Relativistically invariant quantum information

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We show that quantum information can be encoded into entangled states of multiple indistinguishable particles in such a way that any inertial observer can prepare, manipulate, or measure the encoded state independent of their reference frame. Such relativistically invariant quantum information is free of the difficulties associated with encoding into spin or other degrees of freedom in a relativistic context.

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Information encoded into the states of quantum systems allows for powerful new computational and communication tasks [1]. It is perhaps in situations involving extremely long distances that quantum information will find its most useful applications: quantum teleportation [2], entanglement-enhanced communication [3], quantum clock synchonization [4, 5] and reference frame alignment [6], and quantum-enhanced global positioning [5] are just some of the ways that quantum physics offers an advantage over classical methods. In these long-distance situations, relativistic effects can be expected to arise [7]. Consider the canonical example of a qubit encoded into the angular momentum state of a massive spin-1/2 particle. The spin entropy, which quantifies the purity of the encoded information, is not a covariant quantity [8]: under a Lorentz transformation, the spin state becomes entangled with the momentum of the particle. The effect of Lorentz transformations is to *decohere* the qubit, reducing the applicability of such systems to perform quantum information processing tasks in a relativistic setting [7, 8]. Photon polarization qubits behave similarly, with additional effects arising from the transversality of polarization [7, 8].

We show that *relativistically invariant* quantum information can be encoded into entangled states of multiple, indistinguishable particles. This encoding allows any inertial observer to prepare and manipulate quantum information in a way that is independent of their particular frame of reference. In particular, two observers can share entanglement and thus perform any quantum information processing task (teleportation, communication, etc.) without sharing a reference frame. We do this by showing that, under a general Lorentz transformation Λ_{AB}, the spin state of a particle will be transformed due to three distinct effects: (i) a Wigner rotation due to the Lorentz boost Λ_{BA}, which occurs even for momentum eigenstates, (ii) a decoherence due to the entangling of the spin and momentum under the Lorentz transformation Λ_{AB} because the particle is not in a momentum eigenstate, and (iii) a decoherence due to Bob's lack of knowledge about the transformation relating his reference frame to Alice's frame. Then, to construct encodings that are protected from all these forms of decoherence, we construct states of multiple indistingishable particles with well-defined momenta and use the techniques of noiseless subsystems [9, 10]. We begin by considering massive spin-1/2 particles; massless photons are then given a separate treatment.

*Single spin-*1/2 *particle.* Consider two inertial observers, Alice and Bob, who wish to exchange spin-1/2 particles (e.g., protons) for the purposes of some quantum information processing task. First, we consider the exchange of a single particle and outline the associated difficulties. To fix our notation, momentum eigenstates $|\mathbf{0}m\rangle$ of a single spin-1/2 particle in the rest frame ($\mathbf{p} = \mathbf{0}$) are defined by [11],

$$P^\mu|\mathbf{0}m\rangle = p_0^\mu|\mathbf{0}m\rangle\,, \tag{1}$$

$$\mathbf{J}^2|\mathbf{0}m\rangle = \tfrac{3}{4}|\mathbf{0}m\rangle\,, \quad J_z|\mathbf{0}m\rangle = m|\mathbf{0}m\rangle\,, \tag{2}$$

and are given in a boosted frame as $|\mathbf{p}m\rangle = L(\xi_\mathbf{p})|\mathbf{0}m\rangle$ for $L(\xi_\mathbf{p})$ a pure Lorentz boost. The Lorentz transformation Λ acts via the one-particle representation T_1 as

$$T_1(\Lambda)|\mathbf{p}m\rangle = \sum_{m'}|(\Lambda\mathbf{p})m'\rangle D^{1/2}_{m',m}(\Omega(\Lambda,\mathbf{p}))\,, \tag{3}$$

where $\Omega(\Lambda,\mathbf{p}) = L(\xi_{\Lambda\mathbf{p}})^{-1}T_1(\Lambda)L(\xi_\mathbf{p}) \in \mathrm{SO}(3)$ is a Wigner rotation, and $D^{1/2}_{m',m}(\Omega)$ is its the spin-1/2 representation. Thus, on the spin degrees of freedom, the Lorentz transformation acts as a rotation.

