Optimal measurements for relative quantum information

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We provide optimal measurement schemes for estimating relative parameters of the quantum state of a pair of spin systems. Specifically, we consider the task of estimating the angle between the spin directions of a pair of systems in SU(2) coherent states. We prove that the optimal measurements are non-local, in the sense that they cannot be achieved by local operations and classical communication. We also demonstrate that in the limit where one or both of the spins becomes macroscopic, our results reproduce those that are obtained by treating the macroscopic spin classically.

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Whenever a system can be decomposed into parts, a distinction can be made between global and relative degrees of freedom. Global degrees of freedom describe the system's relation to something external to it, while the relative ones describe the relations between its parts. Encoding information preferentially into these relative degrees of freedom can be very useful in situations where the parts are subject to an environmental interaction that does not distinguish the parts (collective decoherence), or where they are subjected to an unknown unitary, or if the external reference frame with respect to which they were prepared is unknown. Indeed, such relative encodings have been shown to have applications in quantum computation [1], communication [2, 3] and cryptography [4, 5].

Determining the optimal measurement schemes [6] for estimating global parameters of a quantum state has been the subject of several recent investigations [7, 8]. This Letter is concerned with a complementary problem: determining the optimal measurement schemes for estimating *relative* parameters of a quantum state. A physical example is given by two uncorrelated (i.e., unentangled) systems, the states of which have small variance in the relevant degree of freedom. Estimation tasks for such an example include estimating the distance between two minimum uncertainty wavepackets of a massive particle, or the phase between a pair of coherent states of the electromagnetic field, or estimating the angle between the directions defined by a pair of SU(2) spin coherent states [9]. It is this last example which shall be the focus of this Letter, although our results apply to a larger class of states, and can be applied to other variables.

One scheme for measuring such relative quantities is to measure each system independently with respect to an external RF, e.g., to perform an optimal estimation of each spin direction and to then calculate the angle between these estimates. We prove that such (local) schemes are not optimal. In fact, we find that possessing an external RF provides no advantage. On the other hand, the ability to perform *non-local* measurements *is* necessary to achieve the optimum.

Specifically, for estimating a relative angle, we prove that the optimal measurement can be chosen to be rotationally-invariant, and we use this fact to solve the optimization problem completely. We then investigate the information gain that can be achieved as different aspects of the estimation task are varied, such as the prior over the relative angle or the magnitude of the spins. Finally, we show that local measurements, which can be implemented using local operations and classical communication (LOCC), perform worse than those that make use of entanglement. Throughout, we explore what occurs in the limit where one of the spins becomes large. We find that, in this limit, our optimal relative measurement gives the same information gain as does the optimal measurement for estimating a single spin's direction relative to a classical RF, and that the need for non-local measurements disappears. These results contribute to our understanding of how the macroscopic systems that act as RFs can be treated within quantum theory, and more specifically how global degrees of freedom, which are defined relative to a classical RF, can be treated as a relative ones between quantized systems. Such an understanding is likely to be critical for quantum gravity and cosmology, wherein all degrees of freedom are expected to be relative [10].

Consider states in the joint Hilbert space $\mathbb{H}_{j_1} \otimes \mathbb{H}_{j_2}$ of a spin-j_1 and a spin-j_2 system. This Hilbert space carries a global tensor representation $R(\Omega) = R_{j_1}(\Omega) \otimes R_{j_2}(\Omega)$ of a rotation $\Omega \in \mathrm{SU}(2)$ where each system is rotated by the same amount. We can parametrise the states in $\mathbb{H}_{j_1} \otimes \mathbb{H}_{j_2}$ by two sets of parameters, α and Ω, such that a state $\rho_{\alpha,\Omega}$ transforms under a global rotation $R(\Omega')$ as

$$R(\Omega')\rho_{\alpha,\Omega}R(\Omega')^\dagger = \rho_{\alpha,\Omega'\Omega} \,. \quad (1)$$

Defining a global parameter as one whose variation corresponds to a global rotation of the state, and a relative parameter as one that is invariant under such a rotation, we see that α is relative and Ω is global. In the example we investigate, a product of two SU(2) coherent states, there is only a single relative parameter α characterising the angle between the two spins.

