

Quantum phase transitions in a linear ion trap

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We show that the quantum phase transition of the Tavis-Cummings model can be realised in a linear ion trap of the kind proposed for quantum computation. The Tavis-Cummings model describes the interaction between a bosonic degree of freedom and a collective spin. In an ion trap, the collective spin system is a symmetrised state of the internal electronic states of N ions, while the bosonic system is the vibrational degree of freedom of the centre of mass mode for the ions.

I. INTRODUCTION

More than two decades ago, when quantum optics was young, the quantum dynamics of collective spin systems interacting with a single bosonic degree of freedom was a major research problem. The model arose as an attempt to describe the interaction between a collection of two level atoms and a single mode of the radiation field. Walls and co workers [1] were among the first to realise that such models provided ideal examples of the role of quantum fluctuations in the nonlinear interaction between matter and light. Quantum fluctuations were shown to drastically change the predictions of semiclassical theory in such systems. This phenomenon has appeared more recently in the discovery of quantum phase transitions in quantum spin glasses [2] and other many body quantum systems. While the collective spin models did not directly apply to achievable experiments at the time, they did provide insight that subsequently proved important for many other quantum optical experiments including anti-bunching, squeezing [3], and cavity QED [4]. In this paper we show that the models of a collective spin interacting with one or more bosonic modes can now be experimentally realised in modern ion trap systems of the kind proposed for quantum computation [5,6]. An enormous effort has gone into making such systems work at the quantum level, with little interference from classical sources of noise, and a number of such experiments exist today. It would thus appear worthwhile to reconsider the collective spin models, and the associated quantum many-body effects exhibited by such systems, with a view to direct experimental realisation.

In particular we consider the Tavis-Cummings (TC) model [7], which can be realised in a linear ion trap of N ions with the bosonic degree of freedom appearing as the quantised collective centre-of-mass motion. If each ion is coupled to the vibrational motion using an identical external (classical) laser detuned to the first red-sideband transition, the symmetry is such that the electronic degree of freedom for the ions can be described as a collective spin (N) and the reversible dynamics is well described by the TC model. The TC model is known to exhibit important nonlinear quantum effects including a quantum phase transition [2] in which the (zero temperature) ground state undergoes a morphological change as a parameter is varied and averages of intensive quantities undergo a bifurcation.

II. THE TAVIS-CUMMINGS MODEL

The interaction Hamiltonian for N ions interacting with the centre of mass vibrational mode can be controlled by using different kinds of Raman laser pulses. A considerable variety of interactions has already been achieved or proposed [5,6,8]. In this paper we consider the first red-sideband transition. The ion is assumed to be in a three dimensional anisotropic harmonic potential. Two dimensions are very tightly bound and are neglected. In the remaining dimension, an external laser couples the electronic state to the vibrational motion. If the vibrational frequency is large enough and the Lamb-Dicke limit [5] applies the motional sidebands of the absorption of the electronic transition can be resolved and a laser detuned below the electronic resonance by one unit of the trap frequency can excite the electronic transition by absorbing one vibrational phonon, the additional energy required being made up by the laser. We will assume that the laser (or lasers if a Raman process is used) is sufficiently strong that it can be treated classically. Under these assumptions the Hamiltonian, in the interaction picture, is

$$H_I = \hbar\Omega \sum_{i=1}^N (a\sigma_+^{(i)} + a^\dagger\sigma_-^{(i)}) \quad (1)$$

where the coupling constant is $\Omega = \eta\Omega_0$ where $\eta^2 = E_r/(\hbar M\omega_0)$ is the Lamb-Dicke parameter with E_r the recoil kinetic energy of the atom, ω_0 is the trap vibrational frequency, and M is the effective mass for the centre-of-mass mode. The Lamb-Dicke limit assumes $\eta \ll 1$, which is easily achieved in practice. The frequency, Ω_0 is the effective Rabi frequency for the electronic transition involved. The raising and lowering operators for each ion are defined by $\sigma_- = |g\rangle\langle e|$ and $\sigma_+ = |e\rangle\langle g|$. This sideband transition can be used to efficiently cool the ions to the collective centre-of-mass ground state, thus preparing the system in the vibrational ground state [5].