Let Alice prepare a single spin-1/2 particle in a state ρ with respect to her reference frame. This state cannot be an (unphysical) eigenstate of momentum [7]; the spatial state of the particle could be prepared, for example, in a coherent state of minimum uncertainty in both position and momentum. A generic pure state for a single particle is given in terms of the basis above by

$$|\Psi\rangle_1 = \sum_m \int_{-\infty}^{\infty} \psi_m(\mathbf{p})|\mathbf{p}m\rangle\,\mathrm{d}\mu(\mathbf{p})\,, \tag{4}$$

where $\mathrm{d}\mu(\mathbf{p}) = (2\pi)^{-3}(2p^0)^{-1}\mathrm{d}^3\mathbf{p}$. To encode a qubit into this particle, Alice may prepare the spin of this particle in an arbitrary encoded state uncoupled (in a product state) with a localized spatial state, i.e.,

$$|\Psi\rangle_1 = \begin{pmatrix}\zeta\\ \eta\end{pmatrix}\int \psi(\mathbf{p})|\mathbf{p}\rangle\,\mathrm{d}\mu(\mathbf{p})\,, \tag{5}$$

where we take the wave function ψ to be concentrated near zero momentum and with a characteristic spread Δ; i.e., to be of the Gaussian form

$$\psi(\mathbf{p}) = N \exp\left(-\mathbf{p}^2/2\Delta^2\right) , \qquad (6)$$

where N is a normalization constant. The reduced density matrix for the spin component of this state in Alice's frame is

$$\rho_1 = \begin{pmatrix} |\zeta|^2 & \zeta\eta^* \\ \zeta^*\eta & |\eta|^2 \end{pmatrix} , \qquad (7)$$

and in this frame is independent of the form of $\psi(\mathbf{p})$.

Now consider the state of this particle as described by another inertial observer, Bob. Let Λ_{BA} be the element of the Lorentz group that relates Bob's inertial frame B to Alice's frame A; Bob thus assigns the transformed state $T_1(\Lambda_{BA})|\Psi\rangle_1$ to the particle. Even if Bob has the perfect knowledge of the relative orientation and velocity of his reference frame with respect to Alice's, the reduced density matrix for the spin degrees of freedom of this qubit decoheres [7]. For example, if the Lorentz transformation Λ_{BA} is a pure boost along the z-axis to the velocity v, the effective state transformation is [8]

$$\rho_1' \approx (1 - \tfrac{1}{4}\Gamma^2)\rho_1 + \tfrac{1}{8}\Gamma^2(\sigma_x\rho_1\sigma_x + \sigma_y\rho_1\sigma_y) , \qquad (8)$$

where $\Gamma = (1-\sqrt{1-v^2})\Delta/v$.

Moreover, if Bob does not *know* the relation (i.e., the Lorentz transformation Λ_{AB}) that relates his frame to the frame in which the state was prepared, the decohering effects are much more significant. Without this knowledge, he represents the state of the system as a mixture over all possible Lorentz transformations. Specifically, we would represent the state of the particle as

$$\mathcal{E}_1(|\Psi\rangle_1\langle\Psi|) = \int \mathrm{d}\Lambda\, f(\Lambda) T_1(\Lambda)|\Psi\rangle_1\langle\Psi| T_1(\Lambda)^\dagger , \qquad (9)$$

where the integration is over the entire Lorentz group, $\mathrm{d}\Lambda$ is its Haar measure and $f(\Lambda)$ describes Bob's prior estimate of the Lorentz transformation relating the systems [13]. Viewing the quantum state $|\Psi\rangle_1$ as a "catalogue" of predictions for the outcomes of future measurements on the particle (or retrodictions about possible preparations by Alice), the process $\mathcal{E}_1$ describes the loss of predictive power by Bob due to his lack of knowledge about the reference frame in which the state of the particle was prepared [15]. It is useful to view the superoperator $\mathcal{E}_1$ as a form of decoherence. Rather than describing an interaction with an environment, this decoherence represents the resulting decrease in Bob's predictive and retrodictive capacity due to his lack of knowledge.

