

# Decoherence and fidelity in ion traps with fluctuating trap parameters

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We consider two different kinds of fluctuations in an ion trap potential: external fluctuating electrical fields, which cause statistical movement (“wobbling”) of the ion relative to the center of the trap, and fluctuations of the spring constant, which are due to fluctuations of the ac-component of the potential applied in the Paul trap for ions. We write down master equations for both cases and, averaging out the noise, obtain expressions for the heating of the ion. We compare our results to previous results for far-off resonance optical traps and heating in ion traps. The effect of fluctuating external electrical fields for a quantum gate operation (controlled-NOT) is determined and the fidelity for that operation derived.

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## I. INTRODUCTION

The ability to engineer and control pure quantum states of trapped ions is driving a number of new technologies including time and frequency measurements, new measurement schemes and quantum logic manipulations for quantum computation [1–3]. The key requirement for engineering quantum states is the necessity to remove, or at least control, all sources of noise and uncertainty. Laser cooling in particular enables uncertainty about the initial vibrational state to be removed by cooling the ions to the collective vibrational ground state. From that point pure states may be prepared using highly stabilized laser pulses. Despite these achievements, however, noise cannot be entirely eliminated. Residual laser intensity and phase fluctuations in the pulses that are used to shape the quantum states must be taken into account [4,5] as well as noise in the trapping parameters. In this paper we consider various sources of noise in the trap itself and determine their effect on cold trapped ions and the ability to perform reversible logical operations.

Noise is of course the origin of decoherence, the process which limits the ability to maintain pure quantum states. However, we need to be careful in making the connection between noise and decoherence. From a fundamental perspective the dynamics of the ions is always unitary and reversible, even in the presence of noise, but by definition the dynamics of a noisy quantity is uncontrollable and often unknown. Thus the precise unitary dynamics varies from one run of the experiment to the next and the exact motion of the state in Hilbert space may not be known or even precisely predictable. In the course of the experiment we do not have precise control over, or knowledge of, the unitary transformations in state engineering, and thus we cannot be sure we have reached the desired state in Hilbert space. In the case of quantum computation such a result is manifest as an error. Faced with describing such a system we can simply average over the noise, which in practical terms means we combine the data from many experiments all performed with different realizations of the noisy control parameters. Alternatively we can give the sample space of error states in each run, together with their probability of occurrence. In this paper we give both descriptions particularly for the case of fluctuations of the center (equilibrium point) of the trap potential.

In an ion trap an inability to precisely control the motion leads to an unwanted excitation of the vibrational state of the ion, that is ‘heating’. The main source of this heating appears to be due to the ambient fluctuating electrical fields in the trap. There are now a number of experiments that have measured this heating [2,6–8]. Recently James [9] has shown that a simple theory of this source of heating can be given in terms of a harmonic oscillator subject to a fluctuating classical driving field. Our model for this source of heating is similar, although our theoretical description is a little different.

The paper is organized as follows: In section II we first derive the heating rate of the ion due to fluctuating electrical fields. Section III is devoted to the effects of fluctuations in the spring constant of the trap potential on the heating. In the fourth section we look at the effects of fluctuating electrical fields on gate operations. As a specific example we investigate the effects on the so far experimentally realized controlled-NOT gate (NIST gate) [11]. We conclude with a discussion of our results.

## II. HEATING DUE TO FLUCTUATING ELECTRICAL FIELDS

We want to model the effects of fluctuating electrical fields with the same formalism used in [4] and compare our results to those obtained by Savard *et al.* [10]. There they derive the heating rate for a far-off resonance optical trap

due to fluctuations in the location of the trap center which are caused by laser-beam-pointing noise. We apply the formalism here to an ion in a Paul trap. In those traps additional electrical fields cause a replacement of the center of the trap, to which the ion adjusts automatically by finding the minimum of the potential. We assume that those electrical fields which cause this replacement have got an additional white noise component, to which the ion cannot adjust.

The Hamiltonian in this case is

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 + \xi(t)x , \quad (1)$$

where  $x$  and  $p$  are the position and momentum operator of the ion in the trap,  $m$  is the mass of the ion, and  $\omega$  is the trap frequency. The term  $\xi(t)$  describes the fluctuating electrical force due to fluctuating fields which we assume to be due to a white noise process, i.e.

$$\xi(t)dt = \sqrt{\gamma}dW(t) , \quad (2)$$

with  $dW(t)$  a Wiener process and the parameter  $\gamma$  scales the noise. Since we are dealing with a white noise process, we define a stochastic Schrödinger equation in the Ito formalism [12]

$$d\rho = -\frac{i}{\hbar}[H_0, \rho]dt - \frac{i}{\hbar}\sqrt{\gamma}[x, \rho]dW(t) - \frac{\gamma}{2\hbar^2}[x, [x, \rho]] , \quad (3)$$

where

$$H_0 = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 . \quad (4)$$

For a single run with a known (or monitored) behaviour of the noise in time, the above equation gives the evolution of the system density operator conditioned on the entire history of the noise process. Since we are not interested in the effects of the fluctuations in some short time limit or just for one run of the experiment, we follow [4] and average over the noise to get the master equation for the average density operator

$$\frac{d\tilde{\rho}}{dt} = -\frac{i}{\hbar}[H_0, \tilde{\rho}] - \frac{\gamma}{2\hbar^2}[x, [x, \tilde{\rho}]] . \quad (5)$$

