# A Unified Approach to the STFT, TFD's, and Instantaneous Frequency

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*Abstract*—**Spectral analysis of time-varying signals is traditionally performed with the short-time Fourier transform (STFT). In the last few years, many authors have advocated the use of time-frequency distributions for this task. This paper has two main aims. The first is to reformulate Cohen's class of time-frequency representations (TFR's) into a discrete-time, discrete-frequency, computer implementable form. The second aim is to show how, in this form, many of the properties of the continuous-time, continuous-frequency formulation are either lost or altered. Intuitions applicable in the continuous-time case do not necessarily carry over to the discrete-time case examined here. The properties of the discrete variable formulation examined are the presence and form of cross-terms, instantaneous frequency (IF) estimation, and relationships between Cohen's class TFR's. We define a parameterized class of distributions which is a blending between the STFT and Wigner–Ville distribution (WVD). The two main conclusions to be drawn are that all TFR's of Cohen's class implementable in the given form (which includes all commonly used TFR's) possess cross-terms and that IF estimation using periodic moments of these TFR's is purposeless, since simpler methods obtain the same result.**

## I. Introduction

In the last few years there has been an enormous increase in the volume of literature examining TFD's to outperform the short-time Fourier transform (STFT) time-frequency representation (TFR). Time-frequency distributions are a subclass of TFR's which attempt to describe a signal's behavior in time and frequency in a similar manner to the way bivariate joint probability distributions describe the statistical behavior of two random variables. The exponential distribution proposed by Choi and Williams [5] and the Wigner–Ville distribution (WVD) examined by Claasen and Mecklenbräuker [6] or White and Boashash [23] appear to offer some theoretical advantages over the STFT, but drawbacks in these methods limit their adoption as all-purpose replacements.

A major use of TFR's is the tracking of frequency modulated tones in noise. If a signal consists of a single modulated tone we say it is monocomponent. A multicomponent signal is formed by adding several modulated tones. It is a main contention of this paper that all TFR's of multicomponent signals suffer from the presence of cross-terms. These cross-terms are unwanted oscillations caused by interactions between frequency components which often lead to misinterpretation of the signal. The large cross-terms of the WVD [6] cause major difficulties when it is used to analyze multicomponent signals. For this reason, the formulation of a TFD with small cross-terms was a major goal of Choi and Williams [5]. Contrary to popular belief, we show that the STFT also has cross-terms; fortunately their magnitude and position are such that they have negligible effect on our ability to interpret the information presented. An earlier comment to this effect is made in [23]; we derive an exact expression for the shape of these terms (see the Appendix).

Another aspect of the frequency tracking problem is the estimation of the frequencies of the individual components by following the crests of the ridges of a TFR. The STFT has been used in this capacity for many years. Application of TFD's to multicomponent signals has been hampered by the cross-term problem. The first moment or peak of many TFR's can be used in the simple case of a monocomponent signal. However, we show that frequency estimators based on the first moment of a TFR offer no benefit since it is much simpler to compute an equivalent result in the time domain from the derivative of the phase of the analytic signal.

This paper is set out as follows. In Section II, we introduce the discrete-time, discrete-frequency setting for the rest of the paper. We particularly concern ourselves with defining digital computer implementable objects such as Cohen's class of TFR's. Requiring various properties of TFR's places constraints on their formulation and we define a class of such TFR's, called the bowtie class. Section III examines how implementation considerations imposed on our Cohen's class definition mean that all members of this class possess cross-terms (this result for continuous-time TFR's (which satisfy the marginals) of signals of finite extent has already been shown by Wigner [24]). Instantaneous frequency estimation using periodic moments of TFR's is considered in Section IV. Section V compares the STFT and the WVD and then introduces a parameterized blend of these two TFR's. The final section summarizes the results obtained throughout the paper.

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## II. Cohen's Class

The primary goal of time-frequency representations is to allow more informed signal interpretation than would be available from simple time or frequency domain representations. TFR's should not contradict other signal information such as the energy spectral density, instantaneous signal power, and overall signal energy.

### A. Desirable Properties of TFR's

For interpretation as an energy distribution in the time-frequency plane, the discrete-time TFR $\rho_a(n, k)$ of a discrete-time signal $a(n)$ should have the following properties.

*Property* $\mathcal{P}_1$: The TFR should be real valued since a complex-valued two-dimensional function is difficult to interpret as a surface:

$$\rho_a(n, k) \in \mathcal{R} \tag{1}$$

where $\mathcal{R}$ is the set of all real numbers.

*Property* $\mathcal{P}_2$: Summation in time should yield the signal energy spectral density, $|A(k)|^2$. For the application of frequency tracking, it is essential that the TFR behave in a manner that can be related to the standard notion of frequency:

$$\sum_{n=0}^{N-1} \rho_a(n, k) = |A(k)|^2. \tag{2}$$

*Property* $\mathcal{P}_3$: Summation in frequency should yield the instantaneous power, $|a(n)|^2$,

$$\sum_{k=0}^{N-1} \rho_a(n, k) = |a(n)|^2 \tag{3}$$

where

$$A(k) = \underset{n \to k}{F}[a(n)] = \sum_{n=0}^{N-1} a(n) e^{-j2\pi nk/N} \tag{4}$$

is the discrete Fourier transform of $a(n)$.

