Generating optical nonlinearity using trapped atoms

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We describe a scheme for producing an optical nonlinearity using an interaction with one or more ancilla two-level atomic systems. The nonlinearity, which can be implemented using high efficiency fluorescence shelving measurements, together with general linear transformations is sufficient for simulating arbitrary Hamiltonian evolution on a Fock state qudit. We give two examples of the application of this nonlinearity, one for the creation of nonlinear phase shifts on optical fields as required in single photon quantum computation schemes, and the other for the preparation of optical Schrödinger cat states.

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Our ability to perform quantum transformations on optical fields is hampered by the lack of materials with intrinsic optical non-linearities. While it is possible to circumvent this problem with schemes conditioned on photo-detection (see for instance [1, 2, 3, 4, 5, 6]), efficiency problems are encountered both due to the inherent nature of the schemes and efficiencies of current photo-detectors. In this paper we propose the use of the interaction with a simple atomic system, conditioned on high efficiency atomic measurements, to generate a near-deterministic (with probability approaching one) nonlinearity on an optical field. This nonlinearity, together with linear transformations, is sufficient for generating arbitrary Hamiltonian evolution on a qudit formed from a truncated sequence of Fock states.

The interaction between the single mode field and a two-level atom is described by the effective Hamiltonian $\boldsymbol{H} = \kappa(\boldsymbol{a}^\dagger\boldsymbol{\sigma}^- + \boldsymbol{a}\boldsymbol{\sigma}^+)$ where $\boldsymbol{a}^\dagger$, $\boldsymbol{a}$ are the boson creation and annihilation operators for the field mode and the operators $\boldsymbol{\sigma}^+ = |e\rangle\langle g|$, $\boldsymbol{\sigma}^- = |g\rangle\langle e|$ are the raising and lowering operators for the atomic state. The unitary transformation that acts when this interaction is applied for a time t is $\boldsymbol{U}(\tau) = \exp[-i\tau(\boldsymbol{a}^\dagger\boldsymbol{\sigma}^- + \boldsymbol{a}\boldsymbol{\sigma}^+)]$ where $\tau = \kappa t$.
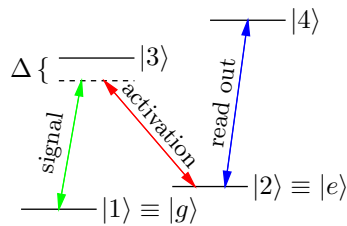

FIG. 1: Level scheme for an effective two-level transition controlled by a stimulated Raman process.

If the atom is prepared in the ground state and found in the ground state after the interaction, the conditional state of the field is given by

$$\boldsymbol{\Upsilon}_g(\tau)|\psi\rangle = \cos(\tau\sqrt{\boldsymbol{a}^\dagger\boldsymbol{a}})|\psi\rangle \qquad (1)$$

On the other hand if the atom is prepared in the excited state and found in the excited state after the interaction, the conditional state is given by $\boldsymbol{\Upsilon}_e(\tau)|\psi\rangle = \cos(\tau\sqrt{\boldsymbol{a}\boldsymbol{a}^\dagger})|\psi\rangle$. There is considerable practical advantage to using the excited state preparation rather than the ground state as there is always a signal for correct operation, however the analysis is the same, and we will concentrate on the operator (1).

We have in mind a quantum computing communication protocol in which the optical field mode is derived from a transform limited pulsed field which is rapidly switched into the cavity mode containing the atomic systems at fixed times determined by the pulse repetition rate. Similar systems have been proposed as a quantum memory for optical information processing [7]. When the atomic measurement yields the required result the field may be switched out again for further analysis or subsequent processing through linear and conditional elements.

Once the cavity field is prepared, we need to switch on the interaction with the atomic system. In order that we can switch this interaction at predetermined times we propose that an effective two level transition connected by a Raman process with one classical field and the quantised signal field, be used. A similar scheme has recently been proposed as the basis of a high efficiency photon counting measurement [8, 9]. The process is also used in the EIT schemes for storing photonic information [10] and for quantum state transfer between distant cavities [11]. The level diagram is shown in figure 1. The nearly degenerate levels $|1\rangle$ and $|2\rangle$ are connected by a stimulated Raman transition to level $|3\rangle$. The detuning of

the Raman pulse from the excited state $|3\rangle$ is Δ, which is approximately the same as the detuning of the signal mode form the same transition. With these parameters the interaction strength is given by $\kappa = \Omega g/2\Delta$ where Ω is the Rabi frequency for the Raman pulse and g is the one photon Rabi frequency for the signal field.