Suppose that Alice prepares a pair of spins in the state $\rho_{\alpha,\Omega}$ and Bob wishes to acquire information about the relative parameter α without having any prior knowledge of the global parameter Ω. The most general measurement that can be performed by Bob is a positive operator valued measure (POVM) [11] represented by a set of operators $\{E_\lambda\}$. Upon obtaining the outcome λ, Bob uses Bayes' theorem to update his knowledge about α, Ω from his prior distribution $p(\alpha, \Omega)$, to his posterior distribution: $p(\alpha, \Omega|\lambda) = \mathrm{Tr}(E_\lambda \rho_{\alpha,\Omega}) p(\alpha, \Omega)/p(\lambda)$, where $p(\lambda) = \int \mathrm{Tr}(E_\lambda \rho_{\alpha,\Omega}) p(\alpha, \Omega) \mathrm{d}\alpha \mathrm{d}\Omega$. Assuming that Bob has no prior knowledge of Ω, we may take $p(\alpha, \Omega)\mathrm{d}\alpha\mathrm{d}\Omega = p(\alpha)\mathrm{d}\alpha\mathrm{d}\Omega$ where $p(\alpha)$ is Bob's prior probability density over α and $\mathrm{d}\Omega$ is the SU(2) invariant measure.

Any measure of Bob's information gain about α can depend only on the prior and the posterior distributions over α for every λ. The latter are obtained by marginalization of the $p(\alpha, \Omega|\lambda)$, and are given by $p(\alpha|\lambda) = \mathrm{Tr}(E_\lambda \rho_\alpha) p(\alpha)/p(\lambda)$, where $\rho_\alpha = \int R(\Omega') \rho_{\alpha,\Omega} R(\Omega')^\dagger \mathrm{d}\Omega'$. For a given POVM $\{E_\lambda\}$, note that any other POVM related by a global rotation (i.e., $E'_\lambda = R(\Omega) E_\lambda R(\Omega)^\dagger$) yields precisely the same posterior distributions over α. This property also holds true for the POVM with elements $\bar{E}_\lambda = \int R(\Omega) E_\lambda R(\Omega)^\dagger \mathrm{d}\Omega$, which is rotationally-invariant, that is,

$$R(\Omega) \bar{E}_\lambda R(\Omega)^\dagger = \bar{E}_\lambda \,, \quad \forall\, \Omega \in \mathrm{SU}(2)\,. \tag{2}$$

We define POVMs that yield the same posterior distribution over α to be *informationally equivalent*. Because every equivalence class contains a rotationally-invariant POVM of the form (2), it is sufficient to consider only rotationally-invariant POVMs in optimizing Bob's choice of measurement. These can be implemented without an external RF for spatial orientation. Moreover, they have a very particular form, as we now demonstrate.

The joint Hilbert space for the two spins decomposes into a multiplicity-free direct sum of irreducible representations (irreps) of SU(2), i.e., eigenspaces $\mathbb{H}_J$ of total angular momentum J. Using Schur's lemma [12], it can be shown that any positive operator satisfying (2) can be expressed as a positive-weighted sum of projectors Π_J onto the subspaces $\mathbb{H}_J$, that is, as $E_\lambda = \sum_J s_{\lambda,J} \Pi_J$, where $s_{\lambda,J} \geq 0$. In order to ensure that $\sum_\lambda E_\lambda = \mathbb{I}$, we require that $\sum_\lambda s_{\lambda,J} = 1$, so that $s_{\lambda,J}$ is a probability distribution over λ. The $\{E_\lambda\}$ can be obtained by random sampling of the projective measurement elements $\{\Pi_J\}$, and such a sampling cannot increase the information (quantified by some concave function such as the average information gain defined below) about the system. Thus, the most informative rotationally-invariant POVM is simply the projective measurement $\{\Pi_J\}$.

We have proved the main result of the letter, which can be summarized as follows: If the prior over global rotations Ω is uniform, then for *any* prior over the relative parameters α, the maximum information gain (by any measure) can be achieved using a measurement of the rotationally-invariant projective measurement $\{\Pi_J\}$.

A useful way to understand this result is to note that our estimation task is equivalent to one wherein Alice prepares a state ρ_α (rather than $\rho_{\alpha,\Omega}$) and Bob seeks to estimate α. Because the ρ_α are rotationally-invariant, they are also positive sums of the Π_J and thus may be treated as classical probability distributions over J. The problem reduces to a discrimination among such distributions, for which Bob can do no better than to measure the value of J.

We now apply this result to several important and illustrative examples of relative parameter estimation. We shall quantify the degree of success in the estimation by the average decrease in Shannon entropy of the distribution over α [11], which is equivalent to the average (Kullback-Leibler) relative information between the posterior and the prior distributions over α, specifically $I_{\mathrm{av}} = \sum_\lambda p(\lambda) I_\lambda$, where $I_\lambda = \int p(\alpha|\lambda) \log_2 \big[p(\alpha|\lambda)/p(\alpha)\big] \mathrm{d}\alpha$. We refer to this quantity as simply the *average information gain*.