This equation has a 'high frequency' limit which may be relevant in experiments for which the time scale is much longer than the period of the trap. For example if the trapped ions form a quantum logic gate the time over which the gate operation is imposed may be much greater than the trap period (which is typically of the order of  $10^{-6}$ s). In that case we can transform to a frame rotating at the trap frequency and time average the rapidly rotating terms to give a master equation in the form

$$\frac{d\rho}{dt} = \frac{\gamma}{2\hbar m\omega} \left( a\rho a^\dagger + a^\dagger \rho a - \frac{1}{2}(a^\dagger a \rho + a a^\dagger \rho + \rho a^\dagger a + \rho a a^\dagger) \right) . \quad (6)$$

To calculate the mean energy

$$\langle H_0 \rangle(t) = \frac{\langle p^2 \rangle}{2m} + \frac{1}{2}m\omega^2 \langle x^2 \rangle , \quad (7)$$

we look at the time derivative of  $\langle x^2 \rangle$  and  $\langle p^2 \rangle$  and get the two equations

$$\langle \dot{x}^2 \rangle = -\frac{1}{m} \langle xp + px \rangle \quad (8)$$

$$\langle \dot{p}^2 \rangle = m\omega^2 \langle xp + px \rangle + \gamma , \quad (9)$$

and thus

$$\frac{d\langle H_0 \rangle}{dt} = \frac{1}{2m} \gamma . \quad (10)$$

So the mean energy is

$$\langle H_0 \rangle(t) = \frac{1}{2m} \gamma t + \langle H_0 \rangle_{(t=0)} . \quad (11)$$

(The same result obtains with the time averaged master equation as well). This result is equivalent to that derived by Savard *et al.* [10] and by James [9] in the limit of white noise and with appropriate changes of notation.

To make the comparison with the results of Wineland *et al.* [2] we write Eq. (11) in terms of the mean vibrational quantum number  $\bar{n}$  as

$$\frac{d\bar{n}}{dt} = \frac{1}{t^*} , \quad (12)$$

where the time constant for decoherence is then given by

$$t^* = \frac{2\hbar m \omega}{\gamma} . \quad (13)$$

The fluctuating linear potentials are caused by fluctuating electric fields,  $E(t)$ , on the trap electrodes thus we expect that the fluctuating term in Eq. (1) is given by  $\xi(t) = qE(t)$ . If the fluctuations in the electric field are treated as white noise the spectral density of these fluctuations (near the trap frequency) is independent of frequency. If we take  $E(t)dt = E_0 dW(t)$  then we have the equivalence  $\gamma = q^2 E_0^2$  with the spectral density of fluctuations in the field given by

$$S_E(\omega) = 4 \int_0^\infty \overline{E(t+\tau)E(t)} d\tau \quad (14)$$

$$= 2E_0^2 . \quad (15)$$

Thus the decoherence time becomes

$$t^* = \frac{4\hbar m \omega}{q^2 S_E(\omega)} , \quad (16)$$

which is the form quoted in reference [2].

### III. NOISE IN THE SPRING CONSTANT

In an ion trap a suitable combination of ac and dc electric fields are used to create an approximate harmonic potential in three spatial dimensions for one or more ions [17]. In the case of more than one ion, the coulomb force couples the motion of the ions and the collective normal mode coordinates undergo harmonic motion. Laser cooling enables the ion, or the collective mode of many ions, to be prepared in or near the ground state of the system. Noise in the spring constant is due to fluctuations in the ac-component and dc-components of the applied potential. In the linear trap of reference [2] with the long axis oriented along the z-axis, and with micromotion ignored, the periodic trap potential at the trap centre is given by

$$\Phi \approx \frac{1}{2} m \omega_z^2 + \frac{1}{2} m \omega_r^2 (x^2 + y^2) , \quad (17)$$

where the harmonic force in the z-direction is formed by a static potential and in the  $x - y$  direction it is formed by an ac potential at frequency  $\Omega_T$  and amplitude  $V_0$ . The resonance frequency is then given by  $\omega_r = qV_0 / (2^{1/2} \Omega_T m R^2)$ , where  $m$  and  $q$  are the ion mass and charge and  $R$  is the distance from the z-axis to the surface of the linear electrodes. Clearly fluctuations in either the dc or ac component will lead to fluctuations in the spring constants of the trap, although in practice ac fluctuations are more significant.

In far-off resonance optical traps, the restoring force is provided by the induced optical dipole force of an applied laser. Typically the atom is confined at the node of a standing wave of a laser tuned above the atomic resonance (blue detuning). The atom then sees a mechanical potential proportional to the intensity of the laser. If the intensity is quadratic near the node, a linear restoring force will be produced. In this case fluctuations in the applied laser intensity lead to fluctuations in the trap frequency [10].

The Hamiltonian for a particle moving in a harmonic potential with fluctuating spring constant is

$$H = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 (1 + \epsilon(t)) x^2 , \quad (18)$$

where again we assume white noise,

$$\epsilon(t)dt = \sqrt{\Gamma}dW(t) , \quad (19)$$

and the noise this time is scaled with the parameter  $\Gamma$ . To make things easier, we introduce dimensionless coordinates

$$x \rightarrow \left(\frac{2\hbar}{m\omega}\right)^{-1/2} x = X \quad (20)$$

$$p \rightarrow (2\hbar m\omega)^{-1/2} p = P , \quad (21)$$

so that the commutation relation in these new coordinates now reads

$$[X, P] = \frac{i}{2} . \quad (22)$$

Thus our Hamiltonian takes on the form

$$\begin{aligned} H &= \hbar\omega (P^2 + X^2) + \epsilon(t)\hbar\omega X^2 \\ &= H_0 + \epsilon(t)\hbar\omega X^2 . \end{aligned} \quad (23)$$

