FIR(q) FILTER DESIGN WITHOUT THE LINEAR PHASE CONSTRAINT

D8.5

Peter J. Kootsookos[†] Robert R.Bitmead[†] Michael Green[‡]

† Systems Engineering Department, R.S. Phys. Sci., Australian National University, GPO Box 4, Canberra 2601, Australia.
‡ Department of Electrical Engineering, Imperial College of Science, Technology and Medicine, Exhibition Road, London SW7 2BT, UK.

ABSTRACT

This paper presents a new approach to the problem of designing a finite impulse response filter of specified length, q, which approximates in uniform frequency (L_∞) norm a given desired (possibly infinite impulse response) filter transfer function. We derive an algorithm-independent lower bound on the achievable approximation error and then present an approximation method which involves the solution of a fixed number of all-pass (Nehari) extension problems and so is called the Nehari Shuffle. An upper bound on the approximation error for the algorithm is derived. As this bound is calculable *a priori* the length of filter required to satisfy a given maximum error can be found before designing the filter. Examples indicate that the method closely approaches the derived global lower bound. We compare the new method with the Parks-McClellan (Remez Exchange) algorithm in some examples.

1 FIR FILTER DESIGN

We present a method that is a direct approach to the problem of approximating a desired IIR transfer function by an FIR(q) design with an error criterion being the maximum magnitude of the error frequency response over all frequencies. Our algorithm has the following properties:

- Upper and lower bounds on the approximation error $E(\omega)$:

$$E(\omega) = \max_{\omega \in (-\pi,\pi]} |G(e^{j\omega}) - \hat{G}(e^{j\omega})| \qquad (1)$$

  are calculable *a priori*.

- The algorithm is given in a state-space form and so is amenable to direct, numerically robust implementation.

- The state-space algorithm description allows direct extension of the algorithm to the multi-input/multi-output (MIMO) case.

Due to the algorithm's extensive use of a concept called the Nehari Extension (see Section 2), we have named the algorithm the Nehari Shuffle.

The remainder of the paper is set out as follows. Sections 2 and 3 contain definitions and some required background material. A statement of the FIR(q) approximation problem and a lower bound on the error associated with FIR(q) approximation is given in Section 4. Section 5 describes the Nehari Shuffle. In Section 6 we derive an upper bound on the approximation error of the algorithm and then, in Section 7, we give some examples using the algorithm which show that this bound is adhered to. The examples also show that, for larger filter lengths, the Nehari Shuffle gives close to the globally minimal error. We also compare our algorithm with the Parks-McClellan algorithm [1]. Finally, Section 8 summarizes the results of the paper.

2 DEFINITIONS

In this paper, we use the following notation and definitions.

The integer, real and complex numbers are denoted **Z**, **R** and **C**, respectively. The transfer function $G(\sigma)$ with *minimal* state-space realization $G(\sigma) = D + C(\sigma I - A)^{-1}B$ will be written as $G(\sigma) = (A, B, C, D)$. We denote by FIR(q) the set of all $G(z)$ which may be written

$$G(z) = \sum_{i=0}^{q-1} g_i z^{-i}$$

where $g_i \in \mathbf{C}$. Given the stable discrete system $G(z) = (A, B, C, D)$ (where $A \in \mathbf{R}^{n\times n}, B \in \mathbf{R}^{n\times p}, C \in \mathbf{R}^{m\times n}$ and $D \in \mathbf{R}^{m\times p}$) then the controllability and observability gramians are given by

$$P = \sum_{k=0}^{\infty} A^k BB^* A^{*k} \qquad (2)$$

$$Q = \sum_{k=0}^{\infty} A^{*k} C^* C A^k \qquad (3)$$

where A^* denotes the Hermitian conjugate of A. A realization of $G(z)$ is called *balanced* [2] if $P = Q = \Sigma$ and Σ is diagonal.