Consider the action of this decoherence on the reduced density matrix ρ_1 of Eq. (7) for the spin component of this particle. While the Lorentz group acts via Eq. (3) on each momentum component as the spin-1/2 representation $D^{1/2}$ of the rotation group, an effective transformation for the reduced density matrix of the state (5) involves averaging over different noisy quantum channels (as the one given in Eq. (8)), and not just rotations. On the other hand, the lack of knowledge of the relative orientation of the reference frames alone is sufficient to completely decohere Bob's qubit [15]. Thus, the decoherence due to entanglement between spin and momentum and the lack of knowledge about the relative motion cannot make matters worse, and the total decoherence on the reduced density matrix for the spin component of a single particle is

$$\mathcal{E}_1(\rho_1) = \int \mathrm{d}\Omega\, D^{1/2}(\Omega)\rho_1 D^{1/2}(\Omega)^\dagger = \tfrac{1}{2}I , \qquad (10)$$

where $\Omega \in \mathrm{SO}(3)$ is a rotation, integration is over the entire group SO(3), and $\frac{1}{2}I$ is the completely mixed density operator on the spin subsystem. The spin state of the particle is decohered in Bob's frame to the completely mixed state, and thus no quantum information can be conveyed to Bob by encoding into the spin of a single particle. When the relative orientation of frames is known, but the relative velocity is not and/or the effects of spin-momentum entanglement are taken into account, Bob's density matrix depends both on $\psi(\mathbf{p})$ and $f(\Lambda)$. This result also proves that Alice and Bob cannot share spin entanglement through the exchange of a single spin-1/2 particle without first sharing a reference frame. We note that Bob may perform a measurement on the particle in an attempt to gain information about the frame in which it was prepared; however, such a measurement necessarily disturbs the state in an unpredictable way.

Creating distinguishable qubits from indistinguishable particles. As we will show, it is possible to use entangled states of multiple particles to combat the deleterious effects of this decoherence. However, first we must demonstrate that it is possible to use elementary *indistinguishable* particles as *distinguishable* qubits through an appropriate preparation of their spatial wavefunctions. Consider the states of N identical particles. To use these particles as qubits to encode quantum information, they must be prepared in such a way that they are (i) distinguishable and (ii) relatively localized and at rest with respect to each other, so that joint (entangling) operations such as preparations and measurements can be performed on them. These conditions are mutually exclusive at first glance: for the particles to all be at rest with respect to each other, they must all be in the eigenstate of zero momentum with respect to some frame, and thus are indistinguishable because they are all in the same spatial state. By preparing particles in minimum-uncertainty states that are well-localized (making them distinguishable) and with a sharp common momentum, we will show that these conditions can be sufficiently satisfied.

Consider a translation of a single particle state $|\Psi\rangle_1$ of

Eq. (5),

$$|\Psi_a\rangle_1 = e^{-\mathrm{i}aP_z}|\Psi\rangle_1 = \begin{pmatrix}\zeta\\ \eta\end{pmatrix}\int e^{-\mathrm{i}p_z a}\psi(\mathbf{p})|\mathbf{p}\rangle\,\mathrm{d}\mu(\mathbf{p})\,, \tag{11}$$

where we arbitrarily choose the translation to be along the z-axis. The overlap between two one-particle states serves as a guide to their distinguishability; thus,