Again we are not interested in a specific noise history, but rather in the overall effect of the fluctuations. So we average out the noise and the corresponding master equation reads

$$\frac{d\tilde{\rho}}{dt} = -\frac{i}{\hbar}[H_0, \tilde{\rho}] - \frac{\Gamma}{2}\omega^2[X^2, [X^2, \tilde{\rho}]] . \quad (24)$$

We want to determine the change of the mean energy with time. The mean energy is given by the expression

$$\langle H_0 \rangle(t) = \hbar\omega (\langle P^2 \rangle(t) + \langle X^2 \rangle(t)) . \quad (25)$$

We wish to get the expressions for  $\langle P^2 \rangle(t)$  and  $\langle X^2 \rangle(t)$ . To do this, we can first derive a system of first order differential equations for  $\langle P^2 \rangle(t)$ ,  $\langle X^2 \rangle(t)$  and  $\langle XP + PX \rangle(t)/2$ :

$$\frac{d}{dt} \begin{pmatrix} \langle X^2 \rangle \\ \langle P^2 \rangle \\ \frac{1}{2}\langle XP + PX \rangle \end{pmatrix} = A \begin{pmatrix} \langle X^2 \rangle \\ \langle P^2 \rangle \\ \frac{1}{2}\langle XP + PX \rangle \end{pmatrix} , \quad (26)$$

where

$$A = \begin{pmatrix} 0 & 0 & 2\omega \\ \Gamma\omega^2 & 0 & -2\omega \\ -\omega & \omega & 0 \end{pmatrix} . \quad (27)$$

So we have to solve this system of differential equations to get the solution for  $\langle X^2 \rangle(t)$  and  $\langle P^2 \rangle(t)$ . The exact solution is

$$\begin{aligned} \langle H_0 \rangle(t) &= \hbar\omega \left\{ \left[ \exp\left(\frac{2(D^2-1)\omega}{\sqrt{3}D}t\right) \frac{(2-D^2+2D^4)(1+D^2+D^4)}{9D^2(1-D^2+D^4)} \right. \right. \\ &\quad + \exp\left(-\frac{(D^2-1)\omega}{\sqrt{3}D}t\right) \left( -\frac{2(D^2-1)^4}{9D^2(1-D^2+D^4)} \cos\left(-\frac{(1+D^2)\omega}{D}t\right) \right. \\ &\quad \left. \left. + \frac{2(1-D^2)(1+2D^2-D^4+D^6+D^8)}{3\sqrt{3}D^2(1+D^6)} \sin\left(-\frac{(1+D^2)\omega}{D}t\right) \right] \langle X^2 \rangle_{t=0} \right. \\ &\quad + \left[ \exp\left(\frac{2(D^2-1)\omega}{\sqrt{3}D}t\right) \frac{1+D^2+D^4}{3(1-D^2+D^4)} \right. \\ &\quad + \exp\left(-\frac{(D^2-1)\omega}{\sqrt{3}D}t\right) \left( \frac{2(D^2-1)^2}{3(1-D^2+D^4)} \cos\left(-\frac{(1+D^2)\omega}{D}t\right) \right. \\ &\quad \left. \left. + \frac{2D^2(D^2-1)}{\sqrt{3}(1+D^2)(1+D^2+D^4)} \sin\left(-\frac{(1+D^2)\omega}{D}t\right) \right] \langle P^2 \rangle_{t=0} \right\} \end{aligned}$$

$$\begin{aligned}
& + \left[ \exp\left(\frac{2(D^2-1)\omega}{\sqrt{3}D}t\right) \frac{2(D^6-1)}{3\sqrt{3}D(1-D^2+D^4)} \right. \\
& + \exp\left(-\frac{(D^2-1)\omega}{\sqrt{3}D}t\right) \left( -\frac{2(D^6-1)}{3\sqrt{3}D(1-D^2+D^4)} \cos\left(-\frac{(1+D^2)\omega}{D}t\right) \right. \\
& \left. \left. + \frac{2(D^2-1)^2(1+D^2+D^4)}{3D(1+D^6)} \sin\left(-\frac{(1+D^2)\omega}{D}t\right) \right) \right] \frac{1}{2} \langle XP + PX \rangle_{t=0} \Bigg\} , \tag{28}
\end{aligned}$$

where

$$D = \sqrt[3]{\frac{3\sqrt{3}\Gamma\omega}{4} \frac{\Gamma\omega}{2} + \sqrt{1 + \frac{27}{16} \left(\frac{\Gamma\omega}{2}\right)^2}} . \tag{29}$$

To simplify things, we assume that the noise is a small effect compared to the free dynamics, i.e.

$$\frac{\Gamma}{2}\omega \ll 1 . \tag{30}$$

Using this approximation we get

$$\langle H_0 \rangle(t) = \hbar\omega \exp\left[\frac{\Gamma\omega^2}{2}t\right] (\langle P^2 \rangle_{(t=0)} + \langle X^2 \rangle_{(t=0)}) . \tag{31}$$

Again this result is equivalent to the one given by Savard *et al.* [10] in the limit of white noise. In Fig. 1 both the exact (solid line) and the approximated (dashed line) solution are plotted for the vibrational frequency  $\omega = 11.2(2\pi)$  kHz and the value of  $\Gamma\omega/2$  is 0.1 in this plot. We choose the initial values to be  $\langle X^2 \rangle_{t=0} = \langle P^2 \rangle_{t=0} = (1/2)\langle XP + PX \rangle_{t=0} = 1/4$ , so that  $\langle H_0 \rangle_{t=0}/\hbar\omega = 1/2$ .

