

Lorentz invariant intrinsic decoherence.

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Abstract

Quantum decoherence can arise due to classical fluctuations in the parameters which define the dynamics of the system. In this case decoherence, and complementary noise, is manifest when data from repeated measurement trials are combined. Recently a number of authors have suggested that fluctuations in the space-time metric arising from quantum gravity effects would correspond to a source of intrinsic noise, which would necessarily be accompanied by intrinsic decoherence. This work extends a previous heuristic modification of Schrödinger dynamics based on discrete time intervals with an intrinsic uncertainty. The extension uses unital semigroup representations of space and time translations rather than the more usual unitary representation, and does the least violence to physically important invariance principles. Physical consequences include a modification of the uncertainty principle and a modification of field dispersion relations, in a way consistent with other modifications suggested by quantum gravity and string theory .

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I. INTRODUCTION

The precision with which intervals of time and length can be measured is limited by intrinsic quantum uncertainties[1]. The limit on precision is determined by fundamental constraints on estimating the parameter of an appropriate time or space translation. These limits arise from the statistical distinguishability of quantum states and reflect the geometry of Hilbert space itself.

In practice however, precision is limited by interactions between the measured system and other degrees of freedom (the environment) over which we have little control. Such interactions limit precision as noise is added to the measurement outcomes reflecting our lack of knowledge of the precise state of the environment. The flip side to added noise is decoherence: the interactions with the environment destroy coherence between superposed quantum states in some specific basis. Studies of environmentally induced decoherence over the last three decades have given a reasonably good picture of the process [2, 3] although detailed comparison to experiment is relatively recent. Quantum decoherence can also arise due to classical fluctuations in the parameters which define the dynamics of the system. Fluctuations in the space-time metric would correspond to a source of intrinsic noise, which would necessarily be accompanied by intrinsic decoherence[4].

Decoherence is often invoked to explain the lack of quantum effects in macroscopic systems. While this is often the case for environment induced decoherence, a number of authors[5, 6, 7, 8] have speculated that an intrinsic decoherence may exist to establish classical behaviour at some level. In this paper we extend a previous approach based on stochastic time [6] to stochastic space. This provides a path to a Lorentz invariant field theoretical formulation of a model of intrinsic decoherence.

These heuristic modifications of the Schrödinger equation should more properly be viewed in a like manner to environmental decoherence, but in which the quantum nature of the environment is left unspecified. Recently Gambini et al.[10], have shown that a recent proposal for quantisation of gravity based on discrete space-time is consistent with the model of intrinsic decoherence discussed in [6]. The extension proposed in this paper like wise has consequences that have been previously considered in the context of quantum gravity. Intrinsic decoherence due to spatial displacements leads to a modification of the uncertainty principle which is of the same general form as considered in the context of quantum gravity by a number of authors[11, 12, 13]. When extended to the relativistic case, in particular the electromagnetic field, we find that the dispersion relation for the free field must be modified. A similar effect has also been suggested for models of quantum space-time[14]

Space and time parameterise fundamental symmetry groups. The action of a group element on a physical state is represented by a unitary operator on Hilbert space. Conventionally we consider continuous representations of these symmetries which reflect the strong classical intuition that space and time are continuous parameters. In this case we can define the unitary representations through their infinitesimal action. The unitary representation is then defined in terms of a hermitian operator which is the generator of the group. In the case of time translations, the generator is the Hamiltonian operator while in the case of spatial translation the generator is the momentum operator. In a relativistic theory these operators are constructed from the quantum fields that define the physical systems under investigation. As Anandan[15] has emphasised, the existence of such unitary representations for space and time translation for any physical system lends a degree of universality to space and time translation, universality that is the core of our understanding of spacetime itself.

Quantum mechanics is an irreducibly statistical theory. The states of a physical system in quantum theory enable one to construct probability measures for the outcomes of all possible measurements made upon the system. Gleason's theorem tells us that the most general way to generate these probability measures in the Hilbert space formulation of quantum mechanics is in terms of a positive trace class operator the density operator ρ . In the nonrelativistic formulation two rules, one kinematical and one dynamical, are required to specify how probability amplitudes change in space and time. These rules were first given implicitly by Schrödinger in terms of a partial differential equation for a complex valued function of space and time. However they can be stated in terms of unitary representations of the space and time translation groups. Space and time translations then correspond to rotations of vectors in Hilbert space.

