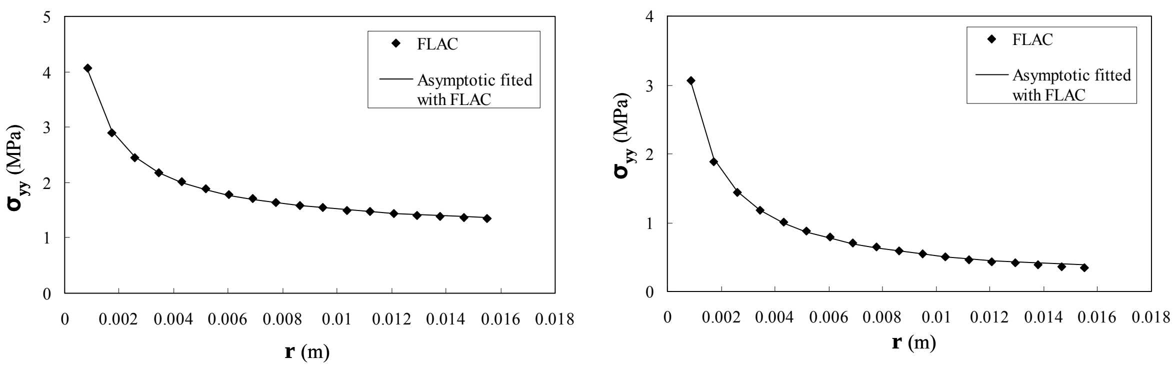
Developing the full solution can be complicated for complex type of loading especially in the presence of finite boundaries. However, when the crack length is still large compared to the process zone length, equation (1) can be used to derive general asymptotic solution for stresses at the crack tip [Dyskin, 1997] in the following form

$$\sigma_y(r) = \frac{K_I}{\sqrt{2\pi r}} + s^{(0)} + s^{(1)}\sqrt{r} + \Delta, \quad \Delta = o\left(\sqrt{r}\right), \quad r = x - a \qquad (2)$$

where K_I is the mode I stress intensity factor.

$$s^{(O)} = \begin{cases} -p & \text{for internal uniform load} \\ 0 & \text{otherwise} \end{cases}$$

$$s^{(1)} = \begin{cases} \dfrac{3}{4}\dfrac{p}{\sqrt{2a}} & \text{for uniform load (external and internal)} \\ -\dfrac{5}{2\pi}\dfrac{P}{(2a)^{3/2}} & \text{for concentrated forces} \end{cases}$$



(a) (b)

Figure 2. Numerically calculated distribution of σ_{yy} along the line of crack continuation (data points) and the fitted asymptotics (solid line) for the case of: (a) remote loading; (b) loading at the crack faces.

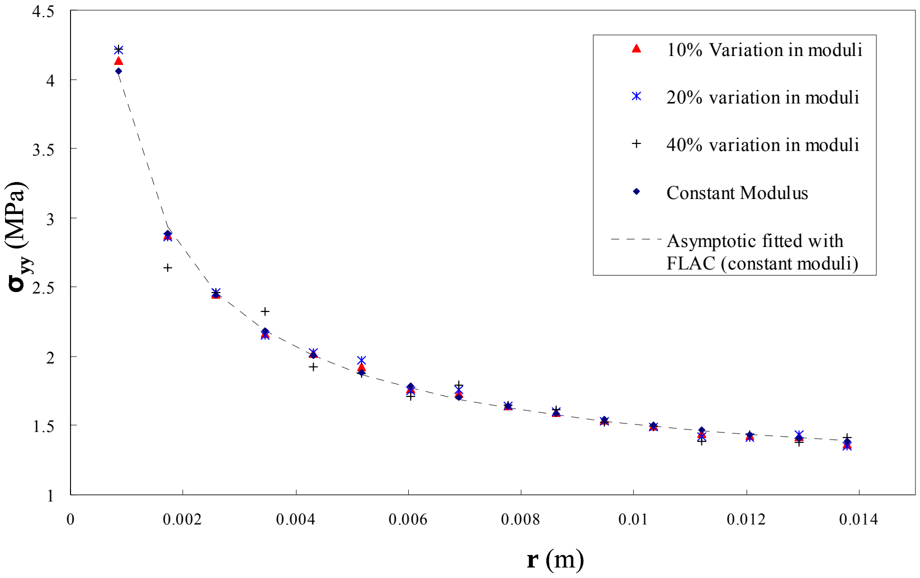


Figure 3. Numerically calculated distribution of σ_{yy} along the line of crack continuation (data points) and the fitted asymptotics (solid line) for the case of remote loading for heterogeneous materials.