An advantage of using a stimulated Raman process of this kind is that the excited state $|2\rangle$ can be a meta-stable, long lived level. We thus do not need to consider spontaneous emission from this level back to the ground state. The readout of the atomic system may be achieved by using a cycling transition between the excited state $|2\rangle$ and another probe level $|4\rangle$. Such measurements are routinely performed in ion trap studies [12] and can have efficiencies greater than 99%.

In the following, we will prove that the ability to perform the conditional transformation (1) together with linear transformations is universal on qudits. We will follow this with two examples.

It is first instructive to consider a unitary operator of the form $\exp(i\theta\sqrt{\boldsymbol{a}^\dagger \boldsymbol{a}})$. A nonlinearity of the form $\sqrt{\boldsymbol{a}^\dagger \boldsymbol{a}}$, which we refer to as the square-root-number operator, turns out to be as good as a Kerr nonlinearity for universal quantum computation in the infinite dimensional Hilbert space of a Harmonic oscillator [13]. The concept of universal computation is isomorphic to the concept of universal simulation, which is the ability to simulate any arbitrary Hamiltonian evolution, to any degree of accuracy, by combining the evolution due to a fixed class of Hamiltonian generators.

To motivate this idea, consider what kind of nonlinear oscillator would correspond to the Hamiltonian $\boldsymbol{H}_{\mathrm{srn}} = \sqrt{\boldsymbol{a}^\dagger \boldsymbol{a}}$. The resulting Heisenberg equation of motion for the field amplitude operator is

$$\dot{\boldsymbol{a}} = i[\boldsymbol{H}_{\mathrm{srn}}, \boldsymbol{a}] = -i(\sqrt{\boldsymbol{a}^\dagger \boldsymbol{a} + 1} - \sqrt{\boldsymbol{a}^\dagger \boldsymbol{a}})\boldsymbol{a} \qquad (2)$$

which clearly indicates an oscillator in which the frequency is state dependent. The right hand side may be expanded as a power series in $(\boldsymbol{a}^\dagger \boldsymbol{a})^{-1/2}$ to give

$$\dot{\boldsymbol{a}} = -i\left(\frac{1}{2}(\boldsymbol{a}^\dagger \boldsymbol{a})^{-1/2} - \frac{1}{8}(\boldsymbol{a}^\dagger \boldsymbol{a})^{-3/2} + \ldots\right)\boldsymbol{a} \qquad (3)$$

This corresponds to an oscillator for which the frequency of oscillation *decreases* with increasing energy. For comparison, the Kerr nonlinear oscillator corresponds to an oscillator in which the frequency increases with increasing energy. It is thus clear that the square-root-number Hamiltonian will result in a rotational shearing of states in the phase plane of the oscillator in much the same way as occurs for the Kerr nonlinearity [14].

We can go further by using the results of Braunstein and Lloyd [13]. They considered the question of what Hamiltonians are universal for quantum computation in an infinite dimensional Hilbert space, such as that for a single mode of the radiation field. Their results show that

Hamiltonians that are at most quadratic in the canonical momentum and position variables are not universal. For instance, in the case of a single mode field with annihilation and creation operators $\boldsymbol{a}, \boldsymbol{a}^\dagger$, successive applications of Hamiltonians from the set of displacements, squeezing and rotations, $\mathcal{H}_{\mathrm{lin}} = \{z\boldsymbol{a} + z^*\boldsymbol{a}^\dagger, z\boldsymbol{a}^2 + z^*(\boldsymbol{a}^\dagger)^2, z\boldsymbol{a}^\dagger \boldsymbol{a}\}$, can generate arbitrary linear canonical transformations in the phase space variables but no other transformations. A universal set is easily obtained by adjoining almost any Hamiltonian that is at least cubic in the canonical variables. A universal set of Hamiltonians could be made up from $\mathcal{H}_{\mathrm{lin}}$ together with the Kerr nonlinearity $\boldsymbol{H}_k = (\boldsymbol{a}^\dagger)^2 \boldsymbol{a}^2$. Another choice is the cubic Hamiltonian $\boldsymbol{H}_c = \boldsymbol{a}^\dagger \boldsymbol{a}(\boldsymbol{a} + \boldsymbol{a}^\dagger) + hc$. We now show that the square-root-number operator $\sqrt{\boldsymbol{a}^\dagger \boldsymbol{a}}$ together with the set $\mathcal{H}_{\mathrm{lin}}$ can be used to simulate a cubic Hamiltonian, and thus is universal for quantum simulations in the Hilbert space of a single mode.