Spatial and temporal translations are then represented by the one parameter unitary transformations,

$$\rho(X) = e^{-iX\hat{p}/\hbar} \rho_0 e^{iX\hat{p}/\hbar} \quad (1)$$

$$\rho(t) = e^{-it\hat{H}/\hbar} \rho_0 e^{it\hat{H}/\hbar} \quad (2)$$

or in differential form,

$$\frac{d\rho(X)}{dX} = -\frac{i}{\hbar} [\hat{p}, \rho(X)] \quad (3)$$

$$\frac{d\rho(t)}{dt} = -\frac{i}{\hbar} [\hat{H}, \rho(t)] \quad (4)$$

The representation of the group of spatial and temporal translations in terms of the unitary operators generated by energy and momentum of course leads directly to the conservation of energy and momentum. The Schrödinger prescription thus reflects how probability amplitudes respect our most important symmetry principles.

Let us now take a different perspective. We will take rotations of vectors in Hilbert space as fundamental. When we make measurements on the corresponding physical systems, these rotations will appear as spatial or temporal translations. The space and time parameters thus inferred are then to be regarded as macroscopically determined parameters that help us coordinate the results of measurements made under different experimental conditions.

Within this perspective let us now ask the following question: how can we change the Schrödinger rules causing the least violence to energy and momentum conservation? To answer this we need to consider in a little more detail the kinds of experiments that enable us to estimate space and time translations. The high precision measurement of space and time intervals are in fact determinations of the statistical distinguishability of quantum states through repeated preparation and measurement.

A. Measurement of time intervals.

How do we make a quantum clock? Answer: superpose two states of definite energy. Indeed this is exactly how time is currently measured with atomic clocks. Let the system be prepared, at time $t = 0$, in the state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|E_1\rangle + |E_2\rangle) \quad (5)$$

where $|E_i\rangle$ are energy eigenstates. The variance of the energy in this state is $\langle\Delta\hat{H}^2\rangle_0 = \Delta_E^2/4$ with $\Delta_E = E_2 - E_1$

After a time t , in accordance with the dynamical Schrödinger rule, the state becomes

$$|\psi\rangle = \frac{1}{\sqrt{2}}(e^{-i\omega_1 t}|E_1\rangle + e^{-i\omega_2 t}|E_2\rangle) \quad (6)$$

where $\omega_i = E_i/\hbar$.

The next step is to measure some quantity represented by an operator that does not commute with the Hamiltonian. The simplest choice is the projection operator $\hat{P}_+ = |+\rangle\langle+|$ onto the state $|+\rangle = \frac{1}{\sqrt{2}}(|E_1\rangle + |E_2\rangle)$. There are two possible values, $x = 0, 1$ for the measurement result, with probability distribution

$$p_1(t) = 1 - p_0(t) = \cos^2\left(\frac{\Delta_E t}{2\hbar}\right) \quad (7)$$

There are two ways this system may be used as a clock. Both cases require us to sample the probability distribution in Eq.(7) and thus both require that we prepare a large number of identical systems in the manner just described and simultaneously measure the quantity \hat{P}_+ on all of them.

The first and most direct method is simply to measure the quantity \hat{P}_+ and thus infer the probability $p_x(t)$ and thus infer t . Of course this inference must come with some error which can easily be determined. There is no escaping this fact for quantum clocks. The second way is to note that the parameter Δ_E/\hbar is a frequency. It may be possible to change this with reference to a given frequency standard, such as a laser, and then use the sampling of the ensemble of systems to keep this frequency locked on a particular value, eg by using a feedback control to ensure that measurements on the ensemble always tend to give the same value x . As in the first method, this is necessarily accompanied by some error in a quantum world.

It is now a simple matter to estimate the uncertainty with which we can infer the parameter t . It suffices to measure the quantity $\hat{P}_+ = |+\rangle\langle+|$ on the state given in Eq.(6). The average value of this quantity is the probability $p_1(t)$. The uncertainty is this measurement is

$$\Delta p_1 = \sqrt{p_1(1 - p_1)} \quad (8)$$

The uncertainty in the inference of the time parameter t , is then given by [16]

$$\delta t = \left|\frac{dp_1(t)}{dt}\right|^{-1} \Delta p_1 \quad (9)$$

Thus we find the well known result

$$\delta t = \frac{\hbar}{|\Delta_E|} \quad (10)$$

Noting that the variance of the energy for the fiducial state is $\langle\Delta\hat{H}^2\rangle_0 = \Delta_E^2/4$, we see that the quality of the inference varies as the inverse of the energy uncertainty. This is the standard result for a parameter based uncertainty principle[1]

To summarise, time measurement in a quantum world requires us to sample a probability distribution by making measurements on an ensemble of identically prepared systems. A single system and a single measurement won't do.