The *Hankel singular values* of G (denoted $\sigma_i(G)$) are given by

$$\sigma_i(G) = \sqrt{\lambda_i(PQ)},\ 1 \le i \le n$$

(where $\lambda_i(A)$ is the i^{th} eigenvalue of A) and are also the singular values of the doubly-infinite (but finite rank) Hankel matrix associated with G [3]. For simplicity, we shall assume that the $\sigma_i(G)$ are distinct (which is generally the case), and that

$$\overline{\sigma}(G) \triangleq \sigma_1(G) > \sigma_2(G) > \cdots > \sigma_n(G) \triangleq \underline{\sigma}(G).$$

We define $\| G \|_H = \overline{\sigma}(G)$ to be the *Hankel-norm* of G. For an excellent treatise on Hankel-norms and Hankel singular values, see [3].

In Section 5 we shall describe the Nehari Shuffle algorithm. To do so, we need to define some transfer function operations. Let G be a stable transfer function (of McMillan degree N), which may be expressed uniquely as

$$G = \sum_{i=0}^{q-1} g_i z^{-i} + z^{-(q-1)} \sum_{j=1}^{\infty} g_{j+q-1} z^{-j} = G^h + z^{-(q-1)} G^t.$$

We then use

$\mathcal{E}G$ to denote formation of the Nehari Extension of G;
$\mathcal{H}_q G$ to denote extraction of G^h from G;
$\mathcal{T}_q G$ to denote extraction of G^t from G.
If $G = (A, B, C, D)$ then $\mathcal{T}_q G = (A, B, CA^{q-1}, 0)$.
Note that $\mathcal{T}_q G$ is of McMillan degree N or less;
$\mathcal{R}G$ to denote the reflection operator $\mathcal{R} : G(z) \mapsto G(z^{-1})$.
If $G = (A, B, C, D)$ then
$\mathcal{R}G = (A^{-1}, A^{-1}B, CA^{-1}, D - CA^{-1}B)$;
$\mathcal{S}_q G$ to denote the shift operation $z^{-(q-1)}G$.
$\mathcal{S}_q^{-1} G$ denotes $z^{q-1}G$.

3 THE NEHARI PROBLEM

The Nehari problem [4] may be stated as follows:

Given $G(z)$, a rational function analytic in $\{|z| > \rho, \ \rho < 1\}$ (i.e. possessing a power series in z^{-1} convergent on the unit circle), find $F(z)$, a rational function analytic in $\{|z| < r, \ r > 1\}$ (i.e. possessing a power series in z convergent on the unit circle) such that

$$\max_{\omega \in (-\pi,\pi]} |G(e^{j\omega}) - F(e^{j\omega})|$$

is minimized. The $F(z)$ so found is called the Nehari Extension of $G(z)$.

The solution to this problem was provided by [5] and Glover[1] [3, Theorem 6.3] gives an explicit state-space expression for the equivalent *continuous time* problem. The discrete solution uses the bilinear transformation and the algorithm of [3].

The main points to note about $F(z)$ (the Nehari extension of $G(z)$) are:

1. $F(z)$ is the closest *anti-causal* sequence to the causal $G(z)$.
2. If $G(z)$ is of McMillan degree N, then $F(z)$ is of degree $N-1$.
3. For $G(z)$ of McMillan degree N,

$$\sigma_i(\mathcal{R}F) = \sigma_{i+1}(G), \ 1 \le i \le N-1 \tag{4}$$

4. $\min_F \max_\omega |G(e^{j\omega}) - F(e^{j\omega})| = \overline{\sigma}(G)$.

4 THE FIR(q) APPROXIMATION PROBLEM

The FIR(q) approximation problem may be stated as follows:

Given $G(z)$, a rational transfer function analytic in $\{|z| > \rho, \ \rho < 1\}$ (i.e. possessing a power series in z^{-1} convergent on the unit circle), find $\hat{G}(z)$ such that $\hat{G} \in FIR(q)$ and

$$\max_{\omega \in (-\pi,\pi]} |G(e^{j\omega}) - \hat{G}(e^{j\omega})|$$

is minimized.