*Property* $\mathcal{P}_4$: The time support of the signal should be preserved in the TFR:

$$\begin{aligned} &\text{If} \quad a(n) = 0 \quad \text{for } n < n_1 \text{ and } n > n_2, \\ &\text{then} \quad \rho_a(n, k) = 0 \quad \text{for } n < n_1 \text{ and } n > n_2. \end{aligned} \tag{5}$$

*Property* $\mathcal{P}_5$: The frequency support of the signal should be preserved in the TFR:

$$\begin{aligned} &\text{If} \quad A(k) = \underset{n \to k}{F}[a(n)] = 0 \quad \text{for } k < k_1 \text{ and } k > k_2 \\ &\text{then} \quad \rho_a(n, k) = 0 \quad \text{for } k < k_1 \text{ and } k > k_2. \end{aligned} \tag{6}$$

*Property* $\mathcal{P}_6$: For interpretation as an energy distribution the TFR must also be nonnegative definite:

$$\rho_a(n, k) \geq 0.$$

Unfortunately, it has been shown [17], [24] that requiring property $\mathcal{P}_6$ means that some of the other properties cannot be satisfied; so we cannot obtain a true energy distribution in the time-frequency plane. Despite this fact, representations which only satisfy some of $\mathcal{P}_1$ to $\mathcal{P}_5$ are usually called distributions. Other TFR's such as the STFT are nonnegative definite but do not satisfy the marginal conditions, $\mathcal{P}_2$ and $\mathcal{P}_3$, or the support properties $\mathcal{P}_4$ and $\mathcal{P}_5$.

### B. Cohen's Class of Time-Frequency Representations

Since one aim of this paper is to show the relationship (in computer implementable form) between the STFT and other TFR's, we now introduce the generalized class of TFR's first presented by Cohen [7]. Most previously defined TFR's are members of Cohen's class. Rather than finding relationships for each TFR individually, Cohen's formulation lets us derive general results. This class was originally formulated using continuous variables. We reformulate the class for the more practical use of discrete-time analytic signals (which contain only positive frequency components).

Use of the analytic signal reduces problems due to aliasing and cross-terms between the positive and negative frequency components in any TFR of Cohen's class. For an examination of this in the particular case of the Wigner distribution, see [3]. Indeed, since we conventionally display only the positive frequency spectrum on any frequency plot, we should use the analytic signal since it only contains positive frequencies.

Note that here, and throughout this paper $P$-periodic discrete signals are assumed (where $P$ will depend on the context). Note also that the definition below is intended merely to show how our implementation (8) relates to the usual continuous-time definition of the class. Problems such as use of circular versus linear convolution, and other standard implementation details are of minor consequence and not relevant to the aims of this paper.

*Definition 1: Cohen's Class of TFR's.* Let $a(n) = \mathcal{A}[x(n)]$ where $x(n)$ is a real signal of odd length $N$ and $\mathcal{A}[\,]$ indicates formation of the analytic signal [3]. Then a Cohen's [7] class time-frequency representation of $a(n)$ is given by

$$\begin{aligned} \rho_a(n, k) = 2 \sum_{m=-L}^{+L} \sum_{p=-L}^{+L} \sum_{l=-L}^{+L} &\exp[j2\pi l(p - n)/N] \\ &\cdot f(l, m) a(p + m) a^*(p - m) \\ &\cdot \exp(-j2\pi km/N) \end{aligned} \tag{7}$$

where $L = (N - 1)/2$ and $f(l, m)$ is the Doppler-lag kernel function which characterizes the individual members of the class. ■

Equation (7) may be more easily understood by rewriting it as

$$\rho_a(n, k) = 2 \sum_{m=-L}^{+L} \sum_{p=-L}^{+L} B(p - n, m) a(p + m)$$

$$\cdot\, a^*(p - m) \exp(-j2\pi km/N) \quad (8)$$
$$= 2 \underset{m \to k}{F} [B(-n, m) \underset{(n)}{*} k_a(n, m)]$$

where $B(n, m) = NF^{-1}_{l \to n}[f(l, m)]$ = the time-lag kernel function, $k_a(n, m) = a(n + m)a^*(n - m)$ = the bilinear product, and $*_{(n)}$ represents a convolution in the $n$ index.

The class can therefore be interpreted as discrete Fourier transform of the bilinear product $k_a(n, m)$ smoothed by an observation window $B(n, m)$. The quantity

$$B(-n, m) \underset{(n)}{*} k_a(n, m)$$

is then just a generalization of the autocovariance estimate of $a(n)$.

We point out that, due to the finite memory considerations of this computer implementable class, $B(n, m)$ is always necessarily finite in extent in both $n$ and $m$ variables. That is, $B(n, m)$ is a finite impulse response (FIR) filter in both $n$ and $m$ variables. While IIR $B(n, m)$ are certainly implementable using recursive filters, property $\mathcal{P}_4$ could not then be met. Finite memory means that infinite order filters cannot be implemented. Thus infinite response (IIR) and infinite degree $B(n, m)$ functions are not allowable under our definition of the computer implementable Cohen's class.