#### IV. EFFECTS OF FLUCTUATING EXTERNAL ELECTRICAL FIELDS DURING QUANTUM GATE OPERATIONS

We have investigated the overall effects of the fluctuations in the spring constant and the position in ion traps. It is of interest to look at the effects of those fluctuations on gate operations used for quantum computation [14]. The gate operations are performed by shining an additional laser, with a specific frequency and for a well determined time, on a two-level transition in an ion. This laser causes an interaction between the internal electronic states of the ion and the CM motion of the ion or of all the ions if there is more than one ion in the trap.

We will denote the electronic qubit as  $|g\rangle$ ,  $|e\rangle$  for the ground state and the excited state respectively. The coding is such that the ground state is logical 0 while the excited state is logical 1. The vibrational state will be denoted by the energy eigenstates  $|0\rangle_v$ ,  $|1\rangle_v$  which are the ground state and the first excited state respectively.

A controlled-NOT gate can be broken down into the circuit shown in figure 2, where a controlled phase shift is sandwiched between two  $\pi/2$  pulses with different phases on the target qubit, which in this case is the electronic qubit. The  $\pi/2$  pulses produce rotations of the electronic qubit:

$$U_R^+ : \begin{cases} |0\rangle \rightarrow 1/\sqrt{2}(|0\rangle - |1\rangle) \\ |1\rangle \rightarrow 1/\sqrt{2}(|0\rangle + |1\rangle) \end{cases} \tag{32}$$

$$U_R^- : \begin{cases} |0\rangle_j \rightarrow 1/\sqrt{2}(|0\rangle + |1\rangle) \\ |1\rangle_j \rightarrow -1/\sqrt{2}(|0\rangle - |1\rangle) \end{cases} . \tag{33}$$

The controlled phase shift  $U_P$  acts to produce a  $\pi$  phase shift only if both the electronic qubit and the vibrational qubit is in the logical state  $|1\rangle_L$ . The total transformation from input to output is then given by

$$|\Psi_{out}\rangle = U_R^- U_P U_R^+ |\Psi_{in}\rangle . \tag{34}$$

We take the most general input state to the controlled-NOT gate as

$$|\Psi_{in}\rangle = (\alpha|g\rangle + \beta|e\rangle) \otimes (\delta|0\rangle_v + \epsilon|1\rangle_v) \tag{35}$$

with  $\alpha, \beta, \gamma$  and  $\delta$  being complex amplitudes satisfying

$$|\alpha|^2 + |\beta|^2 = |\delta|^2 + |\epsilon|^2 = 1 . \quad (36)$$

The first  $\pi/2$  pulse  $U_R^+$  acting on the target (electronic) qubit, then produces the state;

$$|\Psi_1\rangle = \frac{\alpha + \beta}{\sqrt{2}}\delta|g\rangle|0\rangle_v + \frac{\beta - \alpha}{\sqrt{2}}\delta|e\rangle|0\rangle_v + \frac{\alpha + \beta}{\sqrt{2}}\epsilon|g\rangle|1\rangle_v + \frac{\beta - \alpha}{\sqrt{2}}\epsilon|e\rangle|1\rangle_v . \quad (37)$$

where the subscript 1 indicates that this is the state after the first rotation. The controlled phase shift operation just changes the sign of the last term. Then the final rotation  $U_R^-$  takes this state to the output state

$$|\Psi_{out}\rangle = \alpha\delta|g\rangle|0\rangle_v + \beta\delta|e\rangle|0\rangle_v + \alpha\epsilon|e\rangle|1\rangle_v + \beta\epsilon|g\rangle|1\rangle_v . \quad (38)$$

If the vibrational qubit is subject to noise in the trap center position, i.e. there are fluctuating electrical fields, the gate will not operate as required. The noise does not effect the rotations, which only involve the electronic qubit (except for heating up the vibarional state during those rotations), however, it will disrupt the controlled phase shift operation which couples the electronic and vibrational systems. We first make the usual transformation into the interaction picture with

$$U_0 = \exp\left[\frac{i}{\hbar}H_0t\right] , \quad (39)$$

where

$$H_0 = \hbar\omega a^\dagger a + \hbar\omega_A\sigma_+\sigma_- , \quad (40)$$

to give

$$H_I = H_G - \hbar\lambda\xi(t) (a^\dagger e^{i\omega t} + a e^{-i\omega t}) , \quad (41)$$

where  $H_G$  is the interaction picture Hamiltonian causing the gate operation and where

$$\lambda = (2\hbar m\omega)^{-1/2} . \quad (42)$$

To include the noise we need to calculate the effective controlled phase shift operation including the noise term over the time of the gate operation. The delta-correlated nature of white noise enables a simple approach in which noise terms can be separated and treated perturbatively while the gate interaction is treated to all orders. To enable this approximation we first transform to an interaction picture defined by

$$|\Psi(t)\rangle = \exp\left(-\frac{i}{\hbar}H_Gt\right) |\Psi_I(t)\rangle . \quad (43)$$

The total time evolution for the controlled phase-shift gate over the gate time  $T$  is given by  $|\Psi_2'\rangle = U_P'|\Psi_1\rangle$  with

$$\begin{aligned} U_P' &= \exp\left(-\frac{i}{\hbar}H_GT\right) \mathcal{T}\left(\exp\left[-\frac{i}{\hbar}\int_0^T dt' H_{noise}(t')\right]\right) \\ &= U_P U_N[\xi(t)] , \end{aligned} \quad (44)$$

where

$$H_{noise}(t) = \exp\left(\frac{i}{\hbar}H_Gt\right) (-\hbar\lambda\xi(t) (a^\dagger e^{i\omega t} + a e^{-i\omega t})) \exp\left(-\frac{i}{\hbar}H_Gt\right) \quad (45)$$

and where  $\mathcal{T}$  is the time ordering operator and  $T$  the time required for the gate operation in the absence of noise. We have indicated the functional dependence on a particular realization of the noise by  $[\xi(t)]$ . The time-ordered evolution operator appearing as the second factor in Eq. (44) may be treated perturbatively in the stochastic amplitude,  $\xi(t)$ , by carrying the Dyson expansion to second order.