Consider acting on the unitary operator generated by the square-root-number operator with a large displacement α: $\boldsymbol{U}(\alpha, \theta) = \boldsymbol{D}^\dagger(\alpha)\exp(i\theta\sqrt{\boldsymbol{a}^\dagger \boldsymbol{a}})\boldsymbol{D}(\alpha)$ where $\boldsymbol{D}(\alpha)$ is the displacement operator given by $\boldsymbol{D}(\alpha) = \exp(\alpha\boldsymbol{a}^\dagger - \alpha^*\boldsymbol{a})$. The resulting unitary operator may then be written as

$$\begin{aligned}\boldsymbol{U}(\alpha, \theta) &= e^{i\theta|\alpha|}\exp\left[i\theta\left(\frac{\boldsymbol{x}(\phi)}{2} + \frac{\boldsymbol{a}^\dagger \boldsymbol{a}}{2|\alpha|} - \frac{\boldsymbol{x}(\phi)^2}{8|\alpha|}\right.\right.\\ &\left.\left.- \frac{\boldsymbol{a}^\dagger \boldsymbol{a}\boldsymbol{x}(\phi) + \boldsymbol{x}(\phi)\boldsymbol{a}^\dagger \boldsymbol{a}}{8|\alpha|^2} + O(|\alpha|^{-3})\right)\right] \quad (4)\end{aligned}$$

where $\boldsymbol{x}(\phi) = (\boldsymbol{a}e^{-i\phi} + hc)$ with ϕ the phase of α. The first three terms in the argument of the exponential correspond to a second order Hamiltonian and simulate a displacement, a rotation, and a squeezing operation respectively. The fourth term however is a cubic term which is what we required for a universal set of Hamiltonians for a single mode. By using a linear Hamiltonian to mix several modes (for instance a beam-splitter), arbitrary multi-mode Hamiltonians can be constructed [13]. It is thus clear that we can use the square-root-number operator, together with an arbitrary linear transformations to perform universal computation.

Now that we have shown the universality of the operator $\exp(i\theta\sqrt{\boldsymbol{a}^\dagger \boldsymbol{a}})$ what can we say directly about $\cos\left[\theta\sqrt{\boldsymbol{a}^\dagger \boldsymbol{a}}\right]$? The action of this conditional operator on the number state $|n\rangle$ is to multiply the state by the amplitude $\cos(\theta\sqrt{n}) = [\exp(i\theta\sqrt{n}) + \exp(-i\theta\sqrt{n})]/2$. Clearly, if we can restrict the interaction so that this amplitude is ± 1, then it is equivalent to the full unitary operator—and we might expect to exploit the nonlinear interaction in a similar way. Now, as it turns out, it *is* possible [19] to choose θ such that for a finite size computational space, $|0\rangle \ldots |N\rangle$,

$$\cos\left[\theta\sqrt{\boldsymbol{a}^\dagger \boldsymbol{a}}\right]|n\rangle \approx \begin{cases} -|n\rangle & \text{for } n = 2(2m+1)^2 \\ |n\rangle & \text{for } n \neq 2(2m+1)^2 \end{cases} \quad (5)$$

Here m is a non-negative integer and we see a sign shift only on the states $|2\rangle, |18\rangle, |50\rangle \ldots |2(2m+1)^2\rangle \ldots$. We have chosen the first sign shift to occur on the $|2\rangle$ Fock state. This is an arbitrary choice and other terms can be shifted instead.

For most computation schemes in which we are interested, there is a finite number of basis states (there generally always is a natural cutoff even in most CV schemes). In this case it is always possible to choose θ to satisfy the above relations. This indicates that our operator $\cos\left[\theta\sqrt{\boldsymbol{a}^\dagger\boldsymbol{a}}\right]$ is approximating an effective, highly non-linear unitary operator. For instance, if we consider only the qutrit subspace of $|0\rangle$, $|1\rangle$, and $|2\rangle$ then the operator acts equivalently to the Hamiltonian, $H = \frac{\pi}{2}\boldsymbol{a}^{\dagger 2}\boldsymbol{a}^2$ which contains a Kerr nonlinearity.

With this nonlinear Hamiltonian, it is possible in conjunction linear Hamiltonians to generate arbitrary multi-mode Hamiltonians. Hence we can in principle create any of the unitary operators required for universal qutrit computation. This argument extends to higher qudits in a direct manner. The potential problem however, is that the greater the number of states in the qudit space, the larger the value of θ required to satisfy (5).

