

**Title:** UNDERSTANDING METACOGNITIVE FAILURE

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**Running Head:** Metacognitive failure

## UNDERSTANDING METACOGNITIVE FAILURE

This paper reports on a study that investigated patterns of collaborative metacognitive activity in senior secondary school classrooms. Although peers working together on mathematical tasks may enjoy the metacognitive benefits of being able to monitor and regulate each other's thinking, collaboration does not guarantee that they will achieve a mathematically productive outcome. The notion of metacognitive "red flags", or warning signals that problem solving has gone astray, is developed in order to identify three possible scenarios for metacognitive failure. These scenarios, described by the metaphors of blindness, vandalism, and mirage, are illustrated via analysis of videotaped lesson transcripts obtained from a secondary school mathematics classroom. The results provide insights into the interactive constitution of metacognitive activity during small group work, and suggest implications for teachers concerning the fostering of communication and problem solving within a classroom culture of inquiry.

## UNDERSTANDING METACOGNITIVE FAILURE

Research on the role of metacognition in mathematical thinking flourished during the 1980s and into the 1990s, in concert with the emergence of new mathematics curriculum and policy documents that placed increased emphasis on problem solving and mathematical reasoning (e.g. Australian Education Council, 1991; National Council of Teachers of Mathematics, 1989). At the same time, teachers have been urged to engage students in small group or whole class discussion as a means of developing mathematical understanding (National Council of Teachers of Mathematics, 1991). Problem solving and communication remain central to current visions of effective mathematics teaching (National Council of Teachers of Mathematics, 2000). However, our theoretical understanding of problem solving processes, and how students' mathematical thinking is shaped by their interaction with peers, is far from complete (e.g. Lester, 1994; Schoenfeld, 1992), suggesting that new frameworks are needed to bring together fundamentally cognitive and fundamentally social perspectives on human thought and action (Schoenfeld, 1999).

It is widely acknowledged that metacognitive processes, that is, how students monitor and regulate their thinking, are crucial to successful performance on mathematical tasks, and many studies have investigated the metacognitive strategies which secondary school students use in problem solving. Many of these studies have focused on students working individually, in experimental settings, on tasks prescribed by the researcher (Fitzpatrick, 1994; Randhawa, 1994). The few classroom based studies that have investigated the metacognitive potential of small group problem solving have typically used researcher controlled interventions that impose group structures (e.g. based on ability) on students who are unfamiliar with this way of working (Artzt & Armour-Thomas, 1992, 1997; Curcio & Artzt, 1998; Stacey, 1992). Despite increasing interest in the situated nature of mathematics learning (see Lerman, 2000),

there has been little research on the characteristics of collaborative metacognitive activity occurring when students work together in natural classroom settings.

The research discussed in this paper is part of a larger study, carried out in upper secondary school classrooms, that investigated patterns of teacher and student social interactions associated with metacognitive activity, and assumptions about teaching and learning mathematics underlying teachers' and students' actions (Goos, 2000a). A major aim of the study as a whole was to explore links between peer interaction and metacognitive activity in authentic classroom settings. Of particular interest was the potential for peers to act as a sounding board for refining and elaborating students' own ideas (Teasley, 1997).

Results from this study reported previously have indicated that jointly transacted monitoring and regulation can indeed help students to overcome obstacles in their progress towards successful solution of mathematical problems (Goos, 1997; Goos & Geiger, 1995). In the classroom, collaborative metacognitive activity was characterised by students offering their thoughts to peers for inspection, while acting as a critic of their partners' thinking. In addition, the interaction was reciprocal in that students monitored and regulated both their own and each other's thinking when working together on problems. Nevertheless, it would be misleading to claim that peer collaboration always achieves a mathematically productive outcome. Hence the purpose of this paper is to identify circumstances in which collaboration may be metacognitively fruitless, and to suggest reasons for this lack of success. The following section develops a model of metacognitive activity to account for both successful and unsuccessful problem solving outcomes. In the remainder of the paper, findings from the empirical study referred to above are presented in order to illustrate three possible scenarios for metacognitive failure.

### **Metacognitive Success and Failure**

Frameworks for analysing task-oriented mathematical thinking typically identify phases or episodes representing distinctive kinds of problem solving behavior, and describe the ideal characteristics of each episode. Such frameworks can be used to analyse verbal protocols obtained by video or audio taping students as they work on mathematical problems. Perhaps the best known is Schoenfeld's (1985) episode parsing procedure, later adapted by Artzt and Armour-Thomas (1992) to study interactions between students as they worked on problems in small groups. Figure 1 describes characteristic features of small group problem solving episodes identified by Artzt and Armour-Thomas as Reading, Understanding, Analysis, Exploration, Planning, Implementation, and Verification. (Their Watching and Listening episode type was not included in the theoretical and methodological framework of the present study since interest centred on students' verbalisation of problem solving strategies.)