This restriction is of no major consequence because:

1) In the off-line calculation of (7), we must be operating on a finite length sequence. As a result $k_a(n, m)$ is necessarily of finite support, so $B(n, m)$ functions which satisfy the marginals of such signals are still allowable. Thus, off-line implementation of the Choi–Williams, Wigner–Ville, STFT, Born–Jordan–Cohen, and Page and Rihaczek representations [8] is possible.

2) In the on-line calculation of (7) we must (due to finite storage) use $k_a(n, m)$ of finite support, so pseudo (or windowed) distributions must be used (in which case the marginals cannot be satisfied).

Note that we only concern ourselves with the case where $B(n, m)$ is independent of the signal under consideration. This means that we consider only bilinear TFR's. Also, since we have written into our definition of Cohen's class the point that the signal under analysis is analytic, we further restrict the class of TFR's examined. In practice, these restrictions are not of particular consequence. All well-known TFR's have $B(n, m)$ independent of the signal and, where the TFR is based on the real signal, analytic signal-based parallels may be defined.

Our reformulation shows that Amin's [1] nonparametric autocovariance estimator approach to short-term spectral estimation is equivalent to Cohen's approach. Amin [1] assumes *a priori* knowledge of the statistical properties of the signal to be analyzed. This allows an autocovariance estimator to be "designed" for a particular signal. The smoothing window which Amin designs amounts to design of the function $B(n, m)$ in our computer implementable Cohen's class TFR's.

If we wish $\rho_a(n, k)$ to be a TFD we must satisfy properties $\mathcal{P}_1$ to $\mathcal{P}_4$. This places the following restrictions on $B(n, m)$ [10]:

$$\left.\begin{array}{l} \bullet\ \mathcal{P}_1 \Rightarrow B(n, m) = B^*(n, -m). \\ \bullet\ \mathcal{P}_2 \Rightarrow \sum \forall_n B(n, m) = 1. \\ \bullet\ \mathcal{P}_3 \Rightarrow B(n, 0) = \delta(n). \\ \bullet\ \mathcal{P}_4 \Rightarrow B(n, m) = 0, \quad |n| > |m|. \end{array}\right\} \quad (9)$$

The term $\delta(n)$ is the Kronecker delta function defined by

$$\delta(n) = \begin{cases} 1, & n = 0 \\ 0, & \text{otherwise.} \end{cases} \quad (10)$$

Equivalent restrictions placed upon the function $f(l, m)$ appear in [8].

### C. The Bowtie Class of TFR's

Equations (9) and (21) mean the function $B(n, m)$ can only be nonzero in the region depicted in Fig. 1. This allows us to define a subclass of Cohen's class, whose members are the TFD's which yield the IF by their periodic first moment. We shall call this subclass the *bowtie class*, so called due to the shape of the nonzero region in Fig. 1.

Note that a similar result which holds for the continuous-time, continuous-frequency Cohen's class TFR's was obtained by Zhao *et al* [26]. A detailed comparison between the results presented there and the current paper is instructive, but beyond the scope of this paper.

*Definition 2: Bowtie Class of TFD's.* Let $\rho_a(n, k)$ be a member of Cohen's class such that

$$\rho_a(n, k) = \underset{m \to k}{F} [B(-n, m) \underset{(n)}{*} k_a(n, m)]$$

where $k_a(n, m) = a(n + m)a^*(n - m)$ and $a(n)$ is a truncated discrete analytic signal. For $\rho_a$ to be a TFD which yields the IF by its periodic first moment, $B(n, m)$ must satisfy

$$\left.\begin{array}{rl} B(n, m) &= B^*(n, -m) \\ \sum \forall_n B(n, m) &= 1 \\ B(n, 0) &= \delta(n) \\ B(n, m) &= 0, \quad |n| > |m| \\ \text{and} \quad B(n, 1) &= \delta(n). \end{array}\right\} \quad (11)$$

■

So, far, we have introduced Cohen's [7] general class of time-frequency *representations* (TFR's). The subclass of this for which properties $\mathcal{P}_1$ to $\mathcal{P}_4$ hold is called the set of time-frequency *distributions* (TFD's). If we further add the constraint that the TFD must also give the signal instantaneous frequency as its first periodic moment, then this restricted subclass is the *bowtie* class.

The restrictions of (11) and the requirement that the cross-terms (represented by $\rho_{X\text{-term}}(n, k)$) of a TFR be

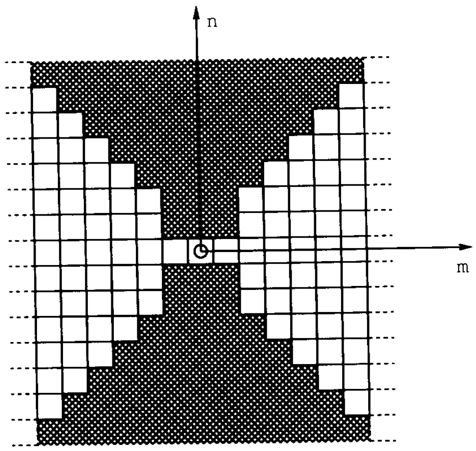


Fig. 1. Restrictions upon $B(n, m)$ shape. In shaded area $B(n, m) = 0$.

small in comparison to the "autoterms" (the true spectral components) give us a set of criteria by which we can compare TFR's.