Two spin-1/2 systems. The simplest example of relative parameter estimation arises in the context of a pair of spin-1/2 systems. Alice prepares the product state $|\mathbf{n}_1\rangle \otimes |\mathbf{n}_2\rangle$, where $|\mathbf{n}\rangle$ is the eigenstate of $\mathbf{J} \cdot \mathbf{n}$ with positive eigenvalue (note that every state of a spin 1/2 system is an SU(2) coherent state). Bob's task is to estimate the relative angle $\alpha = \cos^{-1}(\mathbf{n}_1 \cdot \mathbf{n}_2)$ given no knowledge of the global orientation of the state. Because the joint Hilbert space decomposes into a $J = 0$ and a $J = 1$ irrep, the optimal POVM has the form $\{\Pi_A, \Pi_S\}$, where $\Pi_A = |\Psi^-\rangle\langle\Psi^-|$ is the projector onto the antisymmetric ($J = 0$) subspace and $\Pi_S = \mathbb{I} - \Pi_A$ is the projector onto the symmetric ($J = 1$) subspace. The conditional probability of outcomes A and S given α are simply $p(A|\alpha) = \mathrm{Tr}(\Pi_A \rho_\alpha) = \frac{1}{2}\sin^2(\alpha/2)$ and $p(S|\alpha) = 1 - p(A|\alpha)$. The average information gain and the optimal guess for the value of α depend on Bob's prior over α. We consider two natural choices of prior.

(i) Parallel versus anti-parallel spins: This situation corresponds to a prior $p(\alpha{=}0) = p(\alpha{=}\pi) = 1/2$, yielding $p(A) = 1/4$, $p(S) = 3/4$ and posteriors $p(\alpha{=}0|A) = 0$, $p(\alpha{=}\pi|A) = 1$, $p(\alpha{=}0|S) = 2/3$, $p(\alpha{=}\pi|S) = 1/3$. Upon obtaining the antisymmetric outcome, Bob knows that the spins were anti-parallel, whereas upon obtaining the symmetric outcome, they are deemed to be twice as likely to have been parallel than anti-parallel. We find $I_A = 1$, $I_S = \frac{5}{3} - \log_2 3 \simeq .08\,170$, i.e. 1 bit of information is gained upon obtaining the antisymmetric outcome, and 0.08170 bits for the symmetric outcome. On average, Bob gains $I_{\mathrm{av}} = \frac{1}{4} I_A + \frac{3}{4} I_S \simeq 0.3113$ bits of information.

(ii) Uniform prior for each system's spin direction. In this case, the prior over α is $p(\alpha) = \frac{1}{2}\sin\alpha$. This implies posteriors $p(\alpha|A) = \sin^2(\alpha/2)\sin\alpha$ and $p(\alpha|S) = \frac{1}{3}(2 -$

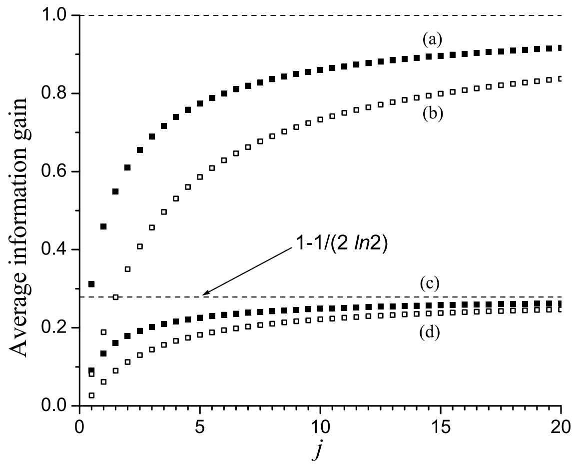

FIG. 1: Average information gain for measurements on a spin-1/2 system and a spin-j system. The curves (a),(b) correspond to the optimal measurement and the optimal local measurement for the case when the spins are prepared parallel or antiparallel with equal probability. The curves (c),(d) correspond to the optimal measurement and the optimal local measurements for the case when the initial direction of each spin is chosen uniformly from the sphere.

$\sin^2(\alpha/2))\sin\alpha$ which are peaked at $2\pi/3$ and 0.4094π respectively. It follows that these are the best guesses for the angle α given each possible outcome. Using the posteriors, we find $I_A \simeq 0.2786$, $I_S \simeq 0.02702$, which yields $I_{\rm av} \simeq 0.08993$. Less information is acquired than in the parallel-antiparallel estimation problem, because angles near $\pi/2$ are more difficult to distinguish.