B. Measurement of space intervals.

How do we make a quantum ruler ? Answer: make a standing wave for the position probability amplitude for a free particle. As we shall see, this will entail doing something very similar to the previous discussion for measuring time. A standing wave for the position probability amplitude requires a superposition of momentum eigenstates of equal and opposite momentum,

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|p_1\rangle + |p_2\rangle) \quad (11)$$

This state is an energy eigenstate of a free particle, and the resulting quantum ruler will not change in time. However it is not an eigenstate of momentum, the momentum uncertainty is given by $\langle \hat{p}^2 \rangle_0 = \Delta_p^2/4$ where $\Delta_p = p_2 - p_1$.

In writing this state with the particular choice of relative phase between the two components, we are assuming that the origin is located at an antinode of the standing wave. We now translate the ruler (ie. 'pick up' the ruler and move it...that is how one uses a metre stick !) to a new position, labelled X . The state now changes in accordance with the kinematical Schrödinger rule as

$$|\psi(X)\rangle = e^{-ik_1X}|p_1\rangle + e^{-ik_2X}|p_2\rangle \quad (12)$$

where $k = p/\hbar$. We now measure some quantity on this state, which in analogy with the time example in the previous section, we choose to be conjugate to momentum, that is to say we measure position. The probability distribution to get a particular result, x , is then

$$P(x) \propto \cos^2\left(\frac{\Delta_p(X-x)}{2\hbar}\right) \quad (13)$$

To determine how much we had translated the ruler we need to sample this distribution. Again we need to consider making simultaneous measurements on an ensemble of identically prepared systems, (or we could simply repeat the sequence of preparation, translation and measurement steps. Of course, the determination of X will be accompanied by some necessary error of a quantum origin. This is roughly the distance between two successive minima of the probability density, $P(x)$. In this case one easily sees that this is a consequence of the Heisenberg uncertainty principle resulting from the initial momentum uncertainty of the state in Eq.(14)[1]. The uncertainty in the inferred value of X is given by

$$\delta X \geq \frac{\hbar}{|\Delta_p|} \quad (14)$$

We have thus seen that determination of temporal duration and space translation require us to sample a probability distribution by making measurements on an ensemble. This leads to the well known intrinsic quantum uncertainty limits for time and position parameter estimation[1]. The process of preparing the ensemble (either by trying to prepare a large number of identical systems, or a process of preparation and re-preparation of a single system) is in practice subject to additional sources of error. Furthermore the measurements are not always perfect and noise may be added from trial to trial. For this reason the actual state used to describe an ensemble may not necessarily be simply a product of identical pure states, but may rather be a density operator reflecting some additional degree of averaging over unknown sources of noise and error. The fact that space and time parameters must be inferred by sampling a probability distribution over an ensemble is an important insight and opens up an additional path for developing a theory of intrinsic decoherence.

II. TOWARDS INTRINSIC DECOHERENCE.

We now modify the unitary representation of space and time translation by using semigroup representations. We first recall that, in estimating a parameter, multiple trials must be performed and in practice it may not be possible to ensure that each trial is identical. Suppose now that *in principle* it is impossible for each trial to be identical, for some fundamental reason. In that case parameter estimation would necessarily be based on an ensemble rather than a pure state. This is the starting point for intrinsic decoherence. From trial to trial, the experimentalist seeks to use the same parameterised state. However suppose that the actual state is different from one trial to the next due to fluctuations in the actual Hilbert space rotations corresponding to a spatial or temporal shift. The overall parameter estimation problem must then necessarily be based on a non pure density operator.

To be more specific we suppose that for each unitary representation of the parameter transformation there is a minimum Hilbert space rotation angle, ϵ , and further that the *number* of such rotations, from one trial to the next, can fluctuate. Let $p_n(\theta, \epsilon)$ be the probability that there are n such phase shifts for a change in the macroscopic parameter from 0 to θ . We obtain different models for each choice of the probability function $p_n(\theta, \epsilon)$. Thus the state $\rho(\theta)$ may be written

$$\rho_\epsilon(\theta) = \sum_{n=0}^{\infty} p_n(\theta, \epsilon) e^{-in\epsilon\hat{g}/\hbar} \rho(0) e^{in\epsilon\hat{g}/\hbar} \quad (15)$$

where $\hat{g} \rightarrow \hat{p}$ for spatial translations and $\epsilon \rightarrow \mu$ with units of length, while $\hat{g} \rightarrow \hat{H}$ for temporal translations, and $\epsilon \rightarrow \nu$ with units of time. These assumptions are equivalent to assuming that space and time are discrete with fundamental scales determined by ν, μ .