## III. Cross-Terms

As mentioned earlier, TFR's suffer from frequency component interaction or cross-terms (i.e., artifacts).

Consider formation of the bilinear product $k_a(n, m)$ when $a(n) = e^{j2\pi f_1 n} + e^{j2\pi f_2 n}$ with $f_1 \neq f_2$:

$$k_a(n, m) = a(n + m)a^*(n - m)$$

$$= e^{j4\pi f_1 m} + e^{j4\pi f_2 m} + 2 \cos(2\pi \Delta f_n) e^{j4\pi \bar{f}_{12} m} \qquad (12)$$

where $\Delta f = f_1 - f_2$, and $\bar{f}_{12} = (f_1 + f_2)/2$.

The first term in (12) produces the frequency component at $f_1$, the second produces the frequency component at $f_2$, and the third term in the above expression gives rise to the cross-terms.

We therefore define the cross-terms in a time-frequency representation of two distinct frequency components at the same time as follows. Not all cross-terms occurring in TFR's are characterized by this description. However, for the purpose of the point we wish to make, involving cross-terms between distinct frequency components at the same time, this restricted class of cross-terms will suffice.

*Definition 3: TFR Cross-Terms.* The cross terms between two distinct frequency components, $f_1$ and $f_2$, which occur in any member of our bilinear computer implementable Cohen's class are given by

$$\rho_{\text{X-term}}(n, k) = \underset{m \to k}{F} \Big[B(-n, m) \underset{(n)}{*} \prod_{2N} (m - N) \cdot 2 \cos(2\pi \Delta f n) \exp(j4\pi \bar{f}_{12} m)\Big] \qquad (13)$$

where

$$\prod_{M} (n) = \begin{cases} 1, & \text{for } |n| \leq M \\ 0, & \text{otherwise.} \end{cases} \qquad (14)$$

■

Henceforth, when we refer to cross-terms in a TFR, we shall be specifically referring to cross-terms of this type.

The shape of the cross-terms produced above is given by the cross section of the function $B(n, m)$ (in the $n$ direction) provided $f(l, m) = F_{n \to l}[B(n, m)]$ is a function of the product $lm$ only [10]. All commonly used TFR's are of this form except for the STFT.

For any nontrivial selection of $B(n, m)$ (e.g., $B(n, m) \neq 0$), $\rho_{\text{X-term}}(n, k)$ will not be zero for all $n$, $k$. This is primarily because we are considering the discrete-time, discrete-frequency case.

If the continuous-time, continuous-frequency case is considered, selection of $B(n, m) = 1$, for all $n$, $m$ will reduce the average power in the cross-terms to zero. Wigner [24] has shown that TFR's (actually quantum-mechanical distribution functions) of finite duration signals which satisfy the marginals must have cross-terms.

In our case, however, the necessarily finite extent of $B(n, m)$ means that only a finite number of frequency zeros are available. Since $B(n, m)$ is finite in extent, it may be considered to be (for fixed $m$) a finite impulse response (FIR) filter, as discussed previously. The frequency response of an FIR filter of length $2N$ can have, at most, $2N - 1$ distinct frequency zeros; the DFT in (13) will be of length $2N$. We can conclude that the DFT of $B(n, m)$ convolved with an nonzero function will itself be nonzero. It is possible that, for a particular input signal, the frequency of the input signal does not correspond to one of the frequency zeros of $B(n, m)$. This will only happen for signals of a particular frequency, in general, the cross-terms of a given TFR will be nonzero.

Hence, any TFR of Cohen's class in the formulation being discussed (the discrete-time/discrete-frequency form) will exhibit nonzero cross-terms.

## IV. Instantaneous Frequency

A concept such as IF must be precisely defined before we can begin to discuss its estimation sensibly and it is important to make a clear distinction between the chosen definition and the practical means of estimation. In the context of an FM communication system the IF can be defined in terms of the instantaneous amplitude of the baseband signal and the voltage-to-frequency conversion law of the modulator. If the FM signal has a moderate modulation index, the estimator based on the phase derivative of the analytic signal (called the analytic derivative or AD estimator) associated with the received signal (which is possibly noisy) yields an accurate estimate of the original baseband signal. This fact has led many authors to use the AD estimator as their definition of IF.

However, if we examine signals with sufficiently large modulation indices, the AD estimator no longer corresponds to the imposed IF law. Moreover, if we have a communication channel carrying two FM signals there will clearly be two IF laws imposed on the channel even though the AD estimator can only yield a single value for its estimate. Therefore, we contend that the IF imposed by the signal generating system is the fundamental concept and that the derivative of the phase of the analytic signal is just a good estimator under certain circumstances.

As a result of this confusion, IF is often estimated by the derivative of the phase of the analytic signal [21]. This estimate is identical to the normalized first moment of the WVD, in the frequency direction, for the continuous case. An equivalent result for discrete-time signals is the estimation of discrete instantaneous frequency by the periodic first moment (defined in Section IV-A) of the discrete WVD. Here discrete instantaneous frequency is estimated by the central finite difference (CFD) of the phase of the analytic signal as follows.