One spin-1/2, one spin-j system. We now consider the estimation of the angle between a spin-1/2 system and a spin-j system for some arbitrary j, where the latter is in an SU(2) coherent state $|j\mathbf{n}\rangle$ (the eigenstate of $\mathbf{J}\cdot\mathbf{n}$ associated with the maximum eigenvalue) [9]. Alice prepares $|\mathbf{n}_1\rangle \otimes |j\mathbf{n}_2\rangle$ and Bob seeks to estimate $\alpha = \cos^{-1}(\mathbf{n}_1\cdot\mathbf{n}_2)$. The joint Hilbert space decomposes into a sum of a $J = j+1/2$ irrep and a $J = j-1/2$ irrep. The optimal measurement is the two outcome POVM $\{\Pi_+, \Pi_-\}$, where $\Pi_\pm$ is the projector onto the $j \pm 1/2$ irrep [18]. Using Clebsch-Gordon coefficients, the probabilities for each of the outcomes are found to be $p(-|\alpha) = \mathrm{Tr}(\Pi_-\rho_\alpha) = \frac{2j}{2j+1}\sin^2(\alpha/2)$ and $p(+|\alpha) = 1 - p(-|\alpha)$. We again consider two possible priors over α.

(i) Parallel versus anti-parallel spins: A calculation similar to the one for two spin-1/2 systems yields the posteriors $p(\alpha{=}0|+) = (2j{+}1)/(2j{+}2)$, $p(\alpha{=}\pi|+) = 1/(2j{+}2)$, $p(\alpha{=}0|-) = 0$, $p(\alpha{=}\pi|-) = 1$. Using these, we can calculate the average information gain as a function of j; the result is curve (a) of Fig. 1. The $j = 1/2$ value is the average information gain for two spin-1/2 systems, derived previously. In the limit $j \to \infty$, $p(\alpha{=}0|+) \to 1$ and $p(\alpha{=}\pi|+) \to 0$ so that the outcome of the measurement leaves no uncertainty about whether the spins were parallel or antiparallel, and the average information gain goes to one bit. Thus, in the limit that one of the spins becomes large, the problem becomes equivalent to estimating whether the spin-1/2 is up or down compared to some classical reference direction, where one expects an average information gain of one bit for the optimal measurement.

(ii) Uniform prior for each system's spin direction: Following the same steps as before, the average information gain can be derived as a function of j; the result is curve (c) of Fig. 1. In the limit $j \to \infty$, we find $I_{\rm av} = 1 - (2\ln 2)^{-1} \simeq 0.2787$ bits, which is precisely the information gain for the optimal measurement of the angle of a spin-1/2 system relative to a classical direction given a uniform prior over spin directions [11].

Optimal local measurements: Consider again the simplest case of a pair of spin-1/2 systems. The optimal measurement in this case was found to be the POVM $\{\Pi_A, \Pi_S\}$. This measurement cannot be implemented by local operations on the individual systems because Π_A is a projector onto an entangled state. We now determine the optimal local measurement. We do so by first finding the optimal separable POVM (one for which all the elements are separable operators), and then showing that this can be achieved by LOCC. (Note the POVMs for LOCC measurements are necessarily separable.) The rotationally invariant states for a pair of spin-1/2 systems, called Werner states [14], have the form $\rho_W = p\Pi_A + (1-p)\Pi_S/3$, and are known to only be separable for $p < 1/2$ [15]. Thus, the greatest relative weight of Π_A to Π_S that can occur in a separable positive operator is 3. The closest separable POVM to the optimal POVM $\{\Pi_A, \Pi_S\}$ is therefore $\{\Pi_A + \frac{1}{3}\Pi_S, \frac{2}{3}\Pi_S\}$. However, it turns out that this POVM is informationally equivalent to measuring the spin of each system along the same (arbitrary) axis and registering whether the outcomes are the same or not, which clearly only involves local operations (and does not even require classical communication). Because the POVM $\{\Pi_A + \frac{1}{3}\Pi_S, \frac{2}{3}\Pi_S\}$ can be obtained by random sampling of the outcome of $\{\Pi_A, \Pi_S\}$, the former is strictly less informative than the latter. Indeed, the maximum average information gain with the optimal local measurement is 0.0817 bits for case *(i)* above, and 0.02702 bits for case *(ii)*, and both of these values are strictly less than those obtained for the optimal (non-local) measurement.