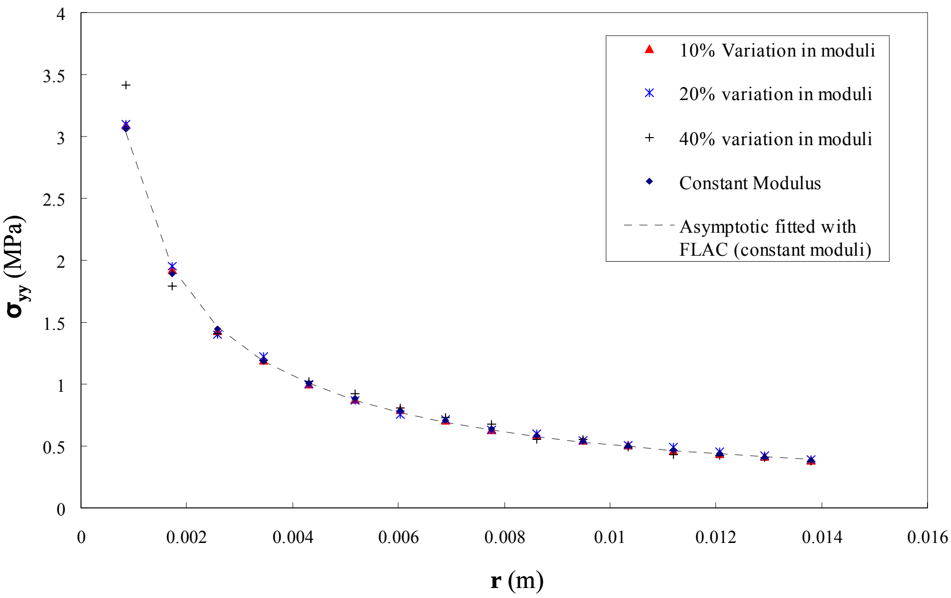


Figure 4. Numerically calculated distribution of σ_{yy} along the line of crack continuation (data points) and the fitted asymptotics (solid line) for the case of loading at crack faces for heterogeneous materials.

Now consider the interaction of this crack with the boundary of the sample. The complete influence of the boundary of a rectangular sample cannot be expressed in mathematically closed form. However, the sample geometry is such that the lateral boundaries are the closest to the crack. Thus, one can hope that by accounting only for these boundaries a sufficiently accurate model can be obtained. This can be done by considering the crack situated in the middle of an infinite strip in x direction, Figure 5.

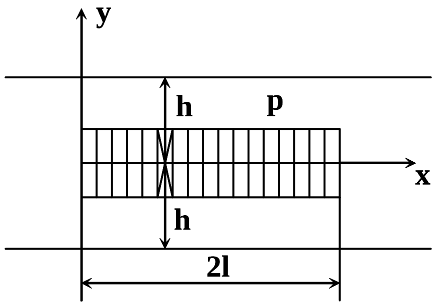


Figure 5. The asymptotic model of crack in an infinite strip.

The next simplification comes from the use of the dipole asymptotics method [e.g., Dyskin and Mühlhaus, 1995] to calculate the interaction between the crack and the boundary of the strip. This method takes into account only the main asymptotic terms characterising the interaction – the terms of the order of l^2/h^2, where $2h$ is the strip width. These terms represent the additional tractions on the crack faces reflecting the influence of the boundary.