We also require that, in some limit, the standard unitary representation is obtained. To this end we require

$$\lim_{\epsilon \rightarrow 0} \rho_\epsilon(\theta) = \rho(\theta) = e^{-i\hat{g}\theta/\hbar} \rho(0) e^{i\hat{g}\theta/\hbar} \quad (16)$$

This condition imposes a restriction on the permissible forms of $p_n(\theta, \epsilon)$.

It is easiest to define the semigroup in terms of its infinitesimal generator. There is great deal of freedom in how we do this corresponding to different choices for the probability distribution for the number of phase shifts. However we stipulate that it must respect the conservation of energy and momentum. We will use the differential form of the parameter transformation

$$\frac{d\rho(\theta)}{d\theta} = \mathcal{D}[\hat{G}]\rho(\theta) \quad (17)$$

where θ is the parameter, $\mathcal{D}[\hat{G}]$ is the generator of a completely positive semigroup map defined by $\mathcal{D}[\hat{G}]\rho = \hat{G}\rho\hat{G}^\dagger - \frac{1}{2}[\hat{G}^\dagger\hat{G}\rho + \rho\hat{G}^\dagger\hat{G}]$. We require that for spatial translations, with generator \hat{S} , $\mathcal{D}[\hat{S}]\hat{p} = 0$ while for temporal translations, with generator \hat{T} , $\mathcal{D}[\hat{T}]\hat{H} = 0$. This ensures that momentum and energy are conserved in the semigroup transformation. As a specific example we will take each generator to be a unitary operator of the form;

$$\hat{G} = \epsilon^{-1/2} e^{-i\frac{\hat{g}\epsilon}{\hbar}} \quad (18)$$

which we shall refer to as the *unital* case. Substituting Eq.(18) into Eq.(17) indicates that

$$\frac{dp_n(\theta, \epsilon)}{d\theta} = \frac{1}{\epsilon} (p_{n-1}(\theta, \epsilon) - p_n(\theta, \epsilon)) \quad (19)$$

$$\frac{dp_0(\theta, \epsilon)}{d\theta} = -\frac{1}{\epsilon}p_0(\theta, \epsilon) \quad (20)$$

The solution is

$$p_n(\theta, \epsilon) = \frac{(\theta/\epsilon)^n}{n!}e^{-\theta/\epsilon} \quad (21)$$

For obvious reasons we call this the Poisson choice. In the limit that $\epsilon \rightarrow 0$ we recover the standard Schrödinger representation from Eq.(17),

$$\frac{d\rho(\theta)}{d\theta} = -\frac{i}{\hbar}[\hat{g}, \rho(\theta)] \quad (22)$$

The condition in Eq.(16) is thus satisfied. We anticipate that ν , μ ultimately take their values from a future quantum theory of spacetime, so here we simply equate them to the Planck time and Planck length, respectively, in which case. $c\nu = \mu$. We now consider the experimental consequences of this modification of the Schrödinger rules.

We begin with temporal translations. The differential change in a state due to a temporal translation, t , is given by[6]

$$\frac{d\rho(t)}{dt} = \frac{1}{\nu} \left[e^{-i\nu\hat{H}/\hbar} \rho(t) e^{i\nu\hat{H}/\hbar} - \rho(t) \right] \quad (23)$$

where ν is a fundamental constant with units of time. (In terms of reference [6] $\gamma = 1/\nu$. The physical consequences of this equation have been explored in [6] and subsequent papers[22]. Firstly the standard limit (Eq.(10)) for the uncertainty in the estimate of a time parameter is changed to include an additional noise source. Secondly, and most relevant for this paper, the dynamics implied by Eq.(23) lead to the decay of coherence in the energy basis.

We will use the example discussed in Section 10. The time parameter uncertainty bound can be conveniently written in terms of the average value of the hermitian operator $\hat{X} = |E_1\rangle\langle E_2| + |E_2\rangle\langle E_1|$ as

$$\sqrt{1 - \langle \hat{X}(t) \rangle^2} \left| \frac{d\langle \hat{X}(t) \rangle}{dt} \right|^{-1} \quad (24)$$

The equation of motion for $\langle \hat{X}(t) \rangle$ can then be found using Eq. [23]. The solution is [6]

$$\langle \hat{X}(t) \rangle = \Re \left\{ \exp \left[-\frac{t}{\nu} (1 - e^{-i\nu\Delta_E}) \right] \right\} \quad (25)$$

The resulting bound on δt is complicated, but can be simplified by the case $\nu\Delta_E \ll 1$, for which we can approximate

$$\langle \hat{X}(t) \rangle \approx e^{-\nu\Delta_E^2 t/2} \cos(\Delta_E t) \quad (26)$$