$$_1\langle\Psi|\Psi_a\rangle_1 = N^2\int \mathrm{d}\mu(\mathbf{p})\,\mathrm{e}^{-\mathbf{p}^2/\Delta^2}\mathrm{e}^{-\mathrm{i}p_z a/\hbar}\,, \tag{12}$$

which should be small. Because $\Delta \ll mc$, we expand the energy as $E = mc^2(1 + p^2/2mc^2 + \ldots)$ and obtain $_1\langle\Psi|\Psi_a\rangle_1 \propto \exp(-a^2\Delta^2/4\hbar^2)$. Thus, the condition for distinguishability is $a \gg \lambda\!\!\!^{-}/\epsilon$, where $\Delta \equiv \epsilon mc$ and $\lambda\!\!\!^{-} = mc/\hbar$ is Compton wavelength of the particle. Now we apply our second condition: that the particles should be nearly at rest in Alice's frame, i.e., they should be cooled down. Using a proton (hydrogen atom) in the millikelvin range as an example, we obtain an upper bound for ϵ to be 10^{-8}, so $\lambda\!\!\!^{-}_p/\epsilon \sim 100$Å. Thus, it is possible to have both relatively sharp momenta and good localization, and so distinguishable qubits can be created from elementary indistinguishable particles in an appropriate momentum state. That is, an N-qubit state can be constructed from N single-particle states as

$$|\Psi\rangle_N = \otimes_{n=1}^{N} e^{-\mathrm{i}naP_z}|\Psi\rangle_1\,, \tag{13}$$

forming a one-dimensional lattice of particles with separation a. In this case, we can loosely define a rest frame of these particles (although they are not precisely in a zero momentum eigenstate), and these particles are sufficiently distinguishable via their spatial wavefunctions so that we can apply labels $1,\ldots,N$. In other inertial frames, these particles will no longer be at rest but are still distinguishable. Alice prepares the N particles is a state $|\Psi\rangle_N$ with respect to her reference frame, where the spatial wavefunctions of the particles are determined by the above localization technique to make distinguishable qubits, but the spin wavefunctions are completely arbitrary. From now on we ignore the effects of momentum spread and consider the particles to be eigenstates of momentum $\mathbf{p}$.

Encoding in multiple particles. We now consider the state of these particles in Bob's reference frame. Let T_N be the (reducible) collective representation of the Lorentz group acting on states of the N particles, i.e., $T_N(\Lambda) = T_1(\Lambda)\otimes T_1(\Lambda)\otimes\cdots\otimes T_1(\Lambda)$. A Lorentz transformation acts on the spin state of each particle as a Wigner rotation via the SU(2) representation $D^{1/2}$. In fact, because these particles posses a common momentum and they were all prepared with respect to a common reference frame (Alice's), the group SU(2) acts *identically* on each spin via the reducible collective representation

$$[D^{1/2}(\Omega)]^{\otimes N} = D^{1/2}(\Omega)\otimes D^{1/2}(\Omega)\otimes\cdots\otimes D^{1/2}(\Omega)\,, \tag{14}$$

for $\Omega \in \mathrm{SO}(3)$. If Bob does not know the Lorentz transformation that relates his frame to Alice's, then he represents the state of the N particles as

$$\mathcal{E}_N(|\Psi\rangle_N\langle\Psi|) = \int_S \mathrm{d}\Lambda\, f(\Lambda)T_N(\Lambda)|\Psi\rangle_N\langle\Psi|T_N(\Lambda)^\dagger\,. \tag{15}$$