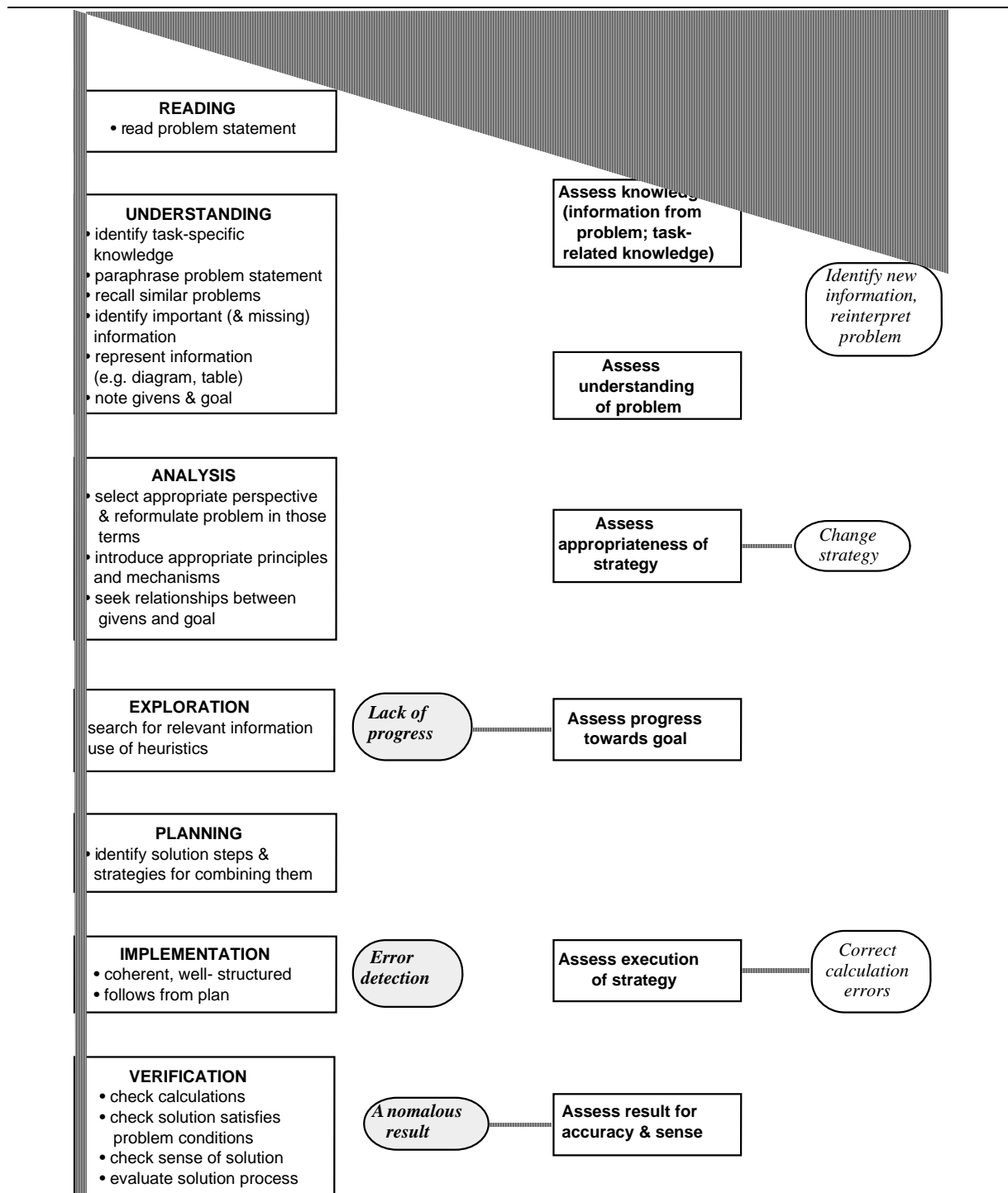


Figure 1. Episode-based model of metacognitive activity during problem solving

While such frameworks acknowledge the central role of metacognitive processes in keeping the solution on track, they nevertheless suffer from a number of limitations that were specifically addressed in the study reported in this paper. To begin with, existing models –

such as that of Artzt and Armour-Thomas – do not consider in detail the types of monitoring and regulatory activities that would be appropriate and expected at each stage of the solution. Hence one worthwhile modification to existing frameworks involves outlining the possible scope of such activities. How this was done in the course of the study reported here is shown in Figure 1, in the columns headed Monitoring and Regulation. For example, during an Understanding episode, *monitoring* activities would include assessing the adequacy of one's knowledge about this particular problem and similar tasks, and *regulatory* activities such as identifying additional information may become necessary in the light of this assessment.

Second, previous research in this area has not distinguished between the routine monitoring that merely serves to confirm that all is well, and the more controlled monitoring and regulatory processes triggered when students become aware of specific difficulties. It is helpful to think of these triggers as metacognitive “red flags” that signal the need for a pause or some backtracking while remedial action is taken. Hence in Figure 1 a distinction is made between routine assessment of activity during each problem solving episode (for example, assessing execution of a strategy), and the conscious actions that may need to be taken in response to three types of “red flags” (shown in shaded boxes).

Recognising the first type of “red flag”, *lack of progress*, should lead students back to analysis of the problem in order to reassess the appropriateness of the chosen strategy and to decide whether to persist, salvage whatever information is useful, or abandon the strategy altogether. In the latter case it is likely that students will need to reassess their understanding of the problem, and search for new information or a new strategy. The second “red flag”, *error detection*, should prompt checking and correction of calculations carried out so far. If attempts to verify the solution reveal that the answer does not satisfy the problem conditions, or does not make sense, then this third “red flag”, an *anomalous result*, should trigger a

calculation check (assess execution of strategy), followed, if necessary, by a reassessment of the strategy.

Of course, metacognitive “red flags” could occur at points other than those indicated in Figure 1. For example, anomalous intermediate results may be discovered during the implementation phase, lack of progress could be recognised at the analysis stage if no suitable strategy can be identified, and errors could be detected during an exploration episode. However, similar regulatory actions would be triggered by “red flags” no matter when they occur.

A third limitation of existing analytical frameworks, and with research on metacognitive processes in mathematics generally, is the lack of explicit attention given to characterising different forms of metacognitive failure, other than to note that students did not exploit useful information, or that checking behavior was absent (Stacey, 1992). Thus the study described in this paper further extended the notion of “red flags” to identify a range of different metacognitive scenarios that could arise when students work on mathematics tasks. These are represented in Figure 2. While metacognitive *success* will occur if students recognise a “red flag” and take appropriate action to deal with the difficulty (or recognise that nothing is wrong and continue on the same solution path), less successful outcomes are likely in at least three other circumstances.



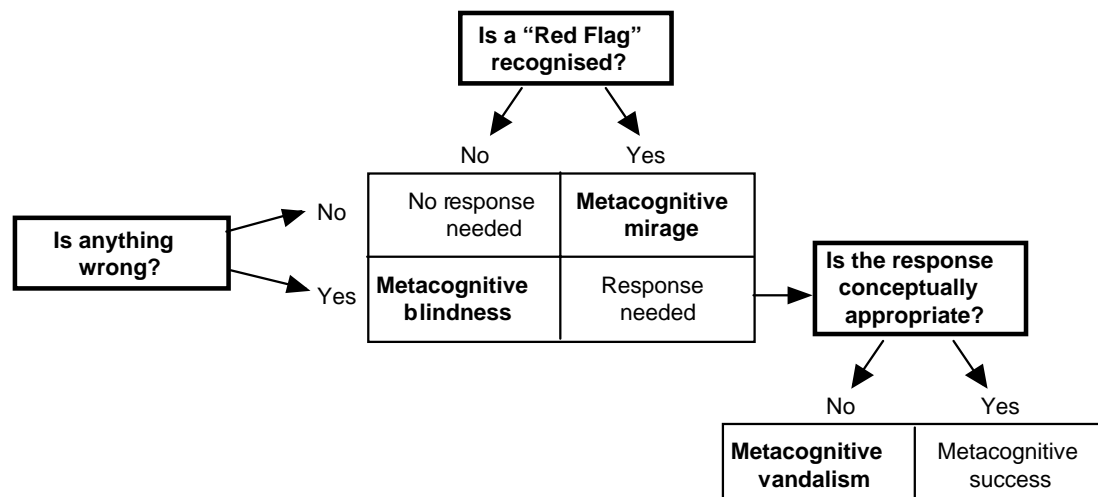


Figure 2. Metacognitive success and failure scenarios

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First, students can be guilty of metacognitive *blindness* if they fail to notice that something is amiss, for example, by persisting with the wrong strategy or overlooking a calculation error. Second, students might commit metacognitive *vandalism* by taking destructive action to deal with an impasse. That is, students could change the problem by imposing an inappropriate conceptual structure to enable them to apply knowledge already available to them. Third, the “red flag” itself may be spurious and represent a metacognitive *mirage* if students “see” difficulties which do not exist, and mistakenly abandon a useful strategy, amend calculations which are not in error, or reject correct answers.

The remainder of the paper draws on observational data from the secondary school mathematics classroom that was the primary research site in the study referred to earlier, in order to illustrate the three scenarios described above and identify implications for small group work on mathematical tasks.

## **The Classroom Study**

### *Background*

The aims of the study were to investigate characteristics of secondary school students' metacognitive activity as they worked together on mathematical tasks in authentic classroom settings, and to examine the teacher's role in creating a classroom culture of inquiry that promotes such mathematical habits of mind. The study was carried out over a three year period from 1994-1996 and involved five secondary school teachers and their Year 11 or Year 12 mathematics classes, all in different schools located in or near a large Australian city.