If the external laser field on each ion is identical (in amplitude and phase) the interaction Hamiltonian is

$$H_I = \hbar\Omega(a\hat{J}_+ + a^\dagger\hat{J}_-) \quad (2)$$

where we have introduced the bosonic annihilation operator a for the centre-of-mass vibrational mode and where we have used the definition of the collective spin operators,

$$\hat{J}_\alpha = \sum_{i=1}^N \sigma_\alpha^{(i)} \quad (3)$$

where $\alpha = x, y, z$. Identical laser fields could easily be obtained by splitting a single, stabilised laser into multiple beams. The interaction Hamiltonian in Eq (2) specifies the Tavis-Cummings model [7]. This model first appeared in quantum optics where the bosonic mode is the quantised field in a cavity. However this realisation is difficult to achieve experimentally. In contrast the vibrational mode realisation should be readily achieved. The dynamics resulting from this Hamiltonian is quite rich. Collective spin models of this kind were considered many decades ago in quantum optics [9,10]. In much of that work however the collective spin underwent an irreversible decay. In the case of an ion trap model however we can neglect such decays due to the long lifetimes of the excited states. On the other hand heating of the vibrational centre-of-mass mode can induce irreversible dynamics in the system in a manner that has not been previously considered, and that is reminiscent of thermal effects in condensed matter physics.

We are interested in the driven Tavis-Cummings model in which the vibrational mode is subject to a linear forcing term which can easily be achieved by a suitable combination of Raman laser pulses, or by appropriate AC voltages applied to the trap electrodes [5]. In this case the Hamiltonian, in the interaction picture, is given by

$$H_I = \hbar\Omega(a\hat{J}_+ + a^\dagger\hat{J}_-) + \hbar E(a + a^\dagger) \quad (4)$$

This may be written in terms of the hermitian canonical oscillator variables $\hat{X} = (a + a^\dagger)/\sqrt{2}$, $\hat{Y} = -i(a - a^\dagger)/\sqrt{2}$, and the canonical angular momentum variables $\hat{J}_x = (\hat{J}_+ + \hat{J}_-)/2$, $\hat{J}_y = -i(\hat{J}_+ - \hat{J}_-)/2$, $\hat{J}_z = [\hat{J}_+, \hat{J}_-]/2$. It takes the form

$$H = \hat{X}\hat{J}_x - \hat{Y}\hat{J}_y + \chi\hat{X} \quad (5)$$

with $\chi = E/\Omega$ and we have scaled the Hamiltonian by $H \rightarrow H/\sqrt{2}\Omega$. This indicates that time is measured in units of $\frac{1}{\sqrt{2}\Omega}$.

Alsing [11] has shown that the ground state of this system, for weak driving, is a product state in which the bosonic mode is squeezed and the electronic states are rotated in the angular momentum space. We provide a direct proof of this statement below. However it is first useful to consider the dynamics of the equivalent semiclassical model as many of the results in the quantum case can be interpreted in terms of the features of the semiclassical model.

A. Semiclassical Tavis-Cummings model

The Tavis-Cummings model represents an interaction between a simple harmonic oscillator and a linear top for which there is a classical model which we now define. We choose the classical model so that the equations of motion are of the same form as the Heisenberg equations of motion for the quantum model. The classical Hamiltonian is defined as

$$\mathcal{H} = X\mathcal{J}_x - Y\mathcal{J}_y + \chi EX \quad (6)$$

where X, Y are respectively the canonical oscillator position and momentum variables with the canonical Poisson bracket $\{X, Y\} = 1$, while \mathcal{J}_k are the three components of angular momentum for a classical top with the canonical Poisson brackets $\{\mathcal{J}_i, \mathcal{J}_k\} = \sum_j \epsilon_{ijk} \mathcal{J}_j$. The equations of motion for a canonical coordinate w is given as usual by Poisson bracket with the Hamiltonian $\dot{w} = \{w, \mathcal{H}\}$. The equations of motion are,

$$\dot{X} = -\mathcal{J}_Y \quad (7)$$

$$\dot{Y} = -\mathcal{J}_X - \chi \quad (8)$$

$$\dot{\mathcal{J}}_x = -Y \mathcal{J}_z \quad (9)$$

$$\dot{\mathcal{J}}_y = -X \mathcal{J}_z \quad (10)$$

$$\dot{\mathcal{J}}_z = X \mathcal{J}_y + Y \mathcal{J}_x \quad (11)$$

Note that these equations have a conservation law $\mathcal{J}_x^2 + \mathcal{J}_y^2 + \mathcal{J}_z^2 = \text{constant}$.