*Definition 4: CFD Discrete Instantaneous Frequency and Estimator.* The discrete instantaneous frequency (central finite difference definition) of the discrete analytic signal $a(n)$ is given by

$$f_i^{CFD}(n) = \frac{f_s}{4\pi}((\arg[a(n+1)] - \arg[a(n-1)]))_{2\pi} \tag{15}$$

where $f_s$ is the sampling frequency and $((\cdot))_{2\pi}$ represents reduction modulo $2\pi$. If $a(n)$ is the analytic signal corresponding to a frequency modulated tone in noise, then the above equation may be used to calculate the CFD IF estimate $\hat{f}_i^{CFD}(n)$ of the tone. ■

We now examine TFR-based IF estimators and compare their performance to those estimators based on finite differencing of the phase of the analytic signal.

### A. Periodic and Linear Moments

Reported problems with estimating IF using direct estimators such as $\hat{f}_i^{CFD}(n)$, led to first moments of TFR's being investigated as estimators [6], [22]. Classen and Mecklenbräuker [6] suggested the normalized linear moment IF estimator defined by

$$\hat{f}_i^{LM}(n) = \frac{f_s}{2N} \frac{\sum_{k=0}^{N-1} k\rho_a(n,k)}{\sum_{k=0}^{N-1} \rho_a(n,k)}. \tag{16}$$

Due to the periodic nature of $\rho_a(n, k)$, this estimator is biased, especially when $f_i(n)$ is close to 0 or $f_s/2$. See [11] and [15] for a full treatment of the errors involved.

When we estimate the IF of a signal by the first moment of a TFR we must account for the periodicity in the frequency variable. Although not widely known, the same problem arises in the study of directional data and is as old as the subject of mathematical statistics itself. The temptation to use conventional linear techniques can lead to paradoxes; for example, the arithmetic mean of the angles 0 and $2\pi$ is $\pi$ whereas by intuition the mean should be 0. Directional data may be visualized by points on a circle whereas linear data may be visualized by points on a line; hence the names circular and linear are used to describe these two distinct types of data. The different algebraic structures of the circle and the line, the circle having only one operation (addition modulo $2\pi$) and the line having two operations (addition and multiplication), produce statistics and operations with quite different behavior.

Taking account of the periodicity of $\rho_a(n, k)$ requires the use of the periodic first moment [19] resulting in the IF estimator $\hat{f}_i^{PM}$:

$$\hat{f}_i^{PM}(n) = \frac{f_s}{4\pi}\left(\left(\arg\left[\sum_{k=0}^{N-1} \rho_a(n,k)e^{j2\pi k/N}\right]\right)\right)_{2\pi}. \tag{17}$$

### B. Relationships Between $\hat{f}_i^{PM}$ and $\hat{f}_i^{CFD}$

Substitution of (8) into (17) gives

$$\begin{aligned}\hat{f}_i^{PM}(n) &= \frac{f_s}{4\pi}\left(\left(\arg\left[\sum_{k=0}^{N-1} \underset{m\to k}{F}[B(-n,m) \underset{(n)}{*} k_a(n,m)]e^{j2\pi k/N}\right]\right)\right)_{2\pi}\\
&= \frac{f_s}{4\pi}((\arg[\underset{k\to m}{F^{-1}}[\underset{m\to k}{F}[B(-n,m) \underset{(n)}{*} k_a(n,m)]]_{m=1}]))_{2\pi}\\
&= \frac{f_s}{4}((\arg[B(-n,1) \underset{(n)}{*} k_a(n,1)]))_{2\pi}\\
&= \frac{f_s}{4\pi}\left(\left(\arg\left[\sum_{\forall p} B(-p,1)|a(n-p+1) \cdot a^*(n-p-1)| \cdot \exp[j4\pi \hat{f}_i^{CFD}(n-p)/f_s]\right]\right)\right)_{2\pi}.\end{aligned} \tag{18}$$

Note that if we select $B(-n, 1) = \delta(n)$ in (18), then

$$\hat{f}_i^{PM}(n) = \hat{f}_i^{CFD}(n)$$

exactly. Thus the periodic first moment of the WVD (which has such a $B(n, m)$) is identical to the discrete IF estimated from $\hat{f}_i^{CFD}$.

If we assume that, over the length of the window defined by $B(-p, 1)$, the term

$$|a(n-p+1)a^*(n-p-1)|$$

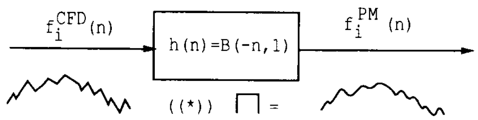


Fig. 2. Periodic first moment of a TFR as a smoothed CFD IF estimator.

is approxmately constant (i.e., we make a constant amplitude, low noise assumption) then we may write

$$\hat{f}_i^{PM}(n) \approx \hat{f}_i^{CFD}(n)((\underset{(n)}{*}))_{f_s/2}B(-n, 1) \tag{19}$$

where $((*_{(n)}))_{f_s/2}$ denotes the modulo convolution operation [2], [11], [13]–[15] in the $n$ index. This is defined as follows.