Since the study was concerned with classroom interaction processes implicated in students' learning, its methods were consistent with naturalistic inquiry (Lincoln & Guba, 1985) and included long term participant observation of classrooms (supplemented by audio and video recording), interviews with students and teachers, and survey questionnaires. Complementary perspectives provided by questionnaire and observational data revealed that in one classroom, more so than others, students seemed to be developing positive metacognitive dispositions and a preference for learning through interaction with peers. Consequently, this classroom was chosen for closer analysis to investigate the research aims outlined above. Although this paper is concerned with students' metacognitive activity rather than the teacher's role in establishing a culture of inquiry, it will be helpful to describe briefly significant features of the classroom in question to provide a backdrop against which the students' actions may be examined. (For a detailed analysis of this classroom see Goos, Galbraith & Renshaw, 1999).

Observations of the teacher and the classes taught by him over a two year period (Year 12 in 1995 and Year 11 in 1996) suggested that he was successful in creating a culture of mathematical inquiry (Borasi, 1992; Schoenfeld, 1989; Schoenfeld, 1994; Weissglass, 1992).

Explanation and justification of ideas featured strongly in classroom social interactions and there was a high incidence of student-student mathematical discussion, defined by Pirie and Schwarzenberger (1988) as purposeful talk on a mathematical subject, with genuine student contributions and interaction. Instead of assigning students to groups structured according to achievement levels, gender, or other student characteristics, the teacher invited them to work together in social groupings of their own choosing. In fact, it was observed that students frequently initiated discussion between themselves without the teacher's prompting. These spontaneous interactions seemed to indicate that they had appropriated the social norms and the modes of reasoning valued by the teacher, which stemmed from his central pedagogical belief in challenging students to make personal sense of the mathematics they were learning.

#### *Data gathering and analysis methods*

Target students within this classroom were chosen for videotaping and interview on the basis of their metacognitive sophistication and preference for working collaboratively with peers, as judged from preliminary observation and responses to questionnaires (see Goos, 1995; Goos, 1999 for details of questionnaires). One lesson was observed each week, and target students were videotaped and audiotaped as they worked together on tasks set by the teacher as part of their regular mathematics program.

Selected portions of the audio and videotapes were transcribed and the resulting verbal protocols parsed into episodes consistent with the framework developed in Figure 1. A finer grained analysis of conversational turns (referred to as *Moves* in the protocol) was then carried out to identify the metacognitive function of the students' dialogue, using a coding scheme developed in an earlier study (Goos & Galbraith, 1996). The first type of metacognitive event, *New Idea*, occurred when potentially useful information came to light or an alternative approach was mentioned. The second type involved making an *Assessment* of the accuracy or sense of a *result*, the execution of appropriateness of a *strategy*, general

*progress* towards a successful solution, or of one's *knowledge* or *understanding*. The coding scheme was therefore suitable for identifying monitoring and regulatory activities outlined in the episode-based problem solving model presented in Figure 1.

Three problem solving transcripts, all obtained from the Year 11 class of 1996, have been selected to illustrate each of the metacognitive failure scenarios described earlier. The excerpts presented here are annotated to indicate New Ideas and Assessments made by the students. Although the focus here is on *unsuccessful* problem solving, it is important to point out that most of the evidence gathered from this class (and others taught by this teacher) is of *successful* metacognitive interactions when students worked together (e.g. see Goos & Geiger, 1995).

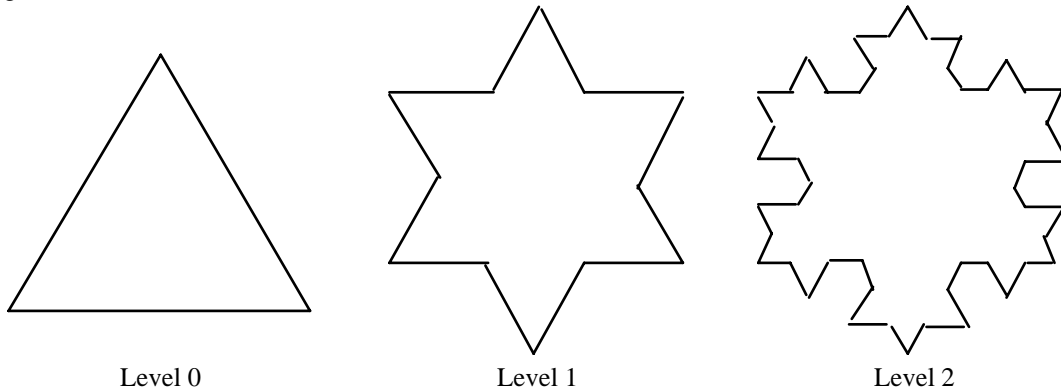
### **Metacognitive Blindness – Area of a Koch Snowflake**

The first example of metacognitive failure illustrates how students can be oblivious to a “red flag”. In this case, *metacognitive blindness* prevented students from recognising a simple error in calculating the area of a Koch Snowflake. The lesson comes from a unit of work which introduced Year 11 students to some of the important ideas of chaos theory, via a teacher-produced booklet containing explanatory text, worked examples, and problems.

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**Area and Length of a Koch Snowflake or an Island of Infinitely Long Coastline.**

The Koch Snowflake has a number of interesting numeric properties related to its geometry. Let's firstly investigate its area.



(\*) Let's start with a Level 0 triangle of side 1 unit in length. The area of this triangle will therefore be  $\frac{\sqrt{3}}{4}$  square units. The Level 1 triangle has an area of  $\frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{12}$  square units. The Level 2 triangle has an area of  $\frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{12} + \frac{\sqrt{3}}{54}$  square units (\*) and Level 3 an area of  $\frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{12} + \frac{\sqrt{3}}{54} + \frac{\sqrt{3}}{243}$ . You should notice that this becomes a geometric progression **after the first term** with  $a = \frac{\sqrt{3}}{12}$  and  $r = \frac{2}{9}$ .

This is a converging series so it is possible to find its sum to infinity.

$$S_{\infty} = \frac{\sqrt{3}}{4} + \frac{a}{1-r} = \frac{\sqrt{3}}{4} + \frac{\frac{\sqrt{3}}{12}}{1-\frac{2}{9}} = \frac{5\sqrt{3}}{14}$$

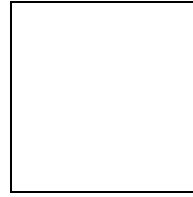
Thus the area of a Koch Snowflake with side length 1 is  $\frac{5\sqrt{3}}{14}$ . It is possible to argue from this example that the area of a Koch Snowflake is finite no matter what the length of the original triangle.

*Figure 3. Finding the area of a Koch Snowflake*

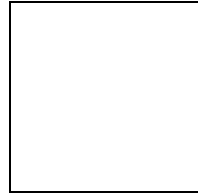
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The teacher began the lesson by asking the class to read the material reproduced in Figure 3, insisting that they should check the calculations in the text to ensure that there were no errors. (The possibility of an error was brought to the teacher's attention as a result of my

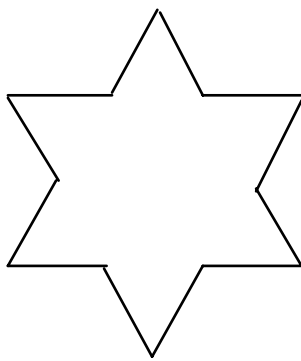
own calculation of the area of the Level 2 snowflake as  square units, and of



the Level 3 area as , results which differ from those given in the text.) The



transcript documents the attempts of two students, Rhys and Sean, to make sense of the passage enclosed by two asterisks (\*) in Figure 3. Although there was not an explicit “problem” to be solved, the students problematised the text by treating the reported results as tentative and requiring validation. Thus, the implicit problem on which they worked could be stated as: *Find the areas of a Level 0, Level 1 and Level 2 Koch snowflake.* A model solution for finding the area of the Level 1 snowflake is provided in Figure 4.



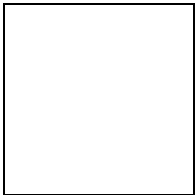
The Level 0 snowflake is an equilateral triangle with sides 1 unit long and hence area  $\frac{\sqrt{3}}{4}$ . The Level 1 snowflake is constructed by adding three equilateral triangles, each of which has sides  $\frac{1}{3}$  units long. Hence the base of these triangles is  $\frac{1}{3}$  and the height  $\frac{1}{3} \cos 30^\circ$ . The area of each of these added triangles is:

$$\begin{aligned} \text{Area } \Delta &= \frac{1}{2} bh \\ &= \frac{1}{2} \times \frac{1}{3} \times \left( \frac{1}{3} \times \cos 30^\circ \right) \\ &= \frac{1}{2} \times \frac{1}{3} \times \left( \frac{1}{3} \times \frac{\sqrt{3}}{2} \right) \\ &= \frac{\sqrt{3}}{36} \end{aligned}$$

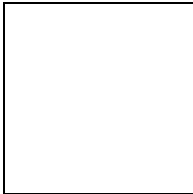
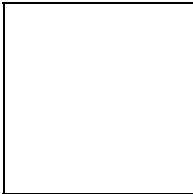
Since there are three added triangles, the additional area is  $\frac{\sqrt{3}}{12}$ . Therefore the total area of a Level 1 snowflake is  $\frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{12}$ .

*Figure 4. Area of Level 1 Koch snowflake*

The boys quickly calculated the area of the Level 0 snowflake (an equilateral triangle

with sides of 1) as , the same result as stated in the text. However, the added

complexity of the Level 1 snowflake caused problems for both students, particularly in handling the many fractions in the calculations. Their use of slightly different versions of the formula for area of a triangle also made it difficult for them to compare their working (shown in Figure 5) and identify any errors the other may have made. Sean's error was to use

 instead of  for the base of the added triangle, and, despite

having many opportunities to make the correction, Rhys did not do so until Move 66, some eleven minutes after the students started work.

Sean's working

$$\begin{aligned} A \text{ of } \Delta &= \left( \frac{B \times H}{2} \right) \\ &= \frac{\frac{1}{6} \times \left( \frac{1}{3} \cos 30 \right)}{2} \quad (*) \end{aligned}$$

Rhys's working

$$\begin{aligned} \text{Area } \Delta &= \frac{1}{2} B \times h \\ &= \frac{1}{2} \times \frac{1}{3} \times \left( \frac{1}{3} \cos 30 \right) \\ &= \frac{1}{2} \times \frac{1}{3} \times \left( \frac{\cos 30}{3} \right) \\ &= \frac{1}{6} \times \left( \frac{\cos 30}{3} \right) \quad (*) \end{aligned}$$

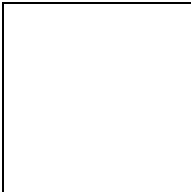
Figure 5. The students' working for the Level 1 snowflake

The first evidence of Sean's error came to light when he queried Rhys's working:

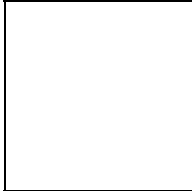
25. S: (Leaning over and pointing to Rhys's working, disputing what he sees) No no no no no. (*Assessment—strategy execution*)
26. R: (disagreeing with Sean) Mm.
27. S: It's a half times *everything*. Look. It's base times height *on two*. (*Assessment—strategy execution*)
28. R: Yeah, or a half times it, multiplication's ... (waves hand, lost for the right word) commutative, it doesn't matter which order you do it in. (*Assessment—strategy execution*)

29. S: Yeah but—
30. R: It's root three on four, it's the same as that—
31. S: Yeah I know but look (pointing to diagram) that, all that, so it's one third of a third cos thirty, over two. (*Assessment—strategy execution*)
32. R: Or a third cos thirty over six. It doesn't really matter. (*Assessment—strategy execution*) Besides I know, I'm pretty sure it's right because I'm getting what Mr G (teacher) got.
33. S: (looks at his own working, shakes his head) But look, I got like this—(Holds up the page of his book to show Rhys. Distracted by teacher's voice, talking to another student. Rhys does not look.)

Sean's intervention was probably prompted by the difference he observed in their respective calculations (marked with an asterisk in Figure 5) – a discrepancy caused by his incorrect use

of  for the triangle's base, not by incorrect manipulation of area formulae as he seems to be arguing. This would explain Sean's argument that "it's a half times everything", if he believed that Rhys had not carried out the step of dividing (base times height) by two.

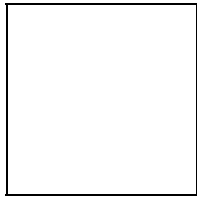
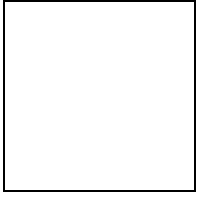
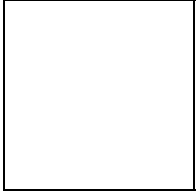
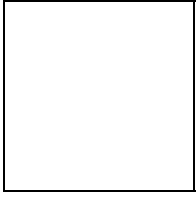
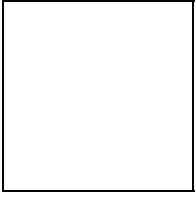
At this point Rhys appealed to the authority of the text and the teacher to verify his

result for the area of the Level 1 snowflake as  (Moves 34 and 38). However,

Sean, who had remained unconvinced by Rhys's argument earlier, was still trying to track down the source of the difference between his own and his partner's calculations (Move 35):

34. R: (raising hands in triumph, his answer matches the result in the workbook) Ayy!
35. S: (Showing Rhys his work) Look. It's base times height over two. Is that a sixth or a third? (Referring to base) (*Assessment—strategy execution*)
36. R: (Looking on) It'll be a sixth.
37. S: A sixth times a third cos thirty.
38. R: (Teacher has walked over. Rhys takes this opportunity to check his work. Sean is left "hanging". He listens to this conversation.) Mr G, I got it.



Unfortunately, Rhys did not spot the error (Move 36), perhaps because he merely compared Sean's working with his own and mistakenly assumed that  had the same meaning in both cases. In fact, Sean's  was meant to represent  $B$ , the length of the triangle's base, while Rhys's  was calculated from the  part of the area formula, .