These additional tractions are found as follows. Firstly, the crack is considered in an infinite plane and stresses are calculated on the lines which correspond to the boundaries of the strip. These stresses are calculated with accuracy of the main asymptotic terns, i.e. only the terms of the order of l^2/r^2, where r is the distance from the crack centre. Then the corresponding components of these stresses, with opposite signs, are applied to the boundaries of the strip without the crack and the stress is calculated at the centre of the strip, i.e. at the place of the crack. Since these tractions are already of the order of l^2/h^2, their variations over the crack length are of the order l^3/h^3, and hence should be neglected within the accuracy limits of the method. This consideration allows one to assume the additional tractions to be uniform, which leads to an especially simple method of calculating these additional tractions. Then this uniform stress is applied to the crack faces and the additional stress intensity factor is calculated. This constitutes the correction to the stress intensity factor associated with the effect of the boundaries [see Dyskin et al., 2000 for details]. As a result, (2) becomes

$$\sigma_y(r) = \frac{K_I^*}{\sqrt{2\pi r}} + s^{(0)} + s_*^{(1)}\sqrt{r} + \Delta, \quad K_I^* = K_I\left(1 + 1.14\frac{l^2}{h^2}\right), \quad s_*^{(1)} = s^{(1)}\left(1 + 1.14\frac{l^2}{h^2}\right) \tag{3}$$

where $2l$ is the crack length and $2h$ is the sample height.

4 RECOVERING OF THE PARAMETERS

Equation (3) gives asymptotic solution for the stress distribution at the crack tip. It is interesting to compare this result with the numerical solution to see whether the parameters K^*_I, $s^{(0)}$ and $s_*^{(1)}$ can be recovered. Technically these parameters are the coefficients of multiple regressions on functions $r^{-1/2}$, 1 and $r^{1/2}$. The results of the fitting are shown in Figures 1-3 with the values of the parameters given in Tables 1-3. It is seen that for the case of homogeneous material the recovered parameters are very close to the actual values in both cases. The error of the recovery however increases with heterogeneity. It is also noted that average values and standard deviations of recovered parameters are higher for the case when cracks are loaded at its faces compared to the case where load is at infinity for the same heterogeneity. This observed discrepancy can be attributed to the heterogeneity acting as local points of stress concentrators.

Table 1. Comparison of recovered and actual parameters when load is applied at: (a) infinity; (b) crack faces.

Parameters	Loaded at boundary			
	Actual	Recovered	Standard error	Coefficient of determination
K_I	0.292	0.288	0.0045	0.9994
$s^{(1)}$	3.521	3.436	0.4827	
$s^{(0)}$	0	0.02	0.06	

(a)

Parameters	Loaded at crack faces			
	Actual	Recovered	Standard error	Coefficient of determination
K_I	0.292	0.2896	0.00478	0.9994
$s^{(1)}$	3.521	3.65	0.534	
$s^{(0)}$	-1.0	-1.0006	0.0685	

(b)

Table 2. Comparison of recovered and actual parameters when load is applied at infinity with varied material properties.

Parameter	Loaded at boundaries					
	10% variation in moduli		20% variation in moduli		40% variation in moduli	
	Average	Standard Deviation	Average	Standard Deviation	Average	Standard Deviation
K_I	0.292	0.006	0.292	0.008	0.298	0.018
$S^{(1)}$	3.457	0.39	3.431	0.442	3.768	0.69
$S^{(0)}$	-0.001	0.065	-0.002	0.078	-0.057	0.151

Table 3. Comparison of recovered and actual parameters when load is applied at crack faces with varied material properties.

Parameter	Loaded at crack faces					
	10% variation in moduli		20% variation in moduli		40% variation in moduli	
	Average	Standard Deviation	Average	Standard Deviation	Average	Standard Deviation
K_I	0.29	0.004	0.296	0.006	0.31	0.007
$S^{(1)}$	3.366	0.267	3.907	0.417	3.976	0.49
$S^{(0)}$	-0.983	0.045	-1.061	0.079	-1.137	0.08

5 CONCLUSIONS

The problem of crack propagation in samples of heterogeneous materials is influenced by three scales: the scale of the sample, the scale of crack and the scale of heterogeneities. Direct modelling technique is used to study stresses at the tip of a mode I crack where the material microstructure is explicitly represented by FLAC-finite elements. Comparison of the results of the simulation with a multiscale asymptotic model shows that the multiscale concept offers an efficient method of handling problems featuring a number of distinctive scales. As a result, the parameters (coefficients) of the expansion of the stress concentration at the crack tip can be efficiently recovered. The recovery becomes more distinct for heterogeneous materials, although the error of recovery also increases with heterogeneity.

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