We now justify this choice of classical Hamiltonian by noting that the Heisenberg equations of motion for the Hamiltonian Eq(4) have the same form as the semiclassical equations of motion with all variables replaced by the corresponding operators. We thus see that the semiclassical equations result from taking moments of the Heisenberg equations and factorising all product moments. The factorisation assumption ignores correlations which scale as $1/N$ for the scaled operators \hat{J}_σ/N . The conservation law $\mathcal{J}_x^2 + \mathcal{J}_y^2 + \mathcal{J}_z^2 = \text{constant}$ is a reflection of the operator relation

$$\hat{J}^2 = \frac{N}{2} \left(\frac{N}{2} + 1 \right) \quad (12)$$

which in the semiclassical limit indicates that $\mathcal{J}_x^2 + \mathcal{J}_y^2 + \mathcal{J}_z^2 = \frac{N^2}{4}$.

The classical equations have one nontrivial fixed point at $X^* = Y^* = \mathcal{J}_y^* = 0$ and $\mathcal{J}_x^* = -\chi$, $\mathcal{J}_z^* = \sqrt{N^2/4 - \chi^2}$. However as the conservation law requires that $|\mathcal{J}_x| \leq N/2$ we see that we must have

$$\frac{2E}{N\Omega} \leq 1 \quad (\text{below threshold}) \quad (13)$$

which corresponds to an energy of $\mathcal{H} = 0$. We will refer to this as the *below threshold* case. As E is increased from zero, the fixed point for the angular momentum system rotates about the \mathcal{J}_y direction eventually reaching the equatorial plane at $\mathcal{J}_x = -N/2$ at the threshold condition. The oscillator system always has zero amplitude below threshold. If we linearise around this fixed point we discover that it is an unstable hyperbolic point with time constant proportional to $\frac{1}{\sqrt{\mathcal{J}_z^*}}$. Note that this time constant goes to infinity as the fixed point is approached as is typical for a hyperbolic fixed point.

We now consider the *above threshold* case

$$\frac{2E}{N\Omega} \geq 1 \quad (\text{above threshold}) \quad (14)$$

Clearly the value of $|\mathcal{J}_x|$ cannot increase above $N/2$. Indeed there is no fixed point above threshold. However there is a special solution curve that continuously joins to the below threshold case for phase curves with $\mathcal{H} = 0$.

To see this we consider making a canonical transformation by a rotation in both the $X - Y$ plane and in the $\mathcal{J}_x, \mathcal{J}_y$ plane (see figure 1). The canonical transformations are

$$X = \bar{X} \cos \theta + \bar{Y} \sin \theta \quad (15)$$

$$Y = \bar{Y} \cos \theta - \bar{X} \sin \theta \quad (16)$$

$$\mathcal{J}_x = \bar{\mathcal{J}}_x \cos \theta - \bar{\mathcal{J}}_y \sin \theta \quad (17)$$

$$\mathcal{J}_y = \bar{\mathcal{J}}_x \sin \theta + \bar{\mathcal{J}}_y \cos \theta \quad (18)$$

The Hamiltonian then takes the form

$$\mathcal{H} = \bar{X}(\bar{\mathcal{J}}_x + \chi \cos \theta) - \bar{Y}(\bar{\mathcal{J}}_y - \chi \sin \theta) \quad (19)$$

The phase curves with $\mathcal{H} = 0$ now correspond to either

$$\bar{X} = 0 \quad ; \quad \bar{\mathcal{J}}_y = \chi \sin \theta \quad (20)$$

or

$$\bar{Y} = 0 \quad ; \quad \bar{\mathcal{J}}_x = -\chi \cos \theta \quad (21)$$

These phase curves smoothly join the fixed point at threshold if $\mathcal{J}_x = -N/2$ which implies

$$\cos \theta = \frac{N\Omega}{2E} \quad (22)$$

These solutions are illustrated in figure 1. Note that as $E \rightarrow \infty$ we have that $\bar{\mathcal{J}}_x$ eventually points in the direction of $-\mathcal{J}_y$ while phase curve in the oscillator phase space points along the Y axis, indicating that for large driving the system is essentially a particle in a linear potential which accelerates at constant rate. These results were first obtained by Alsing and Carmichael [12].