*Definition 5: Modulo Convolution.* The modulo convolution between the modulo quantity $f(n) = ((f(n)))_A$ and the smoothing window $h(n)$ is defined by

$$f(n)((\underset{(n)}{*}))_A h(n) = \frac{A}{2\pi}\left(\left(\arg\left[\sum_{\forall p} h(p) \exp\left[j2\pi f(n-p)/A\right]\right]\right)\right)_{2\pi} \tag{20}$$

■

We use modulo convolution rather than linear convolution to ensure that modulo quantities are averaged sensibly. Although the expression looks awkward it just represents the argument of a phasor sum.

Equation (19) shows that the IF estimate found from the first periodic moment of any TFR is exactly equivalent to a weighted average of phasors. This is analogous to passing the CFD estimate of IF through a filtering operation. In the case of most TFR's, this filtering operation is a smoothing or low-pass filtering of the CFD IF estimator by a filter with impulse response $h(n) = B(-n, 1)$ (this was reported incorrectly as $B(n, 1)$ in [2]) using the new definition of modulo convolution from the above definition. This is illustrated in Fig. 2. In fact, since the TFR periodic moment IF estimate involves the term

$$|a(n-p+1)a^*(n-p-1)|$$

the smoothed CFD estimator actually has a smaller variance.

In summary, if we require the periodic first moment of a TFR to yield the IF we must also have (see Section IV-B)

$$B(-n, 1) = \delta(n). \tag{21}$$

## V. Comparison of TFR's

We consider the relative merits of the STFT, the WVD, and a new TFR which blends the properties of the two. We shall call this blend the short-time Fourier transform Wigner-Ville (FTWV) representation.

### A. The Short-Time Fourier Transform

Consider the TFR based upon the periodogram autocovariance estimator:

$$S_a(n, k) = G_{\text{STFT}}[a(n)] = \sum_{m=-Q}^{+Q} C_{aa}(n, 2m)e^{-j2\pi 2mk/M}$$

where

$$M = \text{the window length} = 2Q + 1$$

and

$$C_{aa}(n, 2m) = \frac{1}{M}\sum_{p=|m|}^{2Q-|m|} a(n+p+m)a(n+p-m).$$

This TFR is the periodogram short-time Fourier transform (STFT). The time-lag kernel function $B(n, m)$ for the STFT is

$$B_{\text{STFT}}(n, m) = \frac{1}{M}\prod_{Q-|m|}(n). \tag{22}$$

$B_{\text{STFT}}(n, m)$ is illustrated in Fig. 3. If we use a nonrectangular window function to form the STFT, the magnitude of $B_{\text{STFT}}(n, m)$ will be nonuniform but it will still retain its characteristic diamond-shaped region of support. Clearly the STFT is not a member of the bowtie class of TFR's. However, it does possess property $\mathcal{P}_6$ ($S_a(n, k) \geq 0$) which is very desirable for interpretation.

The other major factor for the relative popularity of the STFT is that the cross-terms it exhibits, $S_{\text{X-term}}(n, k)$ (defined by (13)), are very small in magnitude compared with the true spectral components.

$S_{\text{X-term}}(n, k)$ between two components of normalized frequencies $f_1$ and $f_2$ is given by [10] (see the Appendix for a complete derivation)

$$S_{\text{X-term}}(n, k) = \frac{2\cos[2\pi\Delta f(n+Q)]}{(2Q+1)\sin[\pi\Delta f]}\left\{\sin\left[2\pi\Delta f\left(Q+\frac{1}{2}\right)\right][\delta(k-2Mf_1)+\delta(k-2Mf_2)] - \cos\left[2\pi\Delta f\left(Q+\frac{1}{2}\right)\right]\left[\frac{\sin[(L+1)\pi(f_1-k/M)]\sin[L\pi(f_1-k/M)]}{\sin[\pi(f_1-k/M)]} + \frac{\sin[(L+1)\pi(f_2-k/M)]\sin[L\pi(f_2-k/M)]}{\sin[\pi(f_2-k/M)]}\right]\right\} \tag{23}$$

STFT window length and $\Delta f = f_1 - f_2$.

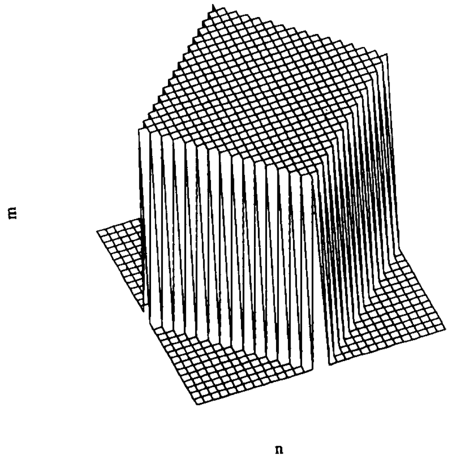


Fig. 3. The function $B_{STFT}(n, m)$ for the short-time Fourier transform.