There are two ways to view the effects of the noise. One way is at the level of a single realization of a gate operation. This view enables us to see what the error states will be in the presence of noise. The second way is to determine the result of a gate operation by ensemble averaging the noise. This view enables us to give an average fidelity for the

gate operation in the presence of the noise. The two views will be referred to as a quantum trajectory picture and an ensemble average picture, respectively. In the quantum trajectory picture the output state is a functional of the particular noise history,  $\xi(t)$ , over the gate operation time  $T$ . The output state is then a pure state of the form

$$|\Psi'[\xi(t)]\rangle = U_R^- U_P U_N[\xi(t)] |\Psi_1\rangle, \quad (46)$$

where, as earlier,  $|\Psi_1\rangle$  is the state after the first rotation of the electronic qubit. To find the likely error states the noise operator can now be expanded in powers of the noise amplitude.

In the ensemble picture we need to average over all possible realizations of the noise over the gate time  $T$ . The output state is now a mixed state given by

$$\rho'_{out} = U_R^- U_P \rho'_1 U_P U_R^- \quad (47)$$

and

$$\rho'_1 = \int U_N([\xi]) |\Psi_1\rangle \langle \Psi_1 | U_N^\dagger([\xi]) P[\xi] d[\xi], \quad (48)$$

where  $P[\xi]d[\xi]$  is the probability functional for each noise realization. To calculate  $\rho'_1$  we expand the unitary operator  $U_N[\xi(t)]$  to second order in the noise and then average over the classical stochastic variables. The evolution of the density operator over the gate time  $T$  is then given by a Dyson-expansion [13] which to second order is

$$\rho'_1 \cong \rho_1 + \frac{1}{i\hbar} \int_0^T [H_{noise}(t_1), \rho_1] dt_1 + \left(\frac{1}{i\hbar}\right)^2 \int_0^T dt_1 \int_0^{t_1} dt_2 [H_{noise}(t_1), [H_{noise}(t_2), \rho_1]]. \quad (49)$$

Since  $E(dW(t)) = 0$ , the average over the second term vanishes and we only have to calculate the average over the third term. This is done by noting that [16]

$$\left\langle \int_0^t G(t') dW(t') \int_0^{t'} G(t'') dW(t'') \right\rangle = \int_0^t G(t')^2 dt'. \quad (50)$$

We can quantify the effect of noise on the average through the fidelity defined by

$$F(\gamma) = \langle \Psi_{out} | \rho'_{out} | \Psi_{out} \rangle, \quad (51)$$

where  $|\Psi_{out}\rangle$  is the output state for a noiseless gate operation while  $\rho'_{out}$  is the output state of the gate averaged over all realizations of the noise. The fidelity is the probability that the system is in the desired state and will depend on the noise correlation strength  $\gamma$ . Substituting Eqs. (34, 47) we find that

$$F(\gamma) = \langle \Psi_1 | \rho'_1 | \Psi_1 \rangle. \quad (52)$$

To proceed we take two examples for realizing a controlled-NOT gate between the vibrational state and the internal state for one ion in the trap.

### A. Mutual phase-shift gate

In the discussion of the perfect controlled-NOT gate we saw that the essential two-qubit operation is a controlled phase shift. We will discuss two different ways by which this can be done. The first way involves a mutual conditional phase shift of the vibrational and electronic degrees of freedom. This operator commutes with the vibrational quantum number. The second way involves an auxiliary electronic level and is used in the NIST scheme to produce a controlled-NOT gate [11]. We first discuss the mutual phase shift gate as this is simpler.

The mutual phase shift gate is defined by the unitary transformation

$$U_P = \exp(-i\pi a^\dagger a \otimes |e\rangle\langle e|). \quad (53)$$

To include the noise we need to calculate the effective controlled phase shift operation including the noise term over the time of the gate operation, where the gate Hamiltonian is

$$H_G = \hbar\kappa a^\dagger a \otimes |e\rangle\langle e| \quad (54)$$

and with the gate operation time such that  $\kappa T = \pi$ , where  $\kappa$  is a constant. An interaction of this kind can be produced by a carrier frequency excitation of the ion [15].

The total time evolution is then given by Eq. (44) with

$$H_{noise} = -\hbar\lambda\xi(t) (a \exp[-i\kappa t|e\rangle\langle e| - i\omega t] + h.c.) , \quad (55)$$

where  $h.c.$  indicates the hermitian conjugate of the preceding term. The effect of the noise is determined by expanding the second time ordered factor in Eq. (44) to second order, thus

$$U_N = 1 + i(a\nu(T) + a^\dagger\nu^\dagger(T)) - \frac{1}{2} \int_0^T dt \left\{ a \frac{d\nu(t)}{dt} + a^\dagger \frac{d\nu^\dagger(t)}{dt} \right\} \{a\nu(t) + a^\dagger\nu^\dagger(t)\} , \quad (56)$$

where the operators  $\nu$  are defined in the following way:

$$\nu(t) = \lambda\sqrt{\gamma} \int_0^t dW(t') e^{-i(\omega+\kappa|e\rangle\langle e|)t'} . \quad (57)$$