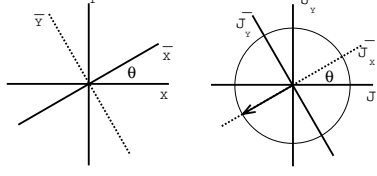


FIG. 1. An illustration of the canonical transformation used in the semiclassical equations above threshold

III. QUANTUM STATES

First note that the ground state when there is no driving is $|j, -j\rangle_z \otimes |0\rangle_v$ with a zero eigenvalue. This ground state corresponds to the fixed point of the semiclassical model with zero oscillator amplitude and angular momentum pointing in the $-\mathcal{J}_z$ direction. We postulate that as the driving is increased from zero the ground state of the Hamiltonian Eq(4) is given by

$$|\mathcal{E}_0\rangle = S(r)R(\theta)|j, -j\rangle_z \otimes |0\rangle_v \quad (23)$$

where $|j, -j\rangle_z \otimes |0\rangle_v$ corresponds to all ions in the ground state and the vibrational mode in the ground state. The operator $S(r)$ is a squeezing operator defined by

$$S^\dagger(r)aS(r) = \mu a + \nu a^\dagger \quad (24)$$

with $\mu = \cosh r$, $\nu = \sinh r$.

The rotation operator $R(\theta)$ is defined by

$$R(\theta) = e^{-\theta(\hat{J}_+ - \hat{J}_-)} \quad (25)$$

and corresponds to a rotation of 2θ around the \hat{J}_y axis. Consider now

$$H_I|\mathcal{E}_0\rangle = SR(R^\dagger S^\dagger HSR)|j, -j\rangle_z \otimes |0\rangle_v \quad (26)$$

If we now transform the Hamiltonian and require that

$$R^\dagger S^\dagger HSR|j, -j\rangle_z \otimes |0\rangle_v = 0 \quad (27)$$

we find the following conditions,

$$\nu(1 + \cos 2\theta) = \mu(1 - \cos 2\theta) \quad (28)$$

$$\Omega j \sin 2\theta = E \quad (29)$$

which requires that

$$\cos 2\theta = e^{-2r} \quad (30)$$

and the ground state energy is taken to be $\mathcal{E}_0 = 0$. The ground state is thus a product of a squeezed state for the vibrational mode and a rotated angular momentum state, rotated about the \hat{J}_y axis.

The above results are consistent with the semiclassical approximation. The mean amplitude of a squeezed vacuum state is zero, corresponding to the semiclassical fixed point at $\bar{X} = \bar{Y} = 0$ while the rotation around the \hat{J}_y axis corresponds to the semiclassical fixed point at $\bar{\mathcal{J}}_x = -\chi$.

If we continue to increase E above the threshold value the system adiabatically follows a zero energy state, although this is no longer a ground state. In fact the canonical transformation used in the semiclassical analysis can be applied

to the quantum operator valued Hamiltonian. The result is the same as the semiclassical case, Eq (19) with all variables replaced with the corresponding operators. The zero energy state then corresponds to the zero energy eigenstate of $\hat{Y} \cos \theta - \hat{X} \sin \theta$ with $\cos \theta = N\Omega/2E$. This is of course just a rotated, infinitely squeezed state. The electronic state is likewise a angular momentum eigenstate rotated from $|j, -j\rangle$ in the equatorial plane (orthogonal to \hat{J}_z). Thus above threshold the zero energy eigenstate deforms continuously from the state at threshold.

Let us summarise these results. For no driving the ground state corresponds to the oscillator in the ground state and all ions in the ground state. As the driving is increased, but kept below threshold, this state deforms to a squeezed oscillator state while the collective spin system begins to rotate about the \hat{J}_y axis. Note that the mean oscillator amplitude $\langle a \rangle$ remains zero as does the mean of the y -component of the collective spin. As the driving increases through the threshold value, this state changes its character so that a non zero value of \hat{J}_y is acquired and the oscillator is infinitely squeezed in a direction at an angle $\cos \theta = N\Omega/2E$ to the below threshold squeezing. This morphological change of the state as the driving passes the semiclassical critical point is a quantum phase transition. The quantum phase transition can be seen in the mean value for \hat{J}_y and \hat{J}_z as shown in figure 2. Below threshold the scaled mean values are given by

$$\frac{\langle \hat{J}_y \rangle}{N/2} = 0 \quad (31)$$

$$\frac{\langle \hat{J}_z \rangle}{N/2} = -\sqrt{1 - x^2} \quad (32)$$

and above threshold we have

$$\frac{\langle \hat{J}_y \rangle}{N/2} = -\sqrt{1 - \frac{1}{x^2}} \quad (33)$$

$$\frac{\langle \hat{J}_z \rangle}{N/2} = 0 \quad (34)$$

where $x = 2E/N\Omega$.