It has recently been shown by Knill, Laflamme and Milburn (KLM) [1], that a conditional nonlinear sign shift (NS) gate on two photon states can be produced with passive linear optical elements and photo-detection. Such conditional nonlinear phase shifts can be used to perform two qubit operations for logical states encoded in photon number states. If such conditional gates are used to prepare entangled states for teleportation, efficient quantum computation can be performed which with suitable error correcting codes can be made fault tolerant [1]. Here we show that if the ancilla modes are replaced with a two level atom similar conditional non-linear phase shifts can be achieved by near deterministic post-selection on atomic measurements. The atomic measurements can be made with fluorescence shelving techniques which are very much more efficient than single photon counting measurements, thus reducing the need for new photon counting technologies inherent in the KLM scheme. In addition, the near-deterministic nature of our gate would also drastically simplify the implementation of the KLM scheme. The ability to induce conditional nonlinear phase shifts on quantum optical fields also has applications to high precision measurements (see for instance [6]).

Our objective for the NS gate is to find a way to produce the nonlinear phase shift defined by

$$c_0|0\rangle + c_1|1\rangle + c_2|2\rangle \to c_0|0\rangle + c_1|1\rangle - c_2|2\rangle \quad (6)$$

where the c_j are complex amplitudes satisfying $\sum|c_j|^2 = 1$. This is in the form of the conditions in equation (5), so consider applying the measurement operator $\boldsymbol{\Upsilon}_g$, on the general two photon state. This will leave us with
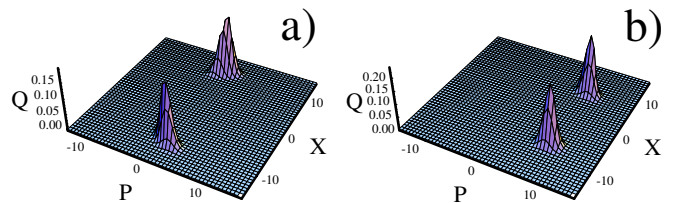

FIG. 2: A plot of the Q-function versus the two canonical phase space variables, for conditional state produced from an initial coherent state using, (a) $\alpha = 10$, $\theta = 10\pi$ and (b) $\alpha = 10$, $\theta = 5\pi$

the state $A_0 c_0|0\rangle + A_1 c_1|1\rangle + A_2 c_2|2\rangle$ where $A_0 = 1$, $A_1 = \cos(\tau)$ and $A_2 = \cos(\sqrt{2}\tau)$. Solutions for which (A_1, A_2) approaches arbitrarily close to $(1, -1)$ can easily be found, for examples see table I. It is also possible

τ	A_0	A_1	A_2
6.5064	1	0.97519	-0.97516
37.73742	1	0.9992663	-0.9992665
219.918	1	0.999979	-0.999978

TABLE I: High-probability results for a single atom initially in a ground state and measured in a ground state after an interaction time τ.

to employ several atoms, each addressed independently, and find solutions of high probability using the same techniques. For example with two atoms, one initially prepared in the ground state and the second prepared in the excited state, the conditional state given that both atoms are found in their initial state after the interaction is $|\phi_{ge}\rangle = \boldsymbol{\Upsilon}_g(\tau_1)\boldsymbol{\Upsilon}_e(\tau_2)|\psi\rangle$ and again values of the interaction times τ_1 and τ_2 can be found which again perform the NS gate with high probability. For instance with $\tau_1 = 37.79300921$ and $\tau_2 = 197.78109842$, then $|A_1| = |A_2| = |A_3| = 0.990321935$ and the required phase shift is performed on $|2\rangle$.

In any real experiment, the desired interaction time can be calibrated by placing the phase shift in one arm of a Mach-Zehnder interferometer, with two single photon inputs, and examining the interference fringes as a function of the interaction time.

Now let us consider the case of the field in an initial coherent state $|\alpha\rangle$. We will show that the conditional transformation, $\cos(\theta\sqrt{\boldsymbol{a}^\dagger\boldsymbol{a}})$, generates a coherent superposition of coherent states, a so called *Schrödinger cat state*. A convenient phase space representation of the conditional state is the Q-function defined by

$$\begin{aligned} Q(\beta) &= |\langle\beta|\cos(\theta\sqrt{\boldsymbol{a}^\dagger\boldsymbol{a}})|\alpha\rangle|^2 \\ &= \frac{1}{4}|\langle\beta|e^{i\theta\sqrt{\boldsymbol{a}^\dagger\boldsymbol{a}}} + e^{-i\theta\sqrt{\boldsymbol{a}^\dagger\boldsymbol{a}}}|\alpha\rangle|^2 \end{aligned}$$