We show that, for any prior distribution $f(\Lambda)$, there exists an efficient encoding scheme that allows for quantum communication. The superoperator $\mathcal{E}_N$ has a decohering effect on the state of the particles, but unlike (9) this decoherence is not complete on the N-particle Hilbert space because T_N does not act irreducibly on the states of N particles. Because all the particles are now considered to have well-defined momentum, so the action on the reduced density operator ρ_N describing the spin states of the N particles is

$$\mathcal{E}_N(\rho_N) = \int \mathrm{d}\Omega\,\tilde{f}(\Omega)[D^{1/2}(\Omega)]^{\otimes N}\rho_N[D^{1/2}(\Omega)^\dagger]^{\otimes N}\,. \tag{16}$$

where $\tilde{f}(\Omega)$ is induced by $f(\Lambda)$. In the following we assume the worst case scenario of a uniform prior $\tilde{f}(\Omega) = 1$. Because $[D^{1/2}(\Omega)]^{\otimes N}$ acts reducibly on the spin states, it is not completely decohering for $N > 1$. By appealing to the techniques of decoherence-free subspaces [9] and noiseless subsystems [10], it is possible use entangled states of multiple particles for encodings that are completely protected against this form of decoherence. Remarkably (and conveniently), the noiseless subsystems for the superoperator $\mathcal{E}_N$ are completely determined by the noiseless subsystems for the spins under collective decoherence [9, 16], i.e., decoherence that acts identically on each particle. The Hilbert space of the N-particle spin states decomposes as

$$\mathcal{H}_{j=1/2}^{\otimes N} = \bigoplus_{j=0}^{N/2}\mathcal{H}_{jR}\otimes\mathcal{H}_{jS}\,, \tag{17}$$

where SU(2) acts irreducibly on each subsystem $\mathcal{H}_{jR}$ (via the irreducible representation of SU(2) labelled by j), and acts trivially on the noiseless subsystems $\mathcal{H}_{jS}$. Thus, states encoded into a noiseless subsystem $\mathcal{H}_{jS}$ are *relativistically invariant*; they appear the same to all inertial observers, regardless of their reference frame.

The following example illustrates how a relativistically-invariant qubit can be encoded into the state of four physical qubits. Let four particles be prepared in the spatial state as described above, making them distinguishable, and let the spin states of these particles be prepared in the $N = 4$ singlet ($j = 0$) subspace spanned by the logical basis

$$|0_L\rangle = \tfrac{1}{2}(|\uparrow\downarrow\rangle_{12} - |\downarrow\uparrow\rangle_{12})(|\uparrow\downarrow\rangle_{34} - |\downarrow\uparrow\rangle_{34}) \tag{18}$$

$$|1_L\rangle = \tfrac{1}{\sqrt{3}}(|\uparrow\uparrow\downarrow\downarrow\rangle_{1234} + |\downarrow\downarrow\uparrow\uparrow\rangle_{1234}) - \tfrac{1}{2\sqrt{3}}(|\uparrow\downarrow\rangle_{12} + |\downarrow\uparrow\rangle_{12})(|\uparrow\downarrow\rangle_{34} + |\downarrow\uparrow\rangle_{34})\,, \tag{19}$$

where $\{|\uparrow\rangle, |\downarrow\rangle\}$ is *any* orthogonal basis for the single qubit spin Hilbert space. Because all states in this subspace possess zero total angular momentum, the group of rotations acts trivially on this subspace. Thus, the superoperator $\mathcal{E}_4$ preserves the two-dimensional subspace spanned by these states, i.e., this subspace is a decoherence-free subspace. Encodings become more efficient for larger N, and also if noiseless subsystems [10] (rather than subspaces) are used. Asymptotically, the number of logical qubits that can be encoded into N spin-1/2 particles in this manner is $N - \log_2 N$ [16].