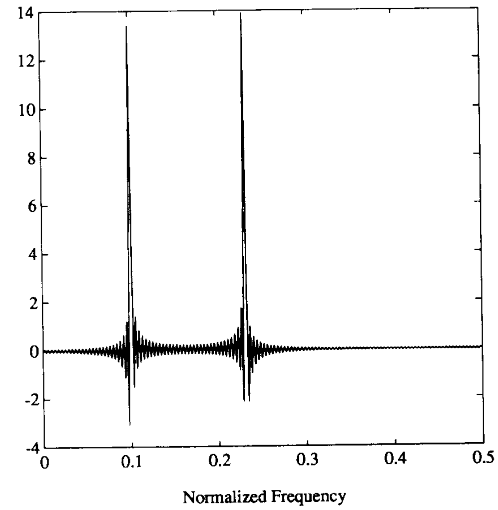


Fig. 4. STFT cross-terms of two tone signal: $G_{STFT}[a(n)] - G_{STFT}[a_1(n)] - G_{STFT}[a_2(n)]$ at $n = 127$.

From (23) we see that the magnitude of the cross-terms is proportional to $1/(2Q + 1)$ and that they occur in the positions where we expect the true spectral components. This somewhat counter-intuitive result means that the cross-terms of the STFT are always swamped by the true components (autoterms).

The reason for this counter-intuitiveness is that, in Cohen's original formulation, the window length, $M$ (and hence $Q$ in (23)) is allowed to approach infinity, so the continuous equivalent to (23) approaches zero.

For the case of an infinite length signal being analyzed with a finite length window, in the continuous-time formulation, this is equivalent to operation of an infinite length window on a finite length signal, so that in this case Wigner's [24] intermediate result shows that cross-terms must occur.

Consider the following illustrative example. Fig. 4 shows the STFT cross-terms of the two-component signal

$$a(n) = a_1(n) + a_2(n) \tag{24}$$

where $a_1(n) = \exp[j(0.2\pi n + 0.5)]$ and $a_2(n) = \exp[j(0.462\pi n + 0.775)]$. Thus, for this example, the cross-terms are seen to peak at the position of the true components as predicted by our derivation, but they are very small in magnitude.

Note that the cross-terms are still centered about the mean frequency $(f_1 + f_2)/2$, but the cross-terms are bimodal rather than unimodal as will be seen later in the Wigner–Ville distribution. The peaks of this bimodal cross-term appear at $f_1$ and $f_2$.

### B. The Wigner–Ville Distribution

The Wigner–Ville distribution (WVD) is defined by

$$W_a(n, k) = G_{WVD}[a(n)] = \sum_{m=-L}^{+L} a(n+m)a^*(n-m)e^{-j2\pi mk/N}$$

where $N = 2L + 1$ is the length of the signal $a(n)$. Here $B(n, m)$ is of the form

$$B_{WVD}(n, m) = \delta(n)$$

which is plotted in Fig. 5. Since convolution in time by $B(n, m)$ can be thought of as a filtering operation, $B_{WVD}(n, m) = \delta(n)$ implies that the Wigner–Ville distribution is in some sense the elemental member of Cohen's class.

Unfortunately, for the Wigner–Ville distribution, the cross-terms (between components of equal magnitude) are up to twice the magnitude of the true components and appear at the arithmetic mean of the two frequencies:

$$\begin{aligned} W_{X\text{-term}}(n, k) &= \underset{m \to k}{F}\left[\delta(n) \underset{(n)}{*} 2\cos(2\pi\Delta f n) \cdot \exp(j4\pi\bar{f}_{12}m/N)\right] \\ &= 2\cos(2\pi\Delta f n)\,\delta(k - \bar{f}_{12}N) \end{aligned} \tag{25}$$

where the components have normalized frequencies $f_1$ and $f_2$, $\Delta f = f_1 - f_2$, and $\bar{f}_{12}$ is the arithmetic mean frequency. Note that the above derivation relies on the term $\bar{f}_{12}N$ being an integer. If $\bar{f}_{12}N$ is not an integer, the standard "noninteger delay" problem [20] must be solved. We do not consider this case here.

For a signal with $n$ distinct time and frequency components (here we allow the case where there are two signals with the same frequencies that occur in disjoint time intervals), there are

$$\binom{n}{2} = \frac{n(n-1)}{2}$$

cross-terms [4].

### C. The FTWV Representation

Following the reasoning of Choi and Williams [5] and incorporating the knowledge we have gained from Section

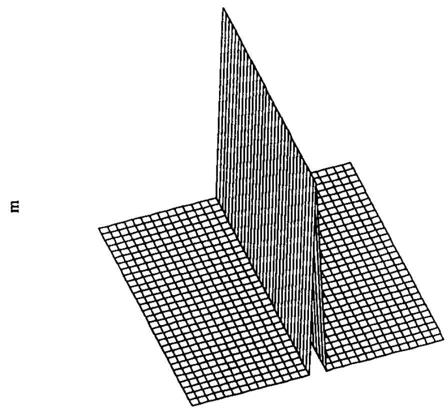


Fig. 5. The function $B_{\mathrm{WVD}}(n, m)$ for the Wigner–Ville distribution.