We expect that the best accuracy for the estimate of time will occur when $\langle \hat{X}(t) \rangle = 0$, as at that point this moment has maximum slope. This corresponds to the condition $\cos(\Delta t) = 0$. At those times we find

$$\delta t \approx \frac{1}{\Delta_E} e^{\nu\Delta_E^2 t/2} \quad (27)$$

For short times, $\Delta t \ll 1$ this agrees with the standard time parameter uncertainty bound for this system. However for long times we see that there is an exponential degradation of

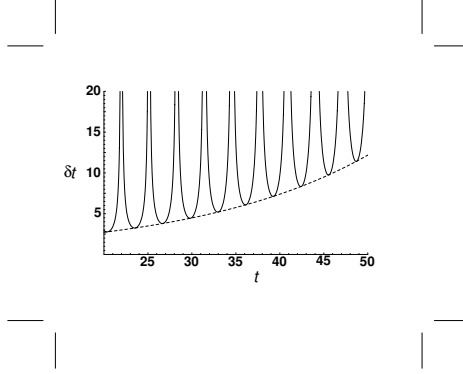


FIG. 1: The modified uncertainty bound for estimating time parameters that follows from Eq.[23] using the two state model in section (solid) and an approximate bound for the minima(dashed curve)

the accuracy. In figure 1 we plot δt versus time using the exact result in Eq.[25]. The dashed curve is the function for the approximation given in Eq.[27]. This result suggests that clocks ‘age’, that is to say long-lived atomic clocks gradually lose accuracy. Of course with ν set to the Planck time, the time scale for this effect is cosmological.

The consequences for estimating a temporal translation are important as they indicate a fundamental limitation on the accuracy of clocks. This is an aspect that has been considered in some detail by Egusquiza, and Garay [23], from a very different starting point.

Now consider the case of spatial translations. Suppose we take as the fiducial state a system in a pure, minimum uncertainty, state, $|\psi_0\rangle$ with a Gaussian position probability density;

$$P_0(x) = |\langle\psi_0|x\rangle|^2 = (2\pi\sigma)^{-1/2}e^{-\frac{x^2}{2\sigma}} \quad (28)$$

where

$$\sigma = \langle\Delta\hat{x}^2\rangle_0 = \frac{\hbar^2}{2\langle\Delta\hat{p}^2\rangle_0} \quad (29)$$

where $\langle\Delta\hat{A}^2\rangle_0$ is the variance of the operator \hat{A} in the fiducial state. Under the conventional Schrödinger rule for displacements this state density after a displacement becomes

$$P^{(c)}(x|X) = (2\pi\sigma)^{-1/2}e^{-\frac{(x-X)^2}{2\sigma}} \quad (30)$$

We can see that the uncertainty, δX with which we can infer the parameter, X is $\delta X \geq \sqrt{\sigma}/2$ or in other words $\delta X^2\langle\Delta\hat{p}^2\rangle_0 \geq \hbar^2/4$, which is the standard result for a parameter based uncertainty principle for position[1].

In the modified Schrödinger rule this uncertainty principle is modified as the width of the position distribution is no longer independent of the displacement but increases linearly with displacement. The change in the state due to the displacement is given by,

$$\frac{d\rho(X)}{dX} = \frac{1}{\mu} \left(e^{-i\mu\hat{p}/\hbar} \rho(X) e^{i\mu\hat{p}/\hbar} - 1 \right) \quad (31)$$

This equation appears to bear a superficial relation to the recent proposal of Shalyt-Margolin and Suarez[19]

To see this how the uncertainty principle is changed we use Eq.(41) to find an equation for rate of change of the mean position and variance with displacement. It is easy to see that

$$\begin{aligned}\frac{d\langle\hat{x}\rangle}{dX} &= 1 \\ \frac{d\langle\hat{x}^2\rangle}{dX} &= 2\langle\hat{x}\rangle + \mu\end{aligned}$$

Thus for the example here with the chosen fiducial state

$$\langle\Delta\hat{x}^2\rangle_X = \langle\Delta\hat{x}^2\rangle_0 + \mu X \quad (32)$$

The uncertainty with which we can estimate the parameter now becomes

$$\delta X^2 \geq \langle\Delta\hat{x}^2\rangle_0 + X\mu \quad (33)$$

which implies the uncertainty principle

$$\delta X^2 \langle\Delta\hat{p}^2\rangle_0 \geq \frac{\hbar^2}{4} + X\mu \langle\Delta\hat{p}^2\rangle_0 \quad (34)$$

This kind of modified uncertainty principle has been suggested in the context of quantum gravity and string theory[12]. We see here it arises as a natural consequence of an intrinsic uncertainty of spatial translations.