We extend this analysis to the spin-1/2, spin-j case. Consider the following LOCC measurement. The spin-j system is measured along the complete basis of SU(2) coherent states $\{|j\mathbf{n}_m\rangle\}_m$ where $m = 0,\ldots,2j$ and $\mathbf{n}_m$ points at an angle $\theta_m = \frac{2\pi m}{2j+1}$ in some fixed but arbitrary plane. Then, conditional on the outcome m of this measurement, the spin-1/2 system is measured along the basis $\{|\mathbf{n}_m\rangle, |{-}\mathbf{n}_m\rangle\}$. The measurement outcome of the spin-j system is then discarded, and all that is registered is whether the outcome for the spin-1/2 system is $\pm\mathbf{n}_m$; i.e., whether the two spins are aligned or anti-aligned. The resulting 2-outcome measurement is informationally equivalent to the rotationally invariant POVM $\{\Pi_1 = \frac{2j+1}{2j+2}\Pi_+,\ \Pi_2 = \Pi_- + \frac{1}{2j+2}\Pi_+\}$. By numerically calculating the partial transpose of the operator $\Pi_- + x\Pi_+$, the negativity of which is a necessary condition for non-separability [16], we find that $\{\Pi_1, \Pi_2\}$ is the optimal separable POVM. Thus, again, the optimal separable POVM can be implemented by LOCC and gives less information than the optimal (non-local) measurement. The average information gain achieved by this measurement, as a function of j, in cases *(i)* and *(ii)* are plotted as curves (b) and (d) of Fig. 1. Note that the optimum *can* be achieved by LOCC measurements in the limit $j \to \infty$.

Discussion: We now briefly discuss some other relative parameter estimation tasks for which our result provides the solution. The case we have yet to address is the estimation of the angle between a spin-j_1 and a spin-j_2 system, both in SU(2) coherent states, for arbitrary j_1, j_2. Assuming $j_2 \geq j_1$, the optimal measurement is the $(2j_1+1)$-element projective measurement which projects onto the subspaces of fixed total angular momentum J. The posterior distributions over α and the average information gain can be calculated in the same manner as before, although in this case they are substantially more complicated. However, in the limit $j_2 \to \infty$, the Clebsch-Gordon coefficients simplify, and one can show that the probability of a measurement outcome J approaches the probabilities obtained using the Born rule for a projective measurement along the classical direction defined by the spin-j_2 system. As a result, the posterior distribution for any measurement result will agree with what would be obtained classically, regardless of the prior over α. If, in addition, we take $j_1 \to \infty$, the information gain for α becomes infinite (for any prior distribution) and thus α can be inferred with certainty from the measurement result, as expected for a measurement of the angle between two classical directions. Our results also indicate that, in the classical limit, a measurement of the magnitude of total angular momentum should be sufficient to estimate the relative angle, which is indeed the case if the magnitude of each spin is known.

It should be noted that estimating the relative angle between a pair of SU(2) coherent states is of particular importance because estimating the eccentricity of an elliptic Rydberg state of a Hydrogen atom is an instance of the same problem [17]. Rydberg states are significant as they can be prepared experimentally. Our results imply that an optimal estimation of eccentricity is in fact straightforward to achieve experimentally because it involves only a measurement of the magnitude of the total angular momentum of the atom.

Our results are also applicable to systems other than spin. For example, for *any* realization of a pair of 2-level systems (qubits), the degree of nonorthogonality between their states (measured by, say, the overlap $|\langle\psi_1|\psi_2\rangle|$) is invariant under global transformations and is thus a relative parameter. Our measurement is thus optimal for estimating this nonorthogonality.

In addition to solving various estimation problems, we have shown that a macroscopic spin in the appropriate limit is equivalent to a classical external RF as far as relative parameter estimation is concerned. This result suggest that it may be possible to express all measurements (and possibly all operations) in a covariant, relative framework that respects the underlying symmetries of the theory. Such a framework is necessary if one wishes to abide by the principle, which has been so fruitful in the study of space and time but has yet to be embraced in the quantum context, that all degrees of freedom must be defined in terms of relations.

There remain many important questions for future investigation. While we have focussed on estimating relative parameters of product states, one can also consider relative parameters of entangled states, and here the landscape becomes much richer. For instance, for a pair of spin-1/2 systems, while the set of product states supports a single relative parameter, the set of all two-qubit states supports three: the angle between the spins in a term of the Schmidt decomposition [11], the phase between the two terms of this decomposition, and the degree of entanglement between the spins. Our measurement scheme is optimal for estimating these relative parameters as well. Given the significance of entanglement for quantum information theory, there is likely much to be learned from investigations of other sorts of relative quantum information.

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