This scheme for encoding quantum information into noiseless subsystems is relativistically invariant because the encoded states (in a noiseless subsystem $\mathcal{H}_{jS}$) are decoupled from any degree of freedom associated with a reference frame (i.e., spatial and angular momentum degrees of freedom). The states describe entirely *relative* properties of the particles[17], evidenced by the fact that the noiseless subsystems carry irreducible representations of the symmetric group for N particles. Thus, it is also interesting to note that, for states of this form, Bob can perform measurements of linear and angular momentum without disturbing the encoded states, and in doing so obtain information about Alice's reference frame. For example, measuring the total linear momentum provides information about the boost that relates Alice's frame to Bob's, whereas performing measurements on the SU(2) representation subsystems $\mathcal{H}_{jR}$ can provide information about the orientation of Alice's frame relative to Bob's (provided that Alice prepared an appropriate state in this subsystem) [6]. Thus, the decomposition (17) of states of N particles into subsystems provides a division between states describing extrinsic (spatial) and intrinsic properties. A key observation about this encoded relativistically invariant quantum information is that it *cannot* be used for tasks such as reference frame alignment because if its fundamentally intrinsic nature.

Photons. Much of the analysis for the massive particles applies to massless photons as well, albeit with a different little group; thus, only the key points of the photonic case will be mentioned. The discrete degrees of freedom for photons transform under a representation of the little group for massless particles, and not under SU(2). The invariant subspaces under this group are the subspaces with zero helicity. Consider two entangled well-separated and therefore distinguishable wave packets, with the same momentum profile centered on p (the construction for creating distinguishable qubits follows the massive case). For example, the states

$$|\Psi_p^\pm\rangle = \tfrac{1}{\sqrt{2}}\big(|p,+\rangle|p,-\rangle \pm |p,-\rangle|p,+\rangle\big)\,, \tag{20}$$

both satisfy $\mathbf{J}\cdot\mathbf{P}|\Psi_p^\pm\rangle = 0$. The little group element for photons in the fiducial state $p^\mu = (k,0,0,k)$ is decomposed as [11, 18]

$$W(\Lambda,p) = S(\alpha,\beta)R_z(\omega(\Lambda,\hat{\mathbf{p}})) \tag{21}$$

where $R_z(\omega)$ is a rotation by $\omega \in [0,2\pi)$ about the z-axis and S acts trivially on the physical states. The unitary representation of the little group is just $U_{\sigma\sigma'}(W(\Lambda,p)) = e^{\mathrm{i}\omega\sigma}\delta_{\sigma\sigma'}$ where $\sigma = \pm 1$ denotes helicity. The states transform as:

$$U(\Lambda)|p,\pm\rangle = e^{\pm\mathrm{i}\omega(\Lambda,\hat{\mathbf{p}})}|\Lambda p,\pm\rangle \tag{22}$$

Thus under a general Lorentz transformation the states $|\Psi_p^\pm\rangle$ will transform as

$$\begin{aligned} U(\Lambda)|\Psi_p^\pm\rangle &= \tfrac{1}{\sqrt{2}}\big(|\Lambda p,+\rangle_1|\Lambda p,-\rangle_2 \pm |\Lambda p,-\rangle_1|\Lambda p,+\rangle_2\big) \\ &= |\Psi_{\Lambda p}^\pm\rangle\,. \end{aligned} \tag{23}$$

Thus one logical qubit can be encoded with two physical qubits (photons) using the states $|\Psi_p^\pm\rangle$ as a basis. Asymptotically, it is possible to encode $N - 2^{-1}\log_2 N$ qubits in N photons. This encoding is analogous to the case of massive particles with one direction shared between Alice and Bob [15], which uses the noiseless subsystems that protect against collective dephasing [19].

For quantum information processing, it is also necessary to perform encoded logical operations. Using the noiseless subsystems for encoded states, the encoded operations are all given by exchange interactions [16]. For elementary spin-1/2 particles confined to a lattice as we describe, one would naturally expect exchange interactions between the qubits; to perform encoded operations, these interactions must be controlled using electromagnetic fields. Finally, measurements may be performed by performing projective measurements pairwise onto singlet states. For photons, recent progress in single photon sources (c.f. [20]) may soon be able to create the entangled encoded states of Eq. (20) with the necessary wavepacket profiles and these advances give promise for experimental realizations in the near future.

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