We first determine the likely error states in the quantum trajectory picture. The desired gate operation, Eq. (53), commutes with the phonon number operator  $a^\dagger a$  and cannot change the vibrational quantum number, and thus the vibrational states always remains in the logical basis of  $|0\rangle, |1\rangle$ . The noise factor  $U_N$ , however, is linear in  $a$  and  $a^\dagger$  and thus does change the phonon number. Keeping terms to second order in the stochastic amplitude means that, in a single realization of the gate, the effect of the noise is to ‘leak’ coherence into the vibrational states  $|2\rangle_v, |3\rangle_v$  as well as changing the weighting of the vibrational qubit basis states  $|g\rangle, |e\rangle$ . For example, in the quantum trajectory picture, a single realization of the noise takes the state Eq. (37) to the state

$$\begin{aligned} |\Psi'_2\rangle = e^{-i\pi a^\dagger a |e\rangle\langle e|} & \left( |\Psi_1\rangle + \frac{\alpha + \beta}{\sqrt{2}} \delta\nu_g^* |g\rangle |1\rangle_v + \frac{\beta - \alpha}{\sqrt{2}} \delta\nu_e^* |e\rangle |1\rangle_v \right. \\ & \left. + \frac{\alpha + \beta}{\sqrt{2}} \epsilon (\nu_g |g\rangle |0\rangle_v + \sqrt{2} \nu_g^* |g\rangle |2\rangle_v) + \frac{\beta - \alpha}{\sqrt{2}} \epsilon (\nu_e |e\rangle |0\rangle_v + \sqrt{2} \nu_e^* |e\rangle |2\rangle_v) + \dots \right) , \end{aligned} \quad (58)$$

where we have suppressed the second order terms, and the subscript 2 indicates that this is the state after the controlled phase shift gate operation, and the prime indicates a state corrupted by noise. The first term is the correct output state and all subsequent terms correspond to an error. The error terms are multiplied by random variables defined by

$$\nu_g = \lambda\sqrt{\gamma} \int_0^T dW(t) e^{-i\omega t} \quad (59)$$

$$\nu_e = \lambda\sqrt{\gamma} \int_0^T dW(t) e^{-i(\omega+\kappa)t} . \quad (60)$$

If the gate time  $T$  is large compared to the vibrational frequency, these random variables have zero mean and the correlation functions are [16]

$$E(\nu_g \nu_g) = E(\nu_e \nu_e) = 0 \quad (61)$$

$$E(\nu_g^* \nu_g) = E(\nu_e^* \nu_e) = \lambda^2 \gamma T \quad (62)$$

$$E(\nu_g^* \nu_e) = \frac{2i\lambda^2 \gamma T}{\pi} . \quad (63)$$

As mentioned above we can calculate the fidelity of the gate operation, Eq. (51, 52). The result up to first in  $\gamma$  is

$$\begin{aligned} F(\Gamma_\kappa) = 1 - 2\Gamma_\kappa(1 + 2|\epsilon|^2) + \Gamma_\kappa(1 - |\epsilon|^2)|\epsilon|^2 & \left[ |\beta - \alpha|^4 \left( 1 + \frac{\kappa}{\pi(\omega + \kappa)} \sin\left(\frac{\omega\pi}{\kappa}\right) \cos\left(\frac{\omega\pi}{\kappa} + 2\Delta\right) \right) \right. \\ & \left. + |\alpha + \beta|^4 \left( 1 + \frac{\kappa}{\pi\omega} \sin\left(\frac{\omega\pi}{\kappa}\right) \cos\left(\frac{\omega\pi}{\kappa} + 2\Delta\right) \right) - |\beta - \alpha|^2 |\alpha + \beta|^2 \frac{4\kappa}{\pi(2\omega + \kappa)} \cos\left(\frac{\omega\pi}{\kappa}\right) \sin\left(\frac{\omega\pi}{\kappa} + 2\Delta\right) \right] \end{aligned} \quad (64)$$

where

$$\Gamma_\kappa = \frac{\pi\gamma}{\hbar m \omega \kappa} \quad (65)$$



is the now dimensionless noise parameter and  $\Delta$  is the phase difference between  $\delta$  and  $\epsilon$

$$\Delta = \phi_\delta - \phi_\epsilon, \quad (66)$$

where we denote the phase of  $\delta$  ( $\epsilon$ ) by  $\phi_\delta$  ( $\phi_\epsilon$ ), respectively.

As expected for an expansion up to second order in  $\sqrt{\gamma}$  the fidelity depends linearly on  $\gamma$ , since we have averaged out the first order term. The inverse dependance on the nonlinear coupling constant  $\kappa$  is easy to understand. If this parameter is large, the gate time can be made very short in which case the noise has less time to act and produce an error. Thus the fidelity should approach one as  $\kappa$  becomes large. Note that in general the fidelity depends on the initial state. The dependence of  $F$  on the initial state is plotted in Figure 3. The plot parameters are  $\omega = 11.0(2\pi)$  MHz,  $\Omega\eta = 1.0(2\pi)$  kHz,  $\Delta = 0$ , and  $\Gamma_a = 0.02$ .

## B. NIST Gate

The Hamiltonian required to describe the controlled phase shift operation in the NIST gate [11] is

$$H_G = \hbar \frac{\Omega_a \eta}{2} (|e\rangle\langle \text{aux}| a^\dagger + |\text{aux}\rangle\langle e| a) \quad (67)$$

where  $\eta$  is the Lamb-Dicke-parameter, and  $a$  and  $a^\dagger$  are the creation and annihilation operators for the quantized CM motion of the ion. This Hamiltonian is used to describe a  $2\pi$  blue-sideband pulse between the excited state of the electronic qubit (logical 1) and an auxiliary state  $|\text{aux}\rangle$ . The rotations of the electronic states are carried out by  $\pi/2$  pulses that are assumed to act without noise. This assumption is reasonable since the effective Rabi frequencies for those transitions are higher than those for the blue sideband pulses, thus leading to much shorter pulse durations.