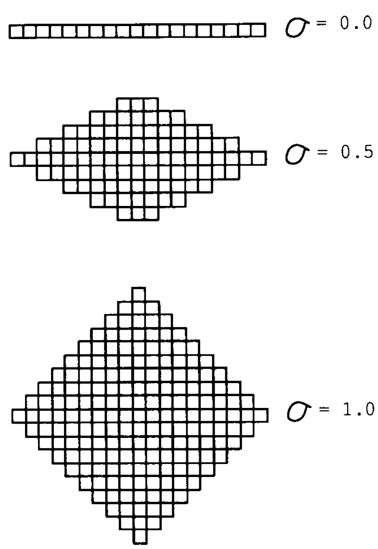


Fig. 6. The function $B_{\mathrm{FTWV}}(n, m)$ for the new representation.

II about TFR structure, we shall now define another parameterized class which is, in effect, a combination of the STFT and the WVD. For this reason, we call the new representation the short-time Fourier transform Wigner–Ville (FTWV) distribution.

Consider the time-lag kernel function

$$B_{\mathrm{FTWV}}(n, m) = \frac{1}{M} \prod_{\sigma(Q-|m|)} (n).$$

Here, for $\sigma = 1$ we have periodogram STFT; for $\sigma = 0$ we have the WVD (provided we define $\Pi_0(n) = \delta(n)$). Cross sections of $B_{\mathrm{FTWV}}(n, m)$ for various values of $\sigma$ are shown in Fig. 6. Fig. 7 shows the cross-terms appearing between two frequency components ($f_1 = 0.1f_s$ and $f_2 = 0.4f_s$) as $\sigma$ varies. Clearly, the cross-term magnitude drops markedly as $\sigma$ increases from 0. The figure also illustrates

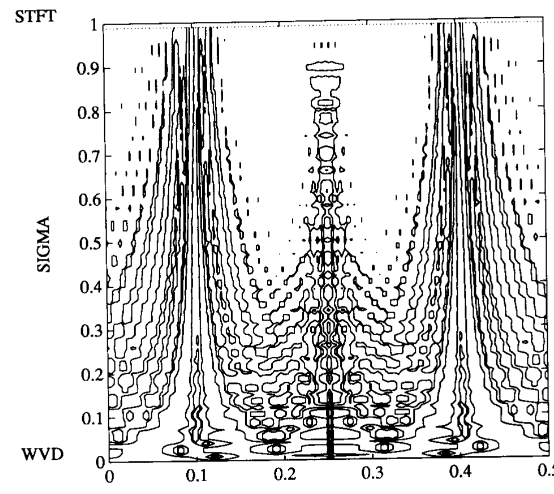


Fig. 7. Contour plot of cross-terms as $\sigma$ varies: Cross-terms between $f_1$ and $f_2$ as $\sigma$: 0.0 → 1.0 for the FTWV representation.

how the cross-term energy migrates from $(f_1 + f_2)/2$ to $f_1$ and $f_2$.

The main purpose of introducing this distribution is to illustrate the close relationship between the WVD and the STFT.

## VI. Conclusions

We have introduced a discrete-time/discrete-frequency formulation of Cohen's class of time-frequency representations (TFR's). This new formulation implies that since any bilinear member of the class performs spectral analysis using a generalization of the signal autocovariance function, then all such TFR's are basically equivalent. The formulation also shows that Cohen's approach is equivalent to the autocovariance smoothing approach taken by Amin [1]. Choice of a "best" smoothing function is very similar to the selection of the best window function in stationary spectral analysis; it is meaningless unless we specify the aspects of the signal which we consider important.

In an attempt to identify what aspects of signal character we believe important, we have introduced a subclass of TFR's called the bowtie class. This places certain restrictions upon the shape and size of the required smoothing function.

The Cohen's class discrete formulation also allows conclusions about TFR cross-terms to be drawn. Since we have shown that all representations in the discrete-time/discrete-frequency class considered here have cross-terms, the question arises "Can we find a bowtie class TFR which has cross-terms which fall on the true signal components just like the short-time Fourier transform?" The answer is "No," because the autocovariance smoothing function, $B(n, m)$, of such a TFR requires the diamond-shaped support as of $B_{\mathrm{STFT}}(n, m)$ illustrated in Fig. 3 to frequency shift the cross-terms onto the true components [10]. Thus it would appear that the STFT is the only suitable TFR for multicomponent signal analysis; every possible bowtie class member has cross-terms positioned between the true components.

A new parameterized class of TFR's (the FTWV representation) which exhibits similar properties to the class of Choi and Williams [5] was introduced. The FTWV representation is a blending of two already well-known TFR's, the Wigner–Ville distribution and the short-time Fourier transform. It illustrates how the large cross-terms of the Wigner–Ville distribution migrate to become the small (almost negligible) cross-terms of the short-time Fourier transform [10].

Using the Cohen's class reformulation, we have also shown that instantaneous frequency (IF) estimation using periodic moments of TFR's is equivalent to applying a suitable "low-pass filtering" function to the more direct estimator based on central finite differencing (CFD) of the phase of the analytic signal. The appropriate smoothing function must be applied using the modulo convolution operation to sensibly smooth the periodic IF estimates. This means that that there is nothing to be gained from using the moments of TFR's to estimate IF; smoothed CFD estimators are computationally simpler and have lower variance. This aspect is considered in further detail in [15] and [16].

## Appendix
## Cross-Terms in the STFT

Consider the autocