

Quantum measurements with a quantum computer

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Abstract

We present a scheme in which an ion trap quantum computer can be used to make arbitrarily accurate measurements of the quadrature phase variables for the collective vibrational motion of the ion. The electronic states of the ion become the ‘apparatus’, and the method is based on regarding the ‘apparatus’ as a quantum computer register which can be prepared in appropriate states by running a Fourier transform algorithm on the data stored within it. The resolution of the measurement rises exponentially with the number of ions used.

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Quantum computing offers promise for greatly enhanced efficiency of implementation for difficult computational problems [1–3]. The most important example to date is the quantum factoring algorithm of Shor [4]. Recent advances in quantum error correction [5–7] and fault tolerant computation [8,9] indicate that there is nothing in principle which will prevent the practical realisation of a quantum computer. The most promising suggestion currently is the ion trap realisation of Cirac and Zoller [10]. In this scheme, the internal electronic states of a string of ions becomes entangled with the lowest centre-of-mass vibrational mode of the ion trap. An experimental demonstration of a fundamental quantum logic gate in a trapped ion realisation was provided by Monroe et al [11].

In this paper we propose that an ion trap quantum computer be viewed as a means of making accurate measurements on the centre-of-mass mode of the ions. The electronic states of the ions forms an ‘apparatus’ coupled to the ‘system’, the vibrational mode. However using the kind of discrete unitary transformations which realise a quantum computing circuit, we can prepare the apparatus in a variety of states to facilitate new quantum limited measurements on the system. This is in the spirit of the suggestion of Wineland and coworkers [12] that highly entangled states may be used to improve frequency measurements.

In recent experiments on trapped ions [13] it was possible to subject the vibrational motion of the ion to a variety of unitary transformations by using sequences of Raman pulses. By selecting the Raman detuning carefully, a variety of oscillator states may be excited. In some case the unitary transformation of the vibrational motion is conditioned on the electronic state of the ion, and may be different for the ground and excited states. This conditional dependence was recently used to generate a Schrödinger cat state for the vibrational mode. In other cases the unitary transformation of the oscillator is independent of the electronic state.

Consider a single ion laser cooled to the ground state of a ion trap in the Lamb-Dicke limit. Assume that the ion is illuminated by laser fields which are well detuned from resonance. Let the interaction Hamiltonian coupling the vibrational and electronic states of a single ion be given by

$$\hat{H}_I^{(i)} = \hbar(\alpha_i a^\dagger + \alpha_i^* a)|e\rangle_i\langle e| \quad (1)$$

where α_i is proportional to a classical laser field acting on the i th-ion in a dispersive regime, a, a^\dagger are the creation and annihilation operators for the vibrational mode of the ion, and $|e\rangle_i$ is the excited electronic state of the i th ion

If we now generalise to the case of N ions in a linear trap, the electronic state of the system may be described by a binary string $S = (S_i = 1, 0; i = 1, N)$, assuming some ordering of the ions, and where a 1,0 represent the excited state and ground state respectively. Alternatively the state may be described by an integer k for which the binary string is the binary code $k = \sum_{i=1}^N S_i 2^i$. There are 2^N possible states for the electronic system so that $k = 0, 1, \dots, K = 2^N - 1$. Assume now that the coupling strengths α_i are adjusted from ion to ion so that $\alpha_i = \alpha 2^i$. The coupling Hamiltonian between the electronic states and the lowest collective vibrational mode of the ions may then be written

$$\hat{H}_I = \hbar(\alpha a^\dagger + \alpha^* a)\hat{Y} \quad (2)$$

where

$$\hat{Y} = \sum_{k=0}^K k|k\rangle\langle k| \quad (3)$$

The phase of α can be adjusted, so for simplicity let us take it to be real. The unitary operator describing the coupling of the vibrational and electronic states at the end of a sequence of pulses is then given by

$$U = e^{ir\hat{x}\hat{Y}} \quad (4)$$

where $\hat{x} = a + a^\dagger$ is the operator describing the in-phase component of the collective vibrational amplitude in an interaction picture rotating at the ion trap frequency, and r is a real parameter.

It is clear that the interaction can realise a measurement of the vibrational quantity \hat{x} . In this interpretation we regard the N ion register as an ‘apparatus’ to measure the

‘system’, the vibrational mode. However if all the ions are initially in the ground state, they will remain in the ground state under this unitary interaction as \hat{Y} commutes with the projection operators for each electronic energy eigenstate. In order for the interaction to change the state of the ions we must first pre-process the electronic register to a state which can be displaced by the unitary interaction. The required states $|\bar{k}\rangle$ are eigenstates of an operator $\hat{\Phi}$ which is canonically conjugate to \hat{Y} . That is to say \hat{Y} must act as a pure differential operator in the $\{|\bar{k}\rangle \ k = 0, \dots, K\}$ basis. This basis is simply a discrete Fourier transform of the original electronic basis $\{|k\rangle \ k = 0, \dots, K\}$. Thus

$$|\bar{l}\rangle = \frac{1}{\sqrt{K+1}} \sum_{k=0}^K \exp\left(\frac{2\pi i k l}{K+1}\right) |k\rangle \quad (5)$$

It is at this point that we remember the quantum computer interpretation of this system. The transformation of Eq (5) may be realised by running a Fourier transform algorithm on the electronic register ions [14,15].

Let the initial state of the system be

$$|\Psi\rangle = |\psi\rangle_v \otimes |0\rangle \quad (6)$$

where $|\psi\rangle_v$ is an arbitrary vibrational state and $|0\rangle$ indicates all the ions are in the ground state. The *first step* is to apply a sequence of $\pi/2$ -pulses to the ions to place them in a symmetric superposition of the ground and excited states. The resulting electronic state is precisely the state $|\bar{0}\rangle$, the 0-state in the Fourier transform basis. This state is equivalently a uniform superposition over all possible electronic energy eigenstates. In the *second step*, the unitary interaction is then implemented to couple the vibrational and electronic states. In the *third step* an inverse Fourier transform is run on the electronic register. At the end of these three steps the state of the system is

$$|\Psi'\rangle = \frac{1}{K+1} \sum_{k,l=0}^K \int_{-\infty}^{\infty} dp \phi(p) e^{-\frac{2\pi i k l}{K+1}} |p + r k\rangle_v \otimes |l\rangle \quad (7)$$

where $\phi(p)$ is the momentum probability amplitude for the initial vibrational state.

In the final step we readout the state of the electronic register. This is very much like the readout of the output register in the Shor algorithm. For trapped ions this can be done

with very high quantum efficiency using quantum jump techniques [13]. The result of this readout is a binary string describing which ions are in the ground state and which are in the excited state. Equivalently the result is the integer l encoded by this binary string. The (unnormalised) conditional vibrational state of the system after a readout result, l , is

$$|\tilde{\psi}\rangle_v = \frac{1}{K+1} \sum_{k=0}^K \int_{-\infty}^{\infty} dp \phi(p) e^{-\frac{2\pi i k l}{K+1}} |p + rk\rangle_v \quad (8)$$

The probability for this result is found to be

$$P(l) = \frac{\sqrt{2\pi}}{(K+1)^2} \sum_{k,k'=0}^K e^{-\frac{2\pi i(k-k')l}{K+1}} \chi(r(k-k')) \quad (9)$$

where

$$\chi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx e^{ikx} P(x) \quad (10)$$

with $P(x)$ being the initial distribution of the in-phase quadrature variable \hat{x} , of the vibrational state. Clearly $\chi(k)$ is just the characteristic function for the ‘position’ distribution for the initial vibrational state. In what follows we will assume that $r = 1$.

The probability distribution for the output l is thus seen to be a kind of discrete approximation to the probability distribution $P(x)$. In the case of a minimum uncertainty Gaussian state with position variance Δ , and zero mean, the readout distribution takes the form

$$P(l) = \frac{1}{K+1} \left(1 + \sum_{m=1}^K (K+1-m) \cos\left(\frac{2\pi ml}{K+1}\right) e^{-m^2 \Delta/2} \right). \quad (11)$$

In the limit $\Delta \rightarrow \infty$ we have an effective uniform position distribution. In this limit it is easy to see that $P(l) = 1$ for all l . In the opposite limit $\Delta \rightarrow 0$ we have a very well defined position at $x = 0$, and $P(l) = \delta_{l,0}$. In general the sum can be truncated at $m_{trunc} = \frac{2\sqrt{2}}{\Delta}$ when subsequent terms contribute less than e^{-4} , which we refer to as the tolerance ϵ . For a large number of ions, the truncation becomes a good approximation. In figure 1 we illustrate the intermediate behaviour for $N = 9$ ions and various values of Δ , (the tolerance used is $\epsilon = 0.01$). In these plots we have reflected the values at $l > N/2$ to negative values so as to ensure the plot is symmetric around 0. We see that as the number of ions increases we get a better and better approximation to the true ‘position’ distribution $P(x)$.

The dimensionless position x is related to the index l by $x = \frac{2\pi}{N}l$, thus the corresponding position distribution $P'(x)$ obtained from $P(l)$ is defined over the domain $[0, 2\pi)$. Thus for a fixed number of ions N , the range of the uncertainty Δ in the initial position distribution $P(x)$ is restricted, if the measured position distribution $P'(x)$ is to be essentially the same as $P(x)$. The lower limit imposed on Δ is determined by the number of ions in the register, whereas the upper limit on Δ is independent of the size of the register and is given by $\Delta_{max} \approx 10$. As the uncertainty approaches this upper limit, the measured electronic distribution $P'(x)$ becomes flat, so that it does not provide any more information about the actual position distribution $P(x)$. However, we are assuming that the collective motion of the ions is in the Lamb-Dicke regime, and hence we require *a priori* that the uncertainty in the position is bounded by $\Delta x \ll \sqrt{N}/\eta$, where η is the Lamb-Dicke parameter of the centre-of-mass vibrational mode. Alternatively, if we are given a lower limit for values of Δ , then we can determine the minimum number of ions N_{min} required to be able to measure the initial position distribution $P(x)$ accurately, and this is given by

$$N_{min} \approx 8.14 - \frac{1}{2} \log_2 \Delta \quad (12)$$

for a tolerance of $\epsilon = 0.01$ in the summation series i.e., this corresponds to $N_{min} = 25$ for $\Delta = 10^{-10}$, and $N_{min} = 10$ ions for $\Delta = 0.1$.

The graphs plotted in Figures 2(a)-(b) demonstrate that the value of the uncertainty Δx in the measured position distribution $P'(x)$ settles to the value of the initial uncertainty Δ as the number of ions increases past the appropriate minimum number N_{min} , thus indicating that the measured position distribution $P'(x)$ is approximately equal to $P(x)$.

We have illustrated a way in which an ion trap quantum computer can be used to measure the distribution of the in-phase and out-of-phase quadrature variables for the collective vibrational motion of the ion. The method is based on regarding the ions as a quantum computer register which can be prepared in appropriate states by running a Fourier transform algorithm on the data stored in the ionic register.

The first point to note is that we are not restricted to any particular quadrature phase

variable. As the phase of the laser pulses coupling the system and apparatus may be varied we can in principle measure any rotated quadrature phase. Given a sufficient set of such distributions a quantum tomographic data inversion could be done to reconstruct the initial vibrational state.

The second point to note is that as the number of ions increases the states available to the computer rises exponentially, providing an increasingly accurate readout of a system variable with a continuous spectrum. This generalises previous schemes to readout vibrational states using only a single ion; a two state apparatus. In that case many repetitions of the measurement must be performed to get an average over particular vibrational variables. In our scheme we can do better (albeit at the considerable expense of running a quantum algorithm), as we have an apparatus with many more states, and thus one which is better adapted to a vibrational degree of freedom.

Finally we note that the scheme is capable of considerable generalisation. For example by choosing different couplings between the ion and vibrational modes we can get access to many other vibrational variables. For example if we choose a sequence of Raman pulses which provided a ‘squeezing’ interaction for the vibrational motion, we can measure the distribution of eigenstates of the squeeze operator [17]. Furthermore, there may be other quantum algorithms that realise different measurement schemes.

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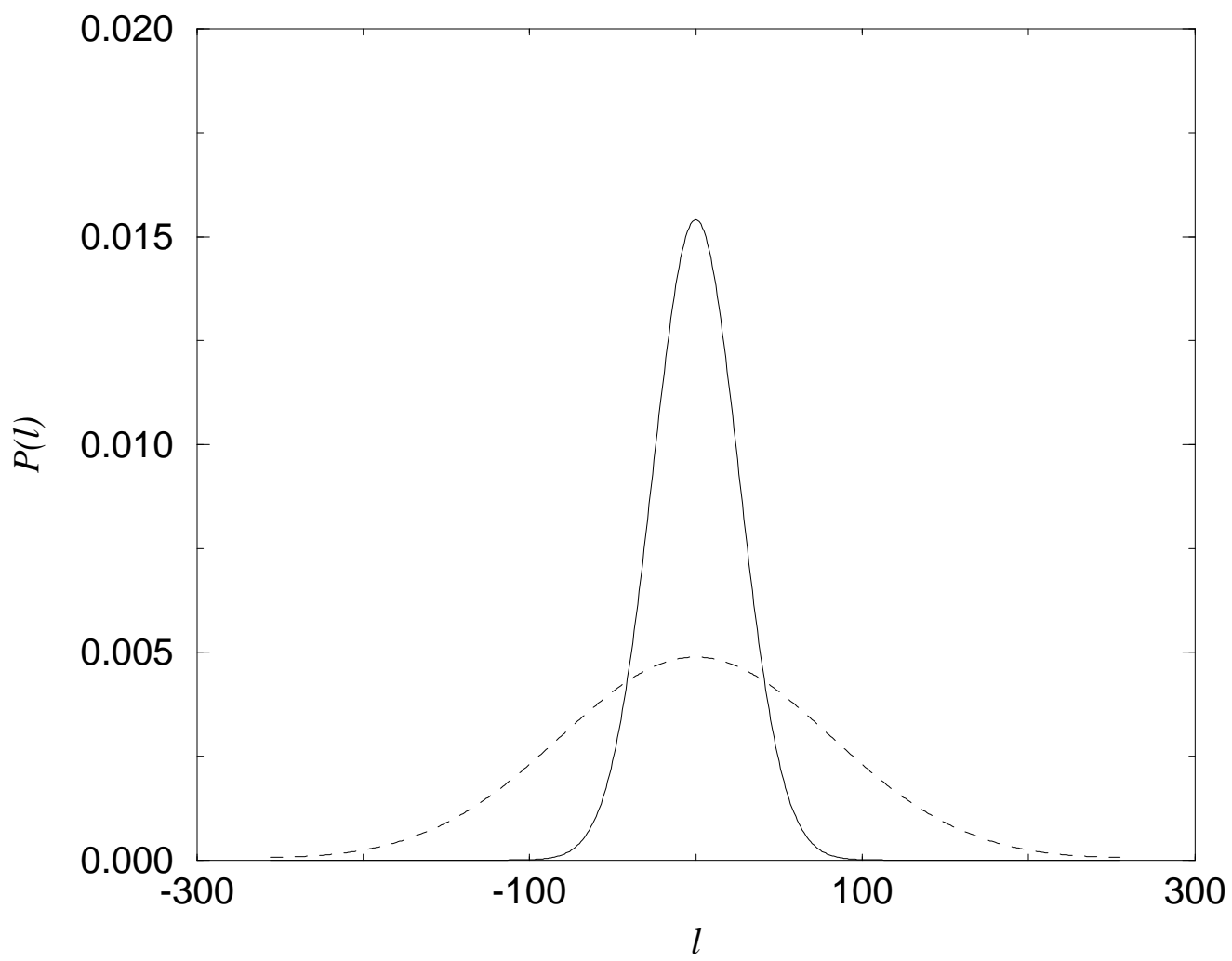
FIGURES

FIG. 1. A plot of the distribution for ion state readouts for an input Gaussian state in the centre-of-mass coordinate, with two different variances Δ . The number of ions is 9. Dashed: $\Delta = 1.0$. Solid: $\Delta = 0.1$