While Rhys moved on to Level 2, Sean persisted with the Level 1 calculation in the hope that his error might be revealed. Consequently the next few minutes were filled with confusion as neither realised that they were now working on different Levels of the snowflake. Only when Sean verbalised the key line in his working was Rhys able to identify the simple error that had hindered Sean's progress:

65. S: Yeah, then look—I got (Rhys not looking) Oi! Oi oi oi! (Rhys looks) All that, over two, right.  
 66. R: (Finally picking up Sean's error) No, it's already over two. (*Assessment—strategy execution*)  
 67. S: Is it?  
 68. R: (Matter-of-factly) You've done a half times a third to get the one sixth, haven't you.  
 69. S: Yeah.  
 70. R: So the half is instead of having it over two. Otherwise you'd have a third times that over two. And a third over two just cancels to become one sixth. (Shows Sean the equivalent point in his own working)

Rhys then guided Sean through his own calculation of the Level 1 area, until the teacher reconvened the class to discuss the section of the text they had been investigating.

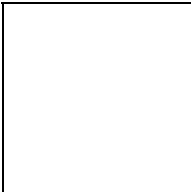
### *Grounds for Metacognitive Failure*

Analysis of the full transcript revealed that the predominant metacognitive function of the students' dialogue was to assess strategy execution, with nine assessments of this type being made by Rhys and five by Sean (out of a total of 19 metacognitive acts in a transcript comprised of 87 conversational Moves). And yet this metacognitive monitoring did not immediately help the students to raise the *Error detection* "red flag" and take corrective action. Instead, both students displayed *metacognitive blindness* in overlooking the error. This mistake was mentioned on five separate occasions (Moves 35 and 37 above; also Moves 43, 49 and 60, not shown here), and yet it was not recognised as such until Move 66, when Rhys casually pointed out that *it's already over two*. Interestingly, the teacher-produced text provided access to a result (i.e. the areas of each successive Level of the snowflake) that acted as a reference point for the students' own area calculations, and thus alerted them to the possible existence of errors in their working. However, this knowledge did not necessarily help the students to locate their own or a partner's error.

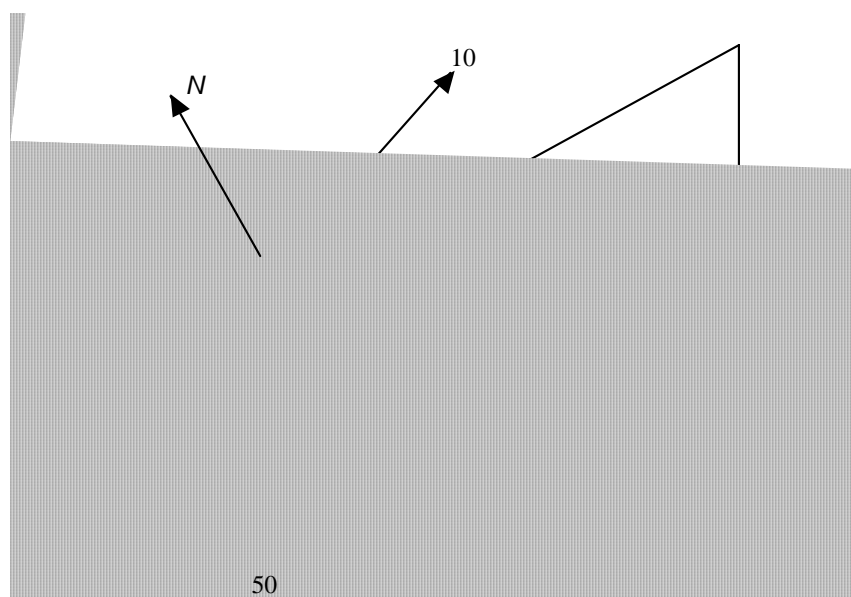
An analysis of metacognitive failure should also attempt to identify opportunities that may have arisen for errors to be corrected. Let us return to the point when Sean's mistake first appeared, when he disagreed with Rhys's calculation for the Level 1 area (Move 27). Rhys's response was to brush off Sean's criticism; indeed, he appeared more interested in ensuring his own working was correct than in engaging with Sean's concerns. Later, instead of challenging Rhys directly, Sean sought feedback on his own calculation of the added triangle's area (Move 35). Rather than taking the time to examine his partner's working and give a reasoned critique, Rhys simply acquiesced to Sean's mistaken interpretation. If either of these moments had been exploited by challenging, rather than accepting, each other's thinking, the boys' metacognitive blindness may have been cured.

### Metacognitive Vandalism – Forces

Another kind of metacognitive failure occurs when students do not respond appropriately to a “red flag”. The label *metacognitive vandalism* can be applied to situations where students deal with an impasse, such as lack of progress, by changing the problem conditions to suit the knowledge they think they can apply. An example of such a situation is analysed below. The transcript comes from a lesson close to the end of Year 11, which was part of a unit on dynamics. The students already had some experience with problems applying Newton’s

Second Law of motion  when all forces are acting in the same straight line, and with resolving forces into components and finding the resultant of forces acting at a point.

To introduce Newton’s Second Law in vector form, the teacher posed a problem involving a body on an inclined plane. Without any preamble, he drew a diagram of a 5 kg body on a plane of inclination  $45^\circ$  to the horizontal. The body was attached to a rope that applied a force acting at an angle of  $20^\circ$  to the plane (see Figure 6 for the diagram and a worked solution). The teacher simply asked students to “describe the motion of that body”. This had the effect of establishing a problem-oriented context for the new work, so that students were expected to draw on their existing knowledge and construct a solution method by themselves instead of passively imitating a standard procedure demonstrated by the teacher. Because the students had never dealt with a problem of this type before, they needed to work together to identify relevant knowledge and monitor their efforts to apply their knowledge to the problem. This need was clearly expressed by one student (Sean), whose immediate response to the teacher’s task instructions was to ask “Can we collaborate?”, to which the teacher replied “Of course – just not with me!”



The forces acting on the body of mass 5 kg are the tension in the rope 10 N, the weight 50 N (taking  $g$  as  $10 \text{ m/s}^2$ ), and (assuming the body stays on the plane) the normal reaction  $N$ , acting in the directions shown in the diagram above.

Resolving normal to the incline, we have

$$\begin{aligned} N + 10 \sin 20^\circ - 50 \cos 45^\circ &= 0 \\ N + 3.4 - 35.4 &= 0 \\ \therefore N &= 32 \end{aligned}$$

Resolving up the incline, and representing the resultant force as  $R$ , we have

$$\begin{aligned} R &= 10 \cos 20^\circ - 50 \sin 45^\circ \\ &= 9.4 - 35.4 \\ &= -26 \end{aligned}$$

Let the acceleration be  $a \text{ m/s}^2$ . From  $R = ma$ , we have

$$\begin{aligned} -26 &= 5a \\ \therefore a &= -5.2 \end{aligned}$$

Hence the body moves down the plane with acceleration  $5.2 \text{ m/s}^2$ .

*Figure 6. Solution to the Forces problem*

While most of the class relied only on their peers for help, some students did “collaborate” with the teacher by asking for clarification of the problem conditions, or for confirmation of their strategy. In these discussions with individual students, the teacher provided only enough information to satisfy the student’s immediate need, so that assistance was contingent upon progress already made. As a result of these private consultations, the teacher judged it necessary to intervene at two points to supply hints to the whole class. The

transcript analysed here records the conversations of three students – Rhys (who featured in the Koch Snowflake example), Alex, and Dylan – before these teacher interventions occurred, and focuses on the first sixteen minutes in which the students worked on the problem.