We present in the following lemma a lower bound on the accuracy with which a given transfer function may be approximated by an FIR(q) system.

Lemma 1. (Global Approximation Error Lower Bound) *Given $G(z)$, a rational transfer function analytic in $\{|z| > \rho, \ \rho < 1\}$ expressed as*

$$\begin{aligned} G(z) &= \sum_{i=0}^{q-1} g_i z^{-i} + z^{-(q-1)} \sum_{j=1}^{\infty} g_{j+q-1} z^{-j} \\ &= \mathcal{H}_q G(z) + \mathcal{S}_q \mathcal{T}_q G(z) \end{aligned} \tag{5}$$

then

$$\min_{\hat{G} \in \mathrm{FIR}(q)} \max_{\omega \in (-\pi,\pi]} |G(e^{j\omega}) - \hat{G}(e^{j\omega})| \ge \overline{\sigma}(\mathcal{T}_q G). \tag{6}$$

The details of the proof are omitted.

Note that this is a *global* lower bound and is irregardless of the algorithm used to produce $\hat{G}$. Lemma 1 says nothing about whether or not the lower bound is achievable.

We now present a result on the approximation error involved in approximation of G by $\mathcal{H}_q G$.

Lemma 2. (Truncation Approximation Error Upper Bound) *Given $G(z)$, a rational transfer function (of McMillan degree N) analytic in $\{|z| > \rho, \ \rho < 1\}$ expressed as*

$$\begin{aligned} G(z) &= \sum_{i=0}^{q-1} g_i z^{-i} + z^{-(q-1)} \sum_{j=1}^{\infty} g_{j+q-1} z^{-j} \\ &= \mathcal{H}_q G(z) + \mathcal{S}_q \mathcal{T}_q G(z) \end{aligned} \tag{7}$$

then

$$\max_{\omega \in (-\pi,\pi]} |G(e^{j\omega}) - \mathcal{H}_q G(e^{j\omega})| \le 2 \sum_{i=1}^{N} \sigma_i(\mathcal{T}_q G) \tag{8}$$

Proof. We know (from the definition) that

$$\mathcal{S}_q \mathcal{T}_q G(z) = G(z) - \mathcal{H}_q G(z)$$

and also

$$|G(e^{j\omega})| = |z^n G(e^{j\omega})|$$

for all ω and all $n \in \mathbf{Z}$. So

$$|G(e^{j\omega}) - \mathcal{H}_q G(e^{j\omega})| = |\mathcal{S}_q \mathcal{T}_q G(z)| = |\mathcal{T}_q G(z)|$$

From [3, Corollary 9.3] (and suitable application of the bilinear transformation) we have that

$$\max_{\omega \in (-\pi,\pi]} |G(e^{j\omega})| \le 2 \sum_{i=1}^{N} \sigma_i(G) \tag{9}$$

which, when applied to $\mathcal{T}_q G$, gives the required result. ■

The system $\mathcal{H}_q G$ is readily obtained from G and, in a sense, this result gives an upper bound on the FIR(q) approximation error. This overbound is easily and exactly computable, as opposed to the effort required to calculate $\max_{\omega \in (-\pi,\pi]} |\mathcal{T}_q G(e^{j\omega})|$ exactly. For any particular G, the overbound is not necessarily achieved.

5 THE NEHARI SHUFFLE

We now present our new algorithm for FIR(q) filter approximation.

Given G, find $\hat{G} \in$ FIR(q).