We turn from intrinsic noise to the complementary process of intrinsic decoherence. In quantum mechanics noise is necessarily accompanied by decoherence. Thus any model that introduces an intrinsic uncertainty due to space time fluctuations must necessarily introduce intrinsic decoherence. In the case of temporal translations, the decoherence occurs in the energy basis as is easily seen by computing the change in the off diagonal elements of the state in the energy basis. From Eq.(23) we see that

$$\frac{d\rho_{i,j}(t)}{dt} = \frac{1}{\nu} \left(e^{-i\nu(E_i - E_j)/\hbar} - 1 \right) \rho_{i,j}(t) \quad (35)$$

where $\rho_{i,j}(t) = \langle E_i | \rho(t) | E_j \rangle$ with $|E_i\rangle$ an energy eigenstate. This equation was discussed extensively in [6]. To see the effect of intrinsic decoherence we expand the right hand side to first order in ν ,

$$\frac{d\rho_{i,j}(t)}{dt} = -i \frac{(E_i - E_j)}{\hbar} \rho_{i,j}(t) - \frac{\nu(E_i - E_j)^2}{2\hbar^2} \rho_{i,j}(t) \quad (36)$$

The last term indicates a decay of off diagonal coherence in the energy basis at a rate that increases quadratically with distance away from the diagonal.

In the case of spatial translations we find a similar equation that causes a decay with, respect to the translation parameter, of off diagonal coherence in the momentum basis,

$$\frac{d\rho_{k,k'}(t)}{dX} = \frac{1}{\mu} \left(e^{-i\mu(k - k')} - 1 \right) \rho_{k,k'}(t) \quad (37)$$

where $\rho_{k,k'}(t) = \langle \hbar k | \rho(t) | \hbar k' \rangle$ with $|\hbar k\rangle$ a momentum eigenstate. Expansion to first order in μ gives a decay of coherence in the momentum basis as the translation parameter increases.

III. LORENTZ INVARIANT FORMULATION

In order to generalise the preceding ideas to include Lorentz invariance we must move to a field theory formulation. Space and time translations are determined by specifying the sources and detectors for the field. We are at liberty to choose any field at all, although in practice the electromagnetic field is the easiest to use. The source determines the fiducial state of the quantum field. Measurements reduce to particle detectors and space and time translation parameters are inferred by the statistics of detection events at such detectors. The relevant unitary translation generators are still position and momentum generators, but now constructed in the usual way from whatever quantum field we wish to use. As in the non relativistic case, spatial translations require a fiducial state with an indefinite momentum while time translations require a fiducial state with an indefinite energy.

We will first discuss how spacetime translations are determined in the standard formulation of quantum field theory. Estimation of a space time translation in quantum parameter estimation theory was considered by Braunstein et al. [1], and we now summarise that treatment. The generator for spacetime translation is the energy-momentum 4-vector

$$\hat{\mathbf{P}} = \hat{P}^\alpha \mathbf{e}_\alpha = \hat{P}^0 \mathbf{e}_0 + \hat{\vec{P}} = \hat{P}^0 \mathbf{e}_0 + \hat{P}^j \mathbf{e}_j. \quad (38)$$

The spacetime translation we seek to estimate can be written as

$$\mathbf{X} = S \mathbf{n} = S n^\alpha \mathbf{e}_\alpha \quad (39)$$

with

$$\mathbf{n} = n^0 \mathbf{e}_0 + \vec{n} \quad (40)$$

is a space-like or time-like unit 4-vector specifying the direction of the spacetime translation and S is the invariant interval that parameterizes the translation. The rotation of the fiducial state $|\psi_0\rangle$, in Hilbert space is then

$$|\psi_S\rangle = e^{iS\mathbf{n}\cdot\hat{\mathbf{P}}/\hbar} |\psi_0\rangle \quad (41)$$

with

$$\mathbf{n} \cdot \hat{\mathbf{P}} = \eta_{\alpha\beta} n^\alpha \hat{P}^\beta = n^\alpha \hat{P}_\alpha = -n^0 \hat{P}_0 + \vec{n} \cdot \hat{\vec{P}} \quad (42)$$

The Minkowski metric is $\eta_{\alpha\beta} = \text{diag}(-1, +1, +1, +1)$ (we use units such that $c = 1$). The three dimensional dot product is written $\vec{n} \cdot \hat{\vec{P}} = n^j \hat{P}^j$. Braunstein et al.[1] show that the parameter based uncertainty principle for estimating the spacetime translation parameter, S is

$$\langle (\delta S)^2 \rangle_S \langle (\mathbf{n} \cdot \Delta \hat{\mathbf{P}})^2 \rangle = \langle (\delta S)^2 n^\alpha n^\beta \langle \Delta \hat{P}_\alpha \Delta \hat{P}_\beta \rangle \rangle \geq \frac{\hbar^2}{4N} \quad (43)$$

for N trials. When \mathbf{n} is time like this is a time-energy uncertainty relation for the observer whose 4-velocity is \mathbf{n} , and when \mathbf{n} is space-like, this is a position-momentum uncertainty relation for an observer whose 4-velocity is orthogonal to \mathbf{n} .