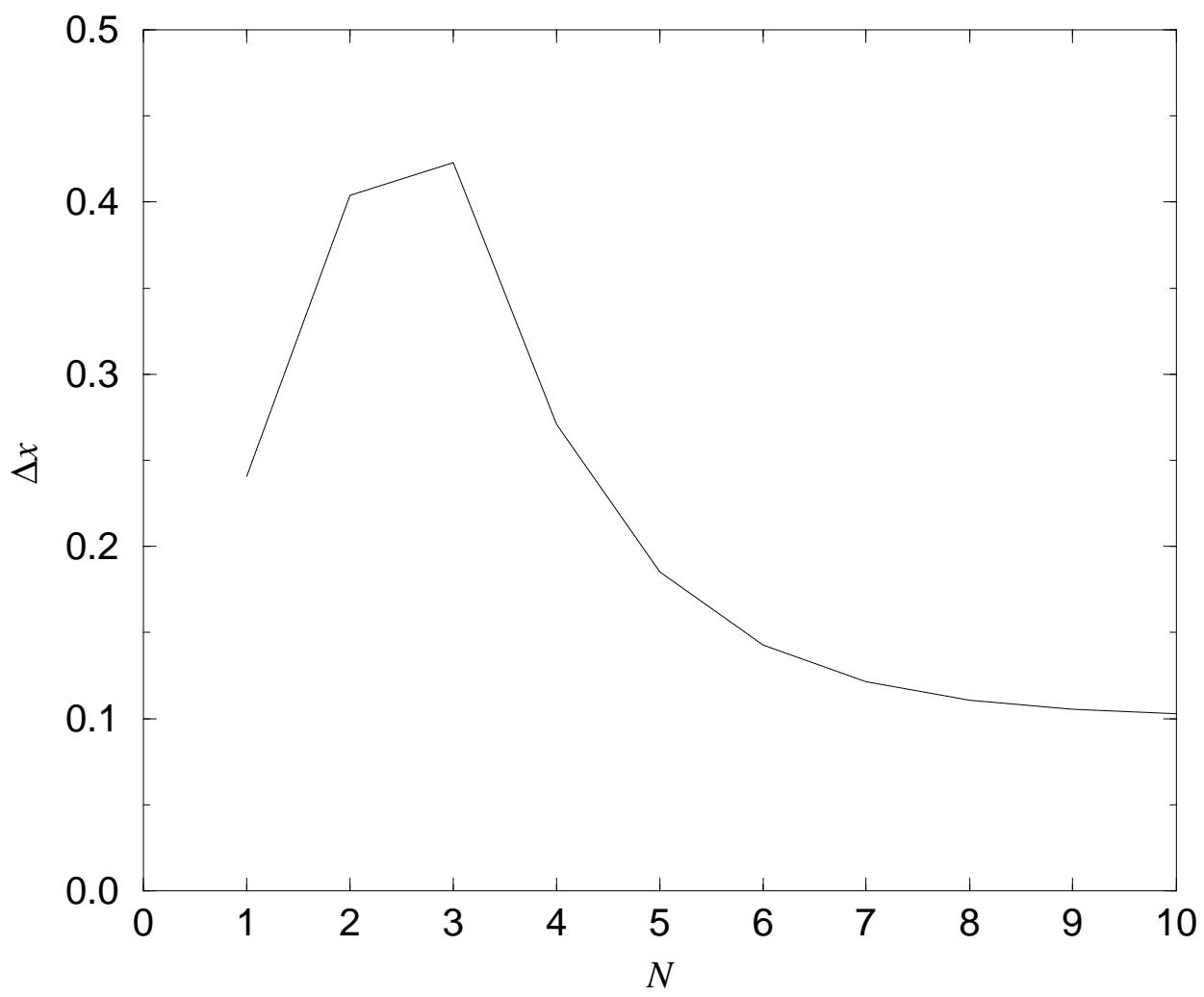
FIG. 2. A plot of the estimated variance in the centre-of-mass coordinate versus ion number for two different values of the initial coordinate variance. (a) $\Delta = 0.1$ (b) $\Delta = 1$.

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(a)



(b)