They faced two difficulties in this time – identifying the forces acting on the body, and choosing appropriate axes for resolving the forces into components. The first difficulty was associated with metacognitive vandalism, in that the students attempted to impose their knowledge of frictional forces on a problem in which friction was to be disregarded. Fortunately, this instance of vandalism did not irretrievably damage the solution process, and the relevant forces were eventually identified. The second difficulty, arising from the students' choice of horizontal and vertical axes when the problem conditions called for axes to be taken parallel and perpendicular to the incline, was not overcome within the time frame for the analysis. While not classed as vandalism, this procedural error clearly interfered with students' attempts to solve the problem.

Signs of the students' first difficulty appeared almost immediately, when Alex mused "I wonder if this *is* the same thing we do in Physics?", and Rhys asked the teacher "Mr G, is there friction?" The boys had already tackled similar problems in Physics lessons, with the added complication of rough surfaces and frictional forces. Consequently, despite the teacher's repeated instructions to ignore friction, the students found it difficult to separate their Mathematics knowledge from their Physics knowledge, and "pretend" they knew nothing about how this force works. In addition, it seems likely that they knew less about friction than they thought they did, since there is evidence later in the transcript which suggests that they believed the normal reaction need only be considered when friction is called into play.

The boys worked separately on their initial calculations and then checked their results with each other before proceeding further. When they arrived at different answers, it became

clear that Alex had chosen horizontal and vertical axes and found components of  $10 \cos 65^\circ$  (4.23) and  $10 \sin 65^\circ$  (9.06) respectively, while Rhys had resolved the 10 N force into components parallel and perpendicular to the incline.

19. R: So what did you guys get for the two components?  
 20. A: Four and nine. (i.e.  $10 \cos 65^\circ$  and  $10 \sin 65^\circ$ ) Although that's just the direction of (inaudible).  
 21. R: What? Are you sure? (*Assessment—strategy execution*)  
 22. A: Four and nine. Ten cos sixty—  
 23. R: Oh I used cos twenty! Bit stupid! (*Assessment—strategy execution*)  
 24. A: So you can work out the velocity. (Correcting slip of the tongue) Oh, you can work out the acceleration. (*New Idea*)  
 25. R: (Changing his working) Sixty-five.

Although Rhys's approach was the more appropriate, he unquestioningly amended his working to match Alex's calculations, and an opportunity to establish a more productive solution path was lost.

The boys then retreated to a lengthy Analysis episode in which they tried to reconcile their knowledge of frictional forces with the teacher's instruction to ignore friction in this problem. So intent were they on identifying forces not originally marked on the teacher's diagram that they overlooked the effect of the 10 N force in all their subsequent calculations. Adding to their concern over which forces to include in their representation of the problem was their inappropriate choice of axes, which made it more difficult for them to make decisions about the direction of the body's motion. The students proposed many New Ideas in this episode, but since few were clarified, elaborated, or assessed, no meaningful progress was made on the problem. Amidst the plethora of suggestions was one that could have improved the students' chances of solving the problem (Move 40) if it had been considered more carefully:

38. R: [...] What I reckon—If we want to know displacement—  
 39. A: Hey! We know that because—  
 40. R: —we take the direction of the force in the same direction as the displacement. (*New Idea*) So we know the force ... that's um cos, cos ...  
 41. D: Four point two three.  
 42. R: Ten cos sixty-five—

Rhys's New Idea proposed a useful strategy – that of finding components parallel to the incline – yet neither he nor his partners explored the idea further, and he returned to his earlier, unhelpful, calculation based on horizontal and vertical axes (Move 42).

At this point, Rhys took the solution process even further off track by reintroducing the possibility of a frictional force (Move 44). Here the boys may have been referring to Physics problem situations represented in diagrams that showed the weight of the body, a frictional force opposing its motion, and the normal reaction (and, possibly, the tension in a rope attached to the body) (Move 45):

44. R: (drawing on Alex's diagram) Has it got any force going that way? (Motioning parallel to, and down, the plane) (*New Idea*)
45. A: That's the thing, I don't think it does. Like in Physics, it's got one going that way, and one going that way, and one going that way. But I don't know if we have to take into account – (*Assessment—understanding*)
46. R: —friction.
47. A: He said not to worry about friction. But, do we do that one? (*New Idea*) (Pause) Because we got to work out whether it's going up or down.
48. R: Besides that, we know that the work done by this –
49. A: I reckon (inaudible). We got to take into account that one. (Motions perpendicular to plane)
50. R: But how big is it?
51. A: Well we can work that out. It's just like, fifty ... oh yeah. It's just like fifty sine ... um ... fifty cos ... (*New Idea*)

Alex refused to be distracted by Rhys's fixation with friction (Move 47), but in doing so he faced another apparent difficulty – should he take into account the normal reaction of the plane on the body? His hesitancy in including this force suggests that the students believed it only acts when a body is moving, or tending to move, on a *rough* surface; that is, when the reaction of the surface on the body has two components, a frictional force opposing the motion and a normal force at right angles to the surface. By this reasoning, if friction is to be ignored, then so is the normal reaction. Eventually, Alex decided that the normal reaction had to be considered (Moves 49 and 55), and Dylan seemed to be in agreement (Move 56):

55. A: I think, you must need that force, surely.
56. D: The thing we don't know, that one's a factor as well (points to Alex's diagram). There's a force from that, and we have to compare the two ... (*New Idea*)

After persisting with the problem for a further five minutes, Rhys's growing frustration at their lack of progress led him to bid, unsuccessfully, for assistance from the teacher:

83. R: Mr G, can you help us with this? (Teacher urges students to think more about it.) We've thought about it!

Six fruitless minutes later, Rhys again decided to approach the teacher for help, and he quizzed him relentlessly on the missing frictional force. On several occasions the teacher repeated that there was *no* friction, and he also confirmed that the normal reaction must be taken into account. The teacher also hinted that horizontal and vertical axes were not appropriate, and decided at this point to reconvene the class in order to clarify the choice of axes.

#### *Grounds for Metacognitive Failure*

In analysing the metacognitive function of the students' dialogue, it emerged that Alex took the lead in proposing New Ideas (nine, compared with four from Dylan and three by Rhys, in a transcript of 121 conversational Moves) and in keeping track of progress (14 assessments, six of which were concerned with strategy appropriateness; compared with three and five assessments offered respectively by Dylan and Rhys). While this level of metacognitive activity was comparable with that observed in other, successful, problem solving sessions, a purely quantitative approach does not necessarily reveal the reasons for success and failure. In the Forces problem, the students' failure to resolve their difficulties can be attributed to poor metacognitive decisions, which distorted the problem to such an extent that the solution could not be obtained.