What fiducial states are appropriate for estimating a spacetime translation? We will discuss the case of space and time translations separately to parallel the discussion in the nonrelativistic case. For specificity we will assume that we are using the electromagnetic field. It should be noted that this is a special case, but will suffice to illustrate the principles

of the more general situation. The energy momentum 4-vector can be written most easily if we decompose the field into plane wave modes,

$$\hat{\mathbf{P}} = \sum_{\vec{k}, \sigma} \hbar \mathbf{k} \hat{a}_{\vec{k}, \sigma}^\dagger \hat{a}_{\vec{k}, \sigma} \quad (44)$$

where $\mathbf{k} = \omega \mathbf{e}_0 + \vec{k} = \omega \mathbf{e}_0 + k^j \mathbf{e}_j$ is a null wave 4-vector with $\omega = |\vec{k}| = k$ and the sum is over all wave 3-vectors \vec{k} and polarisation σ . The generator for spacetime translations is thus determined by the number operator for the field modes which is the generator for phase shifts in the field. Thus determining a spacetime translation via the electromagnetic field reduces to phase parameter estimation. Optimal phase estimation is not a straightforward measurement, particularly in the multimode case[18]. However it will suffice for our purposes to give a simple example based on photon counting. (Ultimately all field measurements reduce to counting field quanta.)

Let us consider just two modes, with wave 4-vectors $\mathbf{k}_1, \mathbf{k}_2$, with the same polarisation. We will designate a Fock state for the mode \mathbf{k}_i as $|n\rangle_i$. Suppose we have a source that produces the single photon state $|1\rangle_1 \otimes |0\rangle_2$, ie one photon in mode \mathbf{k}_1 and the vacuum in mode \mathbf{k}_2 . The first step is to find a unitary transformation, U to give

$$U|1\rangle_1 \otimes |0\rangle_2 = \frac{1}{2}(|1\rangle_1 \otimes |0\rangle_2 + |0\rangle_1 \otimes |1\rangle_2) \quad (45)$$

If the modes have the same frequency, $\omega_1 = \omega_2$, this can be performed with a simple linear optical device know as a beam splitter, but if the modes also have different frequencies we need the nonlinear optical device know as a frequency converter[20]. The state is now subjected to the unitary spacetime translation in Eq.(41), followed by U^\dagger . The final state is

$$|\psi_S\rangle = e^{iS\delta_+/2} (\cos(s\delta_-/2)|0\rangle_1 \otimes |1\rangle_2 + i \sin(S\delta_-/2)|1\rangle_1 \otimes |0\rangle_2) \quad (46)$$

where

$$\delta_\pm = \mathbf{n} \cdot (\mathbf{k}_1 - \mathbf{k}_2) \quad (47)$$

A simple measurement can now be made of the photon number difference between the two modes, with results ± 1 occurring with probabilities

$$P(+1) = 1 - P(-1) = \sin^2(S\delta_-/2) \quad (48)$$

Sampling this distribution enables an inference of the spacetime translation parameter S . Of course such a measurement is not optimal. Using many photon states, and a different kind of output measurement it is possible to do much better[21].

If we seek only a space translation (ie a ruler) then we can chose the modes to have the same frequency, but wave vectors in different directions. Such a state clearly has an indefinite 3-momentum as we found for the non relativistic case. If we seek a time translation (ie a clock) we must chose the wave vectors to have a different frequency, that is to say a different energy as in the non relativistic case.