We thus first consider the amplitude function $\mathcal{A}(\beta) =$

$\langle\beta|\exp(i\theta\sqrt{\boldsymbol{a}^\dagger\boldsymbol{a}})|\alpha\rangle$. Based on the semi-classical expectation that this unitary transformation describes an oscillator with an energy dependent frequency, we anticipate that we need only consider the Q-function on the curve $|\beta|=|\alpha|$. With this in mind we put $\beta=|\alpha|e^{i\phi}$ and the Q-function amplitude is then given by

$$\mathcal{A}(\beta)=\sum_{n=0}^{\infty}p_n(\alpha)e^{i(\theta\sqrt{n}-\phi n)} \qquad (7)$$

where $p_n(\alpha)=e^{-|\alpha|^2}\frac{|\alpha|^n}{n!}$. We now assume that $|\alpha|>>1$ and approximate the Poisson distribution, $p_n(\alpha)$ with a Gaussian

$$p_n(\alpha)\approx(2\pi|\alpha|)^{-1/2}\exp\left[-\frac{(n-|\alpha|)^2}{2|\alpha|}\right]. \qquad (8)$$

We can then replace the sum with an integral over the variable $y=n-|\alpha|$. Under the assumption $|\alpha|>>1$ we find the integrand can be approximated as a general Gaussian and thus the integral is given by

$$\mathcal{A}(\beta)=e^{-i\phi|\alpha|^2+i\theta|\alpha|}\exp\left[-\frac{1}{8}(\theta-2|\alpha|\phi)^2\right]. \qquad (9)$$

Clearly this distribution is peaked at $\phi=\theta/(2|\alpha|)$. If we choose the interaction time so that $\theta=|\alpha|\pi$, we expect the state to be localised on the positive imaginary axis in the phase plane of the Q-function. Similarly for the same parameters $\langle\beta|\exp(i\theta\sqrt{\boldsymbol{a}^\dagger\boldsymbol{a}})|\alpha\rangle$ will be localised on the negative imaginary axis in the phase plane of the Q-function. It then follows that for the full conditional operator, $\cos(\theta\sqrt{\boldsymbol{a}^\dagger\boldsymbol{a}})$, the state has two components localised symmetrically about the origin on the imaginary axis. In figure [2] we show the Q function as a function of the two canonical phase space variables x and p for several choices of α and θ.

We observe that if we load a coherent state into the cavity, and repeat the conditioning measurement procedure outlined previously, the conditional state of the field will be prepared in a state which is close to a cat state. Note however that these cats are not parity eigenstates as the conditional interaction cannot change the photon number distribution. Similar cat states are produced by a Kerr nonlinearity [15, 16].

We now estimate some typical values for the parameters. In a recent experiment a similar stimulated Raman process was observed using single rubidium atoms falling through a high finesse optical cavity [17]. The following parameters are typical of that experiment: $g=2\pi\times4.5$ MHz, $\Omega=2\pi\times30$ MHz and $\Delta=2\pi\times6$ MHz. This gives a coupling constant of the order of 70 Mhz. To achieve effective interaction constants of the order of those in the table I, requires interaction times of the order of $0.1-5$ μs. In this paper we have neglected cavity decay which obviously needs to be kept small over similar time scale, which while difficult is not impossible.

We have shown how the ability to do very efficient measurements on single atoms trapped in an optical cavity can be used to implement nonlinear conditional phase shifts on the intra-cavity field. By carefully choosing the interaction time, a nonlinear interaction can be implemented with near unit probability, that, together with linear transformations, is universal for simulating an interaction on qudits. For small qudits the required interaction time can easily be found numerically. If the field state can be carefully switched in and out of the cavity, the method can be used to implement near-deterministic nonlinear gates for quantum optical computing. For instance, the scheme can be used to implement a nonlinear sign shift gate, which thus provides a path to quantum computation with logical qubits encoded in photon number states. It can also be used to conditionally generate coherent superpositions of coherent states and thus can provide the key resource for quantum computing with coherent states [18].

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[19] With $\cos(\theta\sqrt{2}) = -1$, the fundamental theorem of arithmetic implies that for $\sqrt{n}$ which are rational multiples of $\sqrt{2}$, $\cos(\theta\sqrt{n}) = \pm 1$. Irrational multiples of $\sqrt{2}$ will have incommensurate periods, so it should be possible to find θ for which (5) is true to within a given error.