The algorithm to find $\hat{G}$ in FIR(q) given G proceeds as follows (using the definitions of Section 2):

1. Initialize $i = 1$.
2. $\hat{G}_i = \mathcal{H}_q G$.
3. $G_i = G$.
4. Repeat

$$\begin{aligned} G_{i+1} &= \mathcal{R}\mathcal{E}\mathcal{T}_q G_i, & (10) \\ \hat{G}_{i+1} &= \hat{G}_i + \begin{cases} \mathcal{S}_q \mathcal{R} \mathcal{H}_q G_{i+1} & \text{for } i \text{ odd,} \\ \mathcal{H}_q G_{i+1} & \text{for } i \text{ even.} \end{cases} & (11) \\ i &= i + 1. & (12) \end{aligned}$$

Until $G_i = 0$.

5. So the final approximant is

$$\hat{G} = \hat{G}_{N+1} = \mathcal{H}_q G_1 + \mathcal{S}_q \mathcal{R} \mathcal{H}_q G_2 + \mathcal{H}_q G_3 + \ldots \tag{13}$$

Note that the procedure above may be stopped after any number of steps rather than continued to completion. The penalty for doing this is an increased upper bound on the approximation error.

[1] Glover's paper deals with the more general problem of optimal Hankel-norm approximation of which the Nehari Problem is a special case.

6 NEHARI SHUFFLE ERROR BOUND

We give an upper bound on the Nehari Shuffle approximation error in the following theorem.

Theorem 3. (Nehari Shuffle Approximation Error Upper Bound)
Given $G(z)$, then the $\hat{G}(z) \in FIR(q)$ constructed as detailed in Section 5 has approximation error bounded a priori by

$$\max_{\omega \in (-\pi,\pi]} |G(e^{j\omega}) - \hat{G}(e^{j\omega})| \leq \sum_{i=1}^{N} \sigma_i(\mathcal{T}_q G) \qquad (14)$$

and a posteriori (after the filter has been designed) by

$$\max_{\omega \in (-\pi,\pi]} |G(e^{j\omega}) - \hat{G}(e^{j\omega})| \leq \sum_{i=1}^{N} \overline{\sigma}(\mathcal{T}_q G_i). \qquad (15)$$

In order to prove this theorem, we need the following result:
Lemma 4. (Hankel Singular Values of $\mathcal{T}_q G$)
Given a stable, causal transfer function of McMillan degree N, $G = \mathcal{H}_q G + \mathcal{S}_q \mathcal{T}_q G$ then $\sigma_i(\mathcal{T}_q G) \leq \sigma_i(G)$ for $1 \leq i \leq N$.
We omit the proof of this lemma.
We now proceed to the proof of Theorem 3.
Proof. From (13)

$$\begin{aligned}
&\max_{\omega \in (-\pi,\pi]} |G - \hat{G}| = \\
&\max_{\omega \in (-\pi,\pi]} |\overbrace{\mathcal{H}_q G_1 + \mathcal{S}_q \mathcal{T}_q G_1}^{G_1} \\
&-(\overbrace{\mathcal{H}_q G_1 + \mathcal{S}_q \mathcal{R}\mathcal{H}_q G_2 + \mathcal{H}_q G_3 + \ldots}^{\hat{G}})| \\
= &\max_{\omega \in (-\pi,\pi]} |\mathcal{S}_q \mathcal{T}_q G_1 - \overbrace{\mathcal{S}_q \mathcal{R}(G_2 - \mathcal{S}_q \mathcal{T}_q G_2)}^{\mathcal{S}_q \mathcal{R}\mathcal{H}_q G_2} \\
&-G_3 + \mathcal{S}_q \mathcal{T}_q G_3 - \ldots | \\
= &\max_{\omega \in (-\pi,\pi]} |\mathcal{S}_q[\mathcal{T}_q G_1 - \mathcal{R} G_2] + [\mathcal{R}\mathcal{T}_q G_2 - G_3] + \ldots | \\
\leq &\max_{\omega \in (-\pi,\pi]} \{|\mathcal{T}_q G_1 - \mathcal{E}\mathcal{T}_q G_1| + |\mathcal{R}\mathcal{T}_q G_2 - \mathcal{R}\mathcal{E}\mathcal{T}_q G_2| + \ldots\} \\
= &\sum_{i=1}^{N} \overline{\sigma}(\mathcal{T}_q G_i) \qquad (16)
\end{aligned}$$

which is result (15).
From Lemma 4 and (4)

$$\overline{\sigma}(\mathcal{T}_q G_2) \leq \overline{\sigma}(G_2) = \sigma_2(\mathcal{T}_q G_1)$$

follows directly. Successive application of the lemma yields ($i > 1$)

$$\overline{\sigma}(\mathcal{T}_q G_i) \leq \overline{\sigma}(G_i) = \sigma_2(\mathcal{T}_q G_{i-1}) \leq \ldots \leq \sigma_{i-1}(F_1) = \sigma_i(\mathcal{T}_q G_1)$$

The upper bound on the error (16) of the Nehari Shuffle approximant then becomes

$$\max_{\omega \in (-\pi,\pi]} |G_1 - \hat{G}| \leq \sum_{i=1}^{N} \sigma_i(\mathcal{T}_q G_1). \qquad (17)$$

■

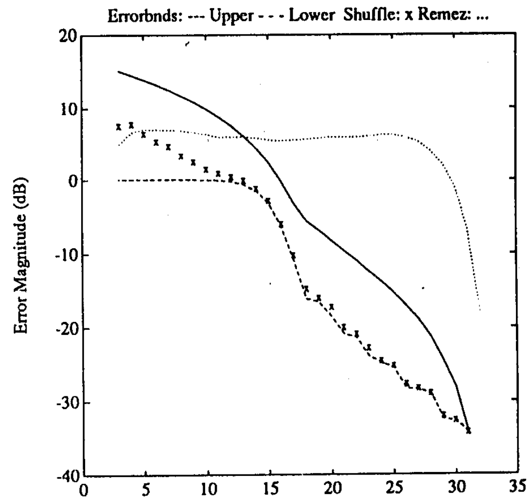

Figure 1: Example 1 — Approximation errors (in dB) vs. Length of Approximant, q.

7 EXAMPLES

Even though the Nehari Shuffle is an approach to a different problem from that addressed by the Parks-McClellan algorithm, comparison of the performance of the two is instructive because the Parks-McClellan approach optimally solves a *constrained* FIR(q) approximation problem whereas the Nehari Shuffle is an approach to the unconstrained problem.
In Figures 1 and 3 the

- solid line is the upper bound of (14),
- dashed line is the lower bound provided by (6),
- dotted line shows the Parks-McClellan approximation errors and
- x points are the Nehari Shuffle approximation errors.

We now consider two examples.

7.1 Example 1: The Linear Phase Case

If G_1 is taken to be a linear phase FIR(32) low-pass filter, and we use the Parks-McClellan and Nehari Shuffle algorithms to design filters of length 3 to 31, the resulting approximation errors are displayed in Figure 1.
Note that the errors lie within the derived bounds.

7.2 Example 2: The Non-linear Phase Case

If G_1 is now taken to be a 6^{th} order, 0.1 dB ripple low-pass Chebyshev filter with magnitude responses shown in Figure 2(a). If the Parks-McClellan and Nehari Shuffle algorithms are used to design FIR(32) filters, then the resulting magnitude responses are shown in Figures 2(b) and 2 (c), respectively.
Both algorithms were used to design filters of length 3 to 32 and the resulting approximation errors are displayed in Figure 3.
The results show that the Nehari Shuffle outperforms the Parks-McClellan algorithm for all tested filter lengths and that the approximation errors again lie within the derived bounds.

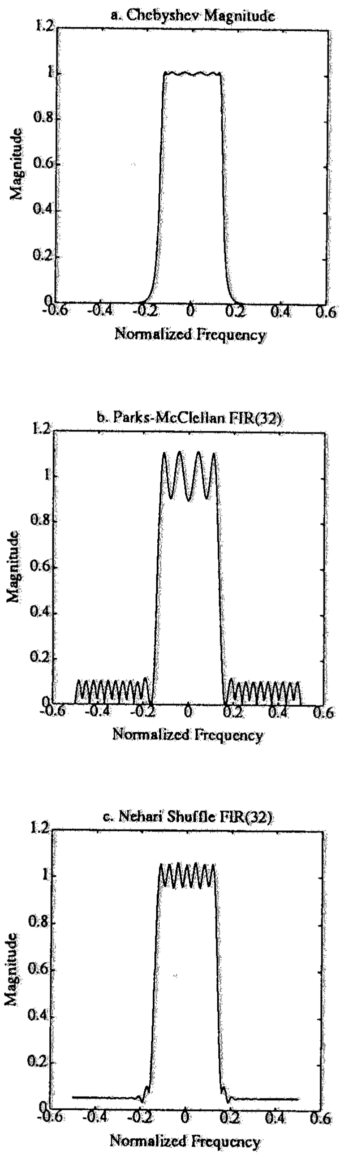

Figure 2: Example 2 — (a) Original Chebyshev Magnitude Response (b) Parks-McClellan FIR(32) Magnitude Response (c) Nehari Shuffle FIR(32) Magnitude Response.

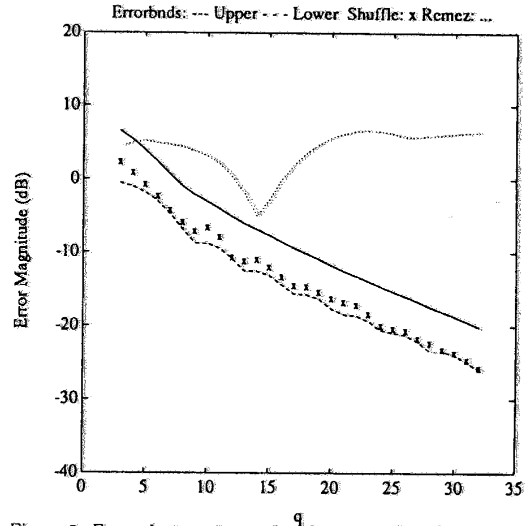

Figure 3: Example 2 — Approximation errors (in dB) vs. Length of Approximant, q.

8 CONCLUSIONS

We have presented the Nehari Shuffle, a novel approach to the design of FIR(q) filters. The main points to note about the method are below.

- It is an approach to the *unconstrained* FIR(q) approximation problem with guaranteed (*a priori* calculable) ∞-norm error bounds. It has *not* been proven to solve the ∞-norm problem optimally.
- The resulting FIR(q) filters are not constrained to have linear phase.
- The implementation method provided by Glover [3] involves only matrix manipulations.
- Extension of the algorithm to the multi-input/multi-output case is direct.

REFERENCES

[1] T.W. Parks & J.H. McClellan, "Chebyshev Approximation for Nonrecursive Digital Filters with Linear Phase," *IEEE Trans. on Circuit Theory* CT-19(2) (March, 1972), pp.189–194.

[2] B.C. Moore, "Principal Component Analysis in Linear Systems: Controllability, Observability and Model Reduction," *IEEE Trans. on Automatic Control* AC-26(1) (February, 1981), pp.17–32.

[3] K. Glover, "All optimal Hankel-norm approximations of linear multivariable systems and their L^∞-error bounds," *International Journal of Control* 39(6) (1984), pp.1115–1193.

[4] Z. Nehari, "On Bounded Bilinear Forms," *Annals of Mathematics* 65(1) (January, 1957), pp.153–162.

[5] V.M. Adamjan, D.Z. Arov & M.G. Krein, "Analytic properties of Schmidt pairs for a Hankel operator and the generalized Schur-Takagi problem," *Mathematics of the USSR, Sbornik* 15(1) (1971), pp.31–73.