As before we separate the pure gate operation from the noise by transforming to an interaction picture through the gate Hamiltonian. We then determine the average fidelity for the gate operation by averaging over the noise.

Calculating  $\rho'_1$  using Eq. (49) and in particular Eq. (45) is a rather tedious, but straightforward process. The fidelity rate turns out to be

$$\begin{aligned} F(\Gamma_a) = & 1 - 2\Gamma_a [1 + |\epsilon|^2] - \Gamma_a |\epsilon|^2 |\alpha + \beta|^2 \\ & + \Gamma_a |\epsilon|^2 (1 - |\epsilon|^2) |\beta - \alpha|^4 \sin\left(\frac{4\pi\omega}{\Omega\eta}\right) \cos\left(\frac{4\pi\omega}{\Omega\eta} + 2\Delta\right) \left[ \frac{\Omega\eta}{8\pi\omega} - \frac{\Omega\eta}{16\pi(\Omega\eta - 2\omega)} + \frac{\Omega\eta}{16\pi(\Omega\eta + 2\omega)} \right] \\ & + \Gamma_a |\epsilon|^2 (1 - |\epsilon|^2) |\alpha + \beta|^4 \left[ 1 + \frac{\Omega\eta}{2\pi\omega} \sin\left(\frac{4\pi\omega}{\Omega\eta}\right) \cos\left(\frac{4\pi\omega}{\Omega\eta} + 2\Delta\right) \right] \\ & + \Gamma_a |\epsilon|^2 (1 - |\epsilon|^2) |\alpha + \beta|^2 |\beta - \alpha|^2 \sin\left(\frac{4\pi\omega}{\Omega\eta}\right) \cos\left(\frac{4\pi\omega}{\Omega\eta} + 2\Delta\right) \left[ \frac{\Omega\eta}{\pi(\Omega\eta - 4\omega)} + \frac{\Omega\eta}{\pi(\Omega\eta + 4\omega)} \right], \quad (68) \end{aligned}$$

where

$$\Gamma_a = \frac{4\pi\gamma}{\hbar m \omega \Omega\eta} \quad (69)$$

is now the new dimensionless noise parameter and  $\Delta$  is again the phase difference between the two vibrational states as defined above in Eq. (66).

Again the fidelity depends linearly on  $\Gamma_a$ . The dependence of  $F$  on the initial state we want to perform the gate on is plotted in Fig. 4. The chosen parameters for the plot are  $\omega = 11.0(2\pi)$  MHz,  $\Omega\eta = 12.0(2\pi)$  kHz,  $\Delta = 0$ , and  $\Gamma_a = 0.02$ .

## V. DISCUSSION

We have determined the heating rates due to a fluctuating trap potential and due to fluctuating electrical fields for the mena motional energy of an ion in a rf Paul trap. The potential use of ion traps as simple quantum computers in view we have calculated the effects of fluctuating electrical fields (considered one of the major sources of noise at the moment) during a controlled-NOT gate operation. We derived fidelities for two different ways of performing the required conditional phase shift needed for those gates. This analysis is particularly useful for the application of ion

traps to quantum computation. It gives an estimate on how strong those fluctuating fields can be to still perform a computation with a certain accuracy.

Taking the current heating rate of the COM mode for the NIST ion trap [6], which is about 19 phonons per ms and assuming that these heating rate is due to fluctuating electrical fields (the reasons for those heating rates are not clear yet, so we just assume at this stage until experimentalists will come up with more elaborate data), we get a rough estimate for  $\Gamma_a \approx 0.02$ . With that value we get fidelities above 90 % for one gate operation, which certainly needs improvement to allow for more than one gate operation.

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## FIGURES

FIG. 1. Plot of the exact (solid line) and approximated (dashed line) solution for the increase of the phonon number in time. This plot is for a vibrational frequency  $\omega = 11.2(2\pi)$  kHz and for  $\Gamma\omega/2 = 0.1$ . We choose the initial values to be  $\langle X^2 \rangle_{t=0} = \langle P^2 \rangle_{t=0} = (1/2)\langle XP + PX \rangle_{t=0} = 1/4$ , so that  $\langle H_0 \rangle_{t=0}/(\hbar\omega) = 1/2$ .

FIG. 2. Schematic representation of a controlled-NOT gate: The controlled phase shift  $U_P$  is sandwiched between the two single qubit rotations  $U_R^+$  and  $U_R^-$ .

FIG. 3. Plot of the dependence of  $F$  on the initial state we want to perform the gate on. The chosen parameters for the plot are  $\omega = 11.0(2\pi)$  MHz,  $\kappa = 1.0$  MHz,  $\Delta = 0$ , and  $\Gamma_\kappa = 0.02$ .

FIG. 4. Plot of the dependence of  $F$  on the initial state we want to perform the gate on for the NIST gate. The chosen parameters for the plot are  $\omega = 11.0(2\pi)$  MHz,  $\Omega\eta = 12.0(2\pi)$  kHz,  $\Delta = 0$ , and  $\Gamma_a = 0.02$ .

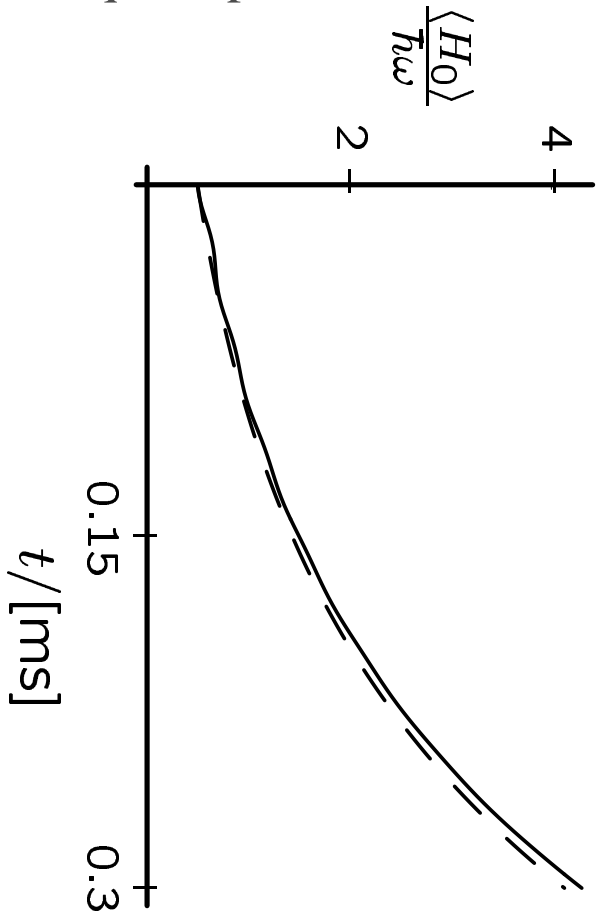


Fig. 1

Fig. 2

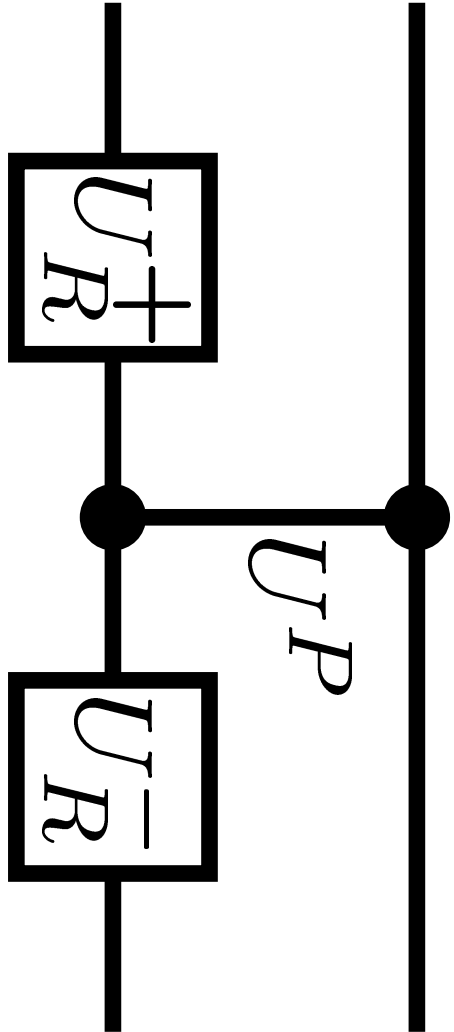


Fig. 3

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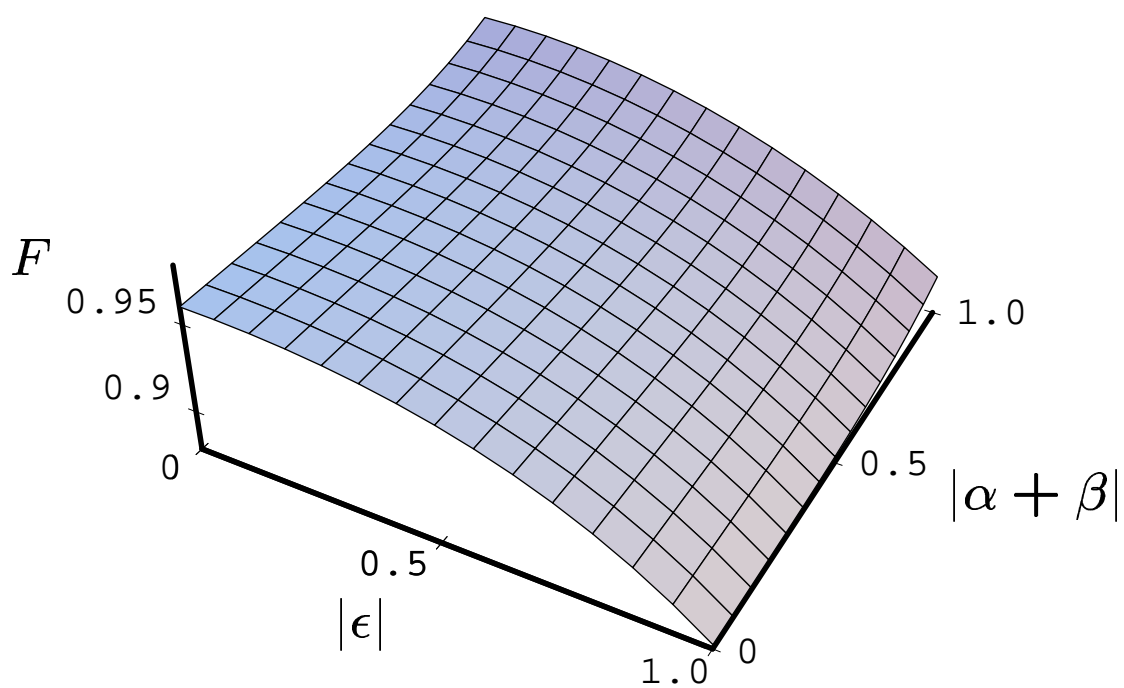


Fig. 4