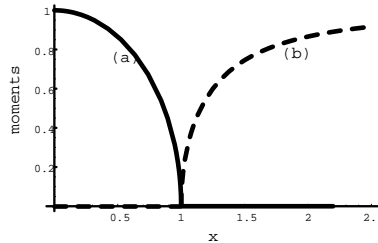


FIG. 2. The scaled moments, (a) $2|\langle J_x \rangle|/N$, and (b) $2|\langle J_y \rangle|/N$ plotted versus the scaled driving strength $x = 2E/(N\Omega)$.

What are the experimental manifestations of this transition ? Needless to say no one is ever going to observe an infinitely squeezed state in an experiment. So what does happens at $2\theta = \pi/2$ when the electronic state is the \hat{J}_x eigenstate $|j - j\rangle_x$ and the vibrational mode appears to be infinitely squeezed ? Is such a state physically possible ? Suppose for example we begin in the ground state of the Hamiltonian with no driving ($E = 0$) which is simply $|j, -j\rangle_z \otimes |0\rangle_v$, and adiabatically increase the driving strength. It would appear that the system would then adiabatically evolve into the squeezed vibrational state described above. If we were ever able to reach the case $2\theta = \pi/2$ we would have reached an infinite energy state for the vibrational mode at a finite driving strength. Clearly this is not possible and to understand why it is useful to reconsider the semiclassical dynamics for this model. The adiabatic approximation requires that we vary the driving strength on a time scale slower than all other time scales in the system. The key time scale for the ground state variation is just the time scale associated with the hyperbolic unstable fixed point, $(N^2/4 - \chi^2)^{-1/2}$, which goes to infinity as we approach $\frac{2E}{\Omega N} = 1$. Thus the adiabatic increase of the driving must proceed infinitely slowly, that is it must be switched to the finite value $E = \frac{\Omega N}{2}$ in an infinite amount of time. This pumps an infinite amount of energy into the system and results in infinite squeezing in the centre of mass vibrational mode. Obviously in practice this cannot be achieved so the totally squeezed ground state is not possible. However it will still be possible to achieve some squeezing of the vibrational mode at smaller values of the driving. This would make an interesting observation for current ion trap experiments even with only a few ions. The squeezing of the vibrational mode can be observed using the dynamical method of reference [13]

In current ion trap experiments, laser cooling techniques allow the centre of mass mode to be prepared in the ground state. Unfortunately it does not stay there. Heating due to a variety of sources, including fluctuating linear potentials, lead to an irreversible evolution away from the ground state. If such heating is present during the coupling of the electronic and vibrational motions, irreversible dynamics will be spread to the collective spin degrees of freedom as well.

As an example we consider what happens if we use the Tavis-Cummings interaction (excitation on first red sideband) in the presence of strong heating. Heating of the centre-of-mass mode due to fluctuating linear potentials may be described in the interaction picture by the master equation,

$$\frac{dW}{dt} = -i\Omega[a\hat{J}_+ + a^\dagger\hat{J}_-, W] + \frac{\gamma}{2} (\mathcal{D}[a] + \mathcal{D}[a^\dagger]) W \quad (35)$$

where W is the density operator for the spin and vibrational degrees of freedom and the superoperator \mathcal{D} is defined by $\mathcal{D}[A]\rho = 2A\rho A^\dagger - A^\dagger A\rho - \rho A^\dagger A$. The irreversible term corresponds to two point processes in which phonons are removed or added from centre of mass mode at the rates $\gamma\langle a^\dagger a \rangle$ and $\gamma\langle aa^\dagger \rangle$ respectively. This does not change any first order moments, however it does lead to a diffusion in energy as $\frac{d\langle a^\dagger a \rangle}{dt} = \gamma$. The effect of heating can be included in the semiclassical analysis by adding an appropriate stochastic term. In the Ito calculus [14] the effect is to add to the equations for X, Y terms of the form

$$dX = (\dots) + \sqrt{\gamma}dW_x(t) \quad (36)$$

$$dY = (\dots) + \sqrt{\gamma}dW_y(t) \quad (37)$$

where $dW_i(t)$ are independent Wiener processes. If the heating rate is small enough these terms can be neglected. However if they are large new steady states can occur in the semiclassical and quantum descriptions which will be described in a future publication.

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