Some of these poor decisions were classed as *metacognitive vandalism*, manifested in the students' tendency to "identify" a frictional force that played no part in the motion of the body. Debate over the presence or absence of friction also included questions as to whether the normal reaction should be considered, and if so, how its magnitude should be calculated. Many metacognitive decisions involved passively accepting, rather than challenging, those



New Ideas and Assessments that hindered progress (e.g. Move 21, when the choice of axes could have been changed), and ignoring, rather than endorsing, others that were potentially useful (e.g. Moves 49, 51, 56, when approaches for finding the normal reaction could have been discussed).

### **Metacognitive Mirage – Combinations**

In an earlier section of this paper, spurious “red flags” were likened to *metacognitive mirages* that mislead students into seeing errors that did not exist. A less obvious form of mirage materialises when students are unsure what they “see” and are unable to make any judgment about the correctness of their work. Such a case is illustrated in a further example of metacognitive failure, where students suspected they had obtained an anomalous result.

This example comes from another Year 11 lesson in the early stages of a unit of work on combinatorics. Knowing that he would be unavoidably absent for this lesson, the teacher had set a series of problems which would give the students their first opportunity to apply their newly gained knowledge of combinations. These problems were contained in a teacher-prepared handout that also included explanations and worked examples, and served as the students’ sole text for the topic. The target students are Alex, Sean, Rhys and Dylan (all of whom have appeared in classroom vignettes presented previously). Figure 7 shows the first problem on which they worked (Question 19 in the problem set), together with model solutions.

- 
- 19** How many selections of five cards can be made from a pack of 52 playing cards so that there are:  
**a** at least three aces?      **b** three hearts?      **c** at least one heart?

**Solutions**

**a** Possible hands could contain either three or four aces.

$$\begin{aligned} \therefore \text{number of selections} &= \binom{4}{C_3} \times \binom{48}{C_2} + \binom{4}{C_4} \times \binom{48}{C_1} \\ &= (4 \times 1128) + (1 \times 48) \\ &= 4512 + 48 \\ &= 4560 \end{aligned}$$

**b** The hand must contain three hearts and two non-hearts.

$$\begin{aligned} \therefore \text{number of selections} &= \binom{13}{C_3} \times \binom{39}{C_2} \\ &= 286 \times 741 \\ &= 211\,926 \end{aligned}$$

**c** The hand may contain either one, two, three, four, or five hearts. (A simpler method is to find the number of hands with no hearts and subtract this from the total number of five card hands.)

$$\begin{aligned} \therefore \text{number of selections} &= \binom{13}{C_1} \times \binom{39}{C_4} + \binom{13}{C_2} \times \binom{39}{C_3} + \binom{13}{C_3} \times \binom{39}{C_2} + \binom{13}{C_4} \times \binom{39}{C_1} + \binom{13}{C_5} \\ &= (13 \times 82\,251) + (78 \times 9139) + (286 \times 741) + (715 \times 39) + 1287 \\ &= 2\,023\,203 \end{aligned}$$

*Figure 7. Combinations problems*

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After reading the stem to Question 19 and making the observation that there were  ${}^{52}C_5$  hands in total, Dylan immediately recognised that this would give too large a number for part (a) of the question, which imposed the constraint of having at least three aces in the hand. Nevertheless, all three boys used the  ${}^nC_r$  buttons on their calculators to gain a feel for the problem and discovered that  ${}^{52}C_5$  does indeed represent a very large number of hands (2,598,960).

Although they had identified the relevant information in the problem, the students struggled to formulate a strategy for taking account of the specified selection of at least three aces.

13. A: How do you do it with three aces? (No response) Maybe we have to work out the probability of aces or something. (*New Idea*)
14. S: Well that's ... four out of fifty-two. That's one out of thirteen chances you've got an ace.
15. A: (doubtfully) Yeah, but how do you work out these three aces?
16. D: No, you've got five cards, so it's only fifty-two, ah ... fifty-two C— (*New Idea*)
17. A: Ohh! Do C two, that's how many won't have— (*New Idea*)
18. D: Yeah, and you *got* to have—
19. A: (simultaneously) —a certain three cards.

20. D: That certain three cards, will be your aces.

Alex and Dylan proposed that  ${}^{52}C_2$  might represent the number of five card hands *without* three aces (Moves 17-20), foreshadowing an approach based on mutually exclusive operations and the addition principle. (Note that they still had not come to grips with the “at least” condition.) Despite their initial enthusiasm for this strategy, it soon became apparent that the boys had no way of knowing whether or not they were on the right track:

24. D: Aarrh I don’t know if I’m doing it right or I’m doing it wrong! (*Assessment—strategy appropriateness*)

Before long, the boys abandoned part (a) of the problem and acknowledged that they were stuck on Question 19 as a whole. However, they were not yet willing to give up completely.

40. D: So how do you do it?  
 41. S: If we had an answer – an answer sheet— (*New Idea*)  
 42. D: Yeah, we could figure it out. (*New Idea*)  
 43. A: You could always think about it without the C rule. And go like, OK, for hearts you’ve got, however many choices, and, the next choice you’ve got however many choices, the next choice you’ve got ... (*New Idea*)  
 44. D: You got a quar—ter ... (hesitation, draws out this word) A quarter of fifty. (*New Idea*)  
 45. A: (Not listening to Dylan) Think about it the long way. Hey, is there an example somewhere? (*New Idea*) (Checks quickly through handout, overlooks Example 12.)

Here the boys considered two potentially useful strategies for dealing with impasses such as the one they faced – working backwards from the answer (Moves 41 and 42), and looking at a similar problem (Move 45). Unfortunately, they were unable to take advantage of either strategy, since the teacher-prepared handout did not provide answers to the problems, and they overlooked a worked example in the text that might have provided some clues. While Alex continued to hunt for a helpful example in the text, Dylan moved on to Question 19 (b), and began hesitantly to reason out a strategy which would lead him to the correct answer (Move 44).

There is evidence here that Dylan was beginning to develop a general understanding of how the choices of cards can be constrained. In the case of Question 19 (b), if a five card hand is to contain three hearts, then the hearts are selected from only one suit (a quarter of fifty-two cards), not the full pack.

54. D: (writes) This is thirteen out of fifty-two ... is ... hearts. So what would you go? Would you go, thirteen ... C ... (inaudible). (*New Idea*)
55. S: So are you still trying to work out something for—?
56. A: No, I'm just going to leave that for now. And wait until *he* comes up with—
57. S: Leave Question 19 altogether?
58. A: Yeah, I don't know how to do it. (*Assessment—understanding*)

Dylan now became absorbed with completing Question 19 (b), and he worked in silence while his friends considered their next move. Eventually his persistence was rewarded:

63. D: (to himself) So should we go ...? I know, I've figured it out! I've figured it out! (*Assessment—Understanding*) (Pause) Multiply that by ... what's the (inaudible)? It's thirteen take fifty-two. (*New Idea*)
64. R: Thirteen *take* fifty-two? (*Assessment—strategy execution*)
65. D: Sorry! Fifty-two take thirteen. Thirty-nine, yeah. (Quietly, to himself) Thirty-nine C two.
66. A: (Reading Question 20) How many committees of five ...?
67. D: (to himself, using calculator) Two hundred and eighty-six times ... seven hundred and forty-one! (Sounds surprised)
68. R: Is that for (a) or (b)?
69. D: That's for (b)! I think (a)'s wrong actually, but anyway ... (*Assessment—accuracy of result*) (Long pause, writing. Goes on to Question 19c.) C ... one ... C four ... is thirteen times ... eight thousand two hundred, no, eighty-two thousand two hundred ... (Long pause, writing. Responds to inaudible question, from student off camera.) Well we don't have any answers, so we don't even know if we're right. (*Assessment—accuracy of result*) (Continues working) Thirteen C two ... (now doing Question 19c)

Although Dylan did not verbalise all his working, it is clear that he was pursuing the correct approach to solving parts (b) and (c) (see Moves 67 and 69, and Figure 7). Nevertheless, the lesson ended with all students still at a loss to know whether they had found the correct way to approach these problems.

### *Grounds for Metacognitive Failure*

Analysis of the metacognitive function of the students' dialogue points to their inability to make valid judgments about their strategies and answers. Three of the four assessments made of their understanding revealed that they did *not* understand how to approach Question

19 (Move 58, also Moves 26 and 27 not shown here). Similarly, Dylan was *unsure* of the appropriateness of the strategy he implemented for Question 19 (a) (Move 24), and he later expressed his *doubts* as to the accuracy of his answers for Questions 19 (a) (which was indeed incorrect) and 19 (c) (Move 69).

As with the transcripts examined previously, opportunities existed for the boys to dispel this metacognitive mirage. For example, Sean, Alex and Rhys all overlooked the significance of Dylan's proposal in Move 44 ("A quarter of fifty"), and of his related query in Move 46, both of which indicated that he now realised the hearts must be selected from one suit, not the whole pack of cards. Instead of asking Dylan to explain *why* he wanted to find the number of cards in a quarter of the pack, his friends merely responded to the superficial aspects of his question.

Similarly, Dylan was left to answer his own tentative question "Would you go, thirteen C ...?" (Move 54). Surprisingly, even Dylan's delight in having worked out how to do the problem attracted no interest (Move 63). No one asked him to explain what he had done, but neither did Dylan ask anyone what they thought of his strategy. Even Dylan's negative Assessments (Move 69) regarding the accuracy of his answers could have provided an opportunity to rehearse his strategy to an audience, if his friends had bothered to ask "Why? – what did you get?", or if Dylan had asked them to check his working. In contrast to the two previous examples, where a lack of *challenges* to justify unhelpful New Ideas and Assessments contributed to metacognitive failure, analysis of the Combinations transcript demonstrates the need to clarify and *endorse* New Ideas which are potentially helpful so that their usefulness becomes apparent to all participants in the interaction.

### **Discussion**

This paper has been concerned with metacognitive failure, and the circumstances under which peer collaboration – often considered to provide a natural context for making thinking

visible, and open to critique and refinement (Teasley, 1997) – does not lead to a mathematically successful outcome. Previous research on metacognitive aspects of individual students' mathematical thinking has suggested that failure is virtually guaranteed by poor metacognitive decisions (e.g. Schoenfeld, 1985), and there is some evidence that these decisions can be adversely affected by peer interactions in small group problem solving (e.g. Stacey, 1992). The results presented here extend and qualify these findings by identifying different forms of metacognitive failure, and also by highlighting ways in which students' verbal interactions may influence each other's thinking. The present study took a naturalistic, long term approach to studying metacognitive activity in authentic classroom contexts, and in this respect it differs from much of the previous research on the role of metacognition in mathematical thinking.

It was argued that existing frameworks for analysing problem solving behavior suffer from a number of limitations, three of which were addressed by the research reported here. First, a theoretical model of metacognitive processes in problem solving was synthesised from the episode-based frameworks of Schoenfeld (1985) and Artzt and Armour-Thomas (1992), to identify specific monitoring and regulatory actions that would be appropriate at different stages of the solution process. Second, the notion of metacognitive “red flags” was developed to highlight the difference between routine monitoring of progress and the more deliberate action needed when specific difficulties, such as an error, lack of progress, or an anomalous result, are recognised. Finally, the “red flag” model was further elaborated to identify different circumstances associated with metacognitive failure, described by the metaphors of blindness, vandalism, and mirage. These scenarios were illustrated by problem solving sessions where students overlooked a “red flag” indicating a calculation error (blindness), responded to a lack of progress “red flag” by imposing an irrelevant conceptual structure on

the problem (vandalism), and imagined an anomalous result “red flag” in mistakenly rejecting a correct answer (mirage).

While analysis of failure scenarios revealed quantitatively similar levels of metacognitive activity – in the form of New Ideas and Assessments of the state of the solution – to that observed in successful problem solving (see Goos, 2000b; Goos & Geiger, 1995), the descriptive metaphors directed attention to the quality of students’ metacognitive judgments. In particular, failure was likely if students passively accepted unhelpful suggestions, or ignored potentially useful strategies proposed by peers. That is, students’ responses to their partners’ New Ideas and Assessments played a critical role in shaping problem solving outcomes. Thus the analysis highlights the interactively constituted nature of metacognitive activity in small group problem solving, and allows metacognition to be conceptualised in terms of social practices as well as individual mental processes. This dual focus on the (meta)cognitive and social aspects of students’ mathematical activity may go some way towards building a theory of acting-in-context that focuses on “human decision making in complex, dynamic social settings” (Schoenfeld, 1999, p. 6). At the level of practice, it may inform teachers’ attempts to foster problem solving and communication, both of which are considered essential to mathematics learning (National Council of Teachers of Mathematics, 2000).

### **Conclusion and Implications**

This study did not initially set out to investigate metacognitive failure. In fact, it was most common to find that students were successful, rather than unsuccessful, in their collaborative problem solving efforts. Also, unlike much previous research on metacognition in mathematical problem solving, the study was not designed to manipulate problem solving outcomes by assigning students to experimental and control groups, or to specific groupings within the classroom, on the basis of ability or task difficulty. Neither were students’

interactions constrained by teacher-imposed rules governing how they should act or speak. Instead, the teacher communicated his expectations regarding discussion and interaction in more subtle ways, for example, by encouraging students to consult a neighbour if they were “stuck” on a task.

In addition, the teacher actively modelled metacognitive habits of mind during whole class instruction by asking students to evaluate strategies for tackling tasks, locate and correct any inadvertent calculation errors, and decide whether the answers made sense. This culture of collaborative inquiry also seemed to facilitate in students a positive attitude towards group work and a willingness to share ideas, both of which are necessary for effective group functioning (see Artzt & Armour-Thomas, 1997). Indeed, the teacher insisted that students exhaust all avenues of assistance before turning to him as a last resort. (The Combinations lesson represents an extreme case as the teacher was absent altogether.) However, this raises questions as to the timing of teacher interventions that may help students avoid metacognitive failure.

For example, when students worked on the Forces problem, the teacher sometimes withheld assistance, judging that students who asked him for help had not fully exploited their own, and their peers', resources. At other times, students deliberately refrained from asking the teacher for help, preferring instead to persevere with their own endeavours. Whether this is interpreted as inappropriate persistence or endorsement of the teacher's personal values of sense-making, students themselves are clearly active agents in selecting and structuring the assistance they obtain. Perhaps the most significant implication for teachers is the need to establish participation structures that facilitate students' active engagement with each other's thinking. In particular, holding students accountable for explaining and justifying their thinking may afford similar forms of discourse and reasoning when students work collaboratively with peers.



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