It is now straightforward to define an intrinsic decoherence model that is Lorentz invariant. The change in the state of a quantum field as a function of the displacement interval is

$$\frac{d\rho(S)}{dS} = \frac{1}{\epsilon} \left(e^{i\epsilon \mathbf{n} \cdot \hat{\mathbf{P}}/\hbar} \rho(S) e^{-i\epsilon \mathbf{n} \cdot \hat{\mathbf{P}}/\hbar} - \rho(S) \right) \quad (49)$$

Equivalently

$$\rho(S) = \sum_{m=0}^{\infty} \frac{(S/\epsilon)^m}{m!} e^{-S/\epsilon} e^{im\epsilon \mathbf{n} \cdot \hat{\mathbf{P}}/\hbar} \rho_o e^{-im\epsilon \mathbf{n} \cdot \hat{\mathbf{P}}/\hbar} \quad (50)$$

As the generator of spacetime translations is already explicitly Lorentz invariant, these equations are Lorentz invariant. In fact $S \mathbf{n} \cdot \hat{\mathbf{P}}$ is nothing more than the action associated with the spacetime interval. The central assumption for this relativistic model of intrinsic decoherence is that the action along some worldline can vary from trial to trial in an experimental determination of a spacetime translation (this observation suggests an equivalent formulation in terms of path integrals). The state ρ is a many particle field state and would typically be specified in the Fock basis for some mode decomposition. The specific form the intrinsic decoherence takes depends on the field under discussion through the energy momentum 4-vector. We now consider some consequences of this equation for the case of the electromagnetic field.

The most obvious modification is to the experimentally observed dispersion relation. The dispersion relation must be determined by making phase dependent measurements on the field amplitude at different spacetime points. We have postulated that such repeated measurements are described by a density operator $\rho(S)$ rather than a pure state, and we have given a rule for how to translate this state to describe measurements made at different spacetime positions. In order to measure a field amplitude that is non zero we must specify a fiducial field state that has a non zero amplitude. We will take this to be a coherent state[20].

In the standard theory of the electromagnetic field, we specify the electric field at position \vec{x} by the operator

$$\hat{E}(\vec{x}) = i \sum_k \omega_k (u_k(\vec{x}) a_k - u_k(\vec{x})^* a_k^\dagger) \quad (51)$$

where a_k, a_k^\dagger are boson annihilation and creation operators, while $u_k(\vec{x})$ are a set of orthonormal mode functions and Let us choose the state of the field on the spacelike hypersurface $t = 0$ to be a coherent state[20] such that

$$\text{tr}[a_k \rho] = \alpha_k \quad (52)$$

This is a semiclassical state for which the field amplitude on $t = 0$ is given by

$$\mathcal{E}(\vec{x}) = \sum_k i \omega_k (u_k(\vec{x}) \alpha_k - u_k(\vec{x})^* \alpha_k^*) \quad (53)$$

We now translate the field along the time-like direction $n^\alpha = (1, 0, 0, 0)$, so that the field amplitude becomes

$$\mathcal{E}(\vec{x}, \tau) = \text{tr}[\hat{E}(\vec{x}) \rho(\tau)] \quad (54)$$

where τ denotes the proper time. Using Eq.(50) for this spacetime path we have that

$$\rho(\tau) = \sum_{n=0}^{\infty} \frac{(\tau/\epsilon)^n}{n!} e^{-\tau/\epsilon} e^{-in\epsilon \sum_k k a_k^\dagger a_k} \rho(0) e^{in\epsilon \sum_k k a_k^\dagger a_k} \quad (55)$$

The field amplitude at $\tau \neq 0$ is then determined by

$$\text{tr}[a_k \rho(\tau)] = \sum_{n=0}^{\infty} \frac{(\tau/\epsilon)^n}{n!} e^{-\tau/\epsilon} \text{tr} \left[a_k e^{-in\epsilon \sum_l l a_l^\dagger a_l} \rho(0) e^{in\epsilon \sum_l l a_l^\dagger a_l} \right] \quad (56)$$

$$= \sum_{n=0}^{\infty} \frac{(\tau/\epsilon)^n}{n!} e^{-\tau/\epsilon} \text{tr} \left[\rho(0) e^{in\epsilon k a_k^\dagger a_k} a_k e^{-in\epsilon k a_k^\dagger a_k} \right] \quad (57)$$

$$= \alpha_k \exp \left[\frac{\tau}{\epsilon} (e^{-i\epsilon k} - 1) \right] \quad (58)$$

Thus

$$\alpha_k(\tau) = \alpha_k e^{-i\omega(k)\tau} e^{-\gamma(k)\tau} \quad (59)$$

where the observed frequency of this mode amplitude is

$$\omega(k) = \frac{\sin(\epsilon k)}{\epsilon} \quad (60)$$

and the amplitude decays due to intrinsic decoherence at the rate

$$\gamma(k) = \frac{1}{\epsilon} (\cos(\epsilon k) - 1) \quad (61)$$

As $\epsilon \rightarrow 0$ we recover the standard dispersion relation with a small modification

$$\omega(k) = k \left(1 - \frac{\epsilon^2 k^2}{6} + \dots \right) \quad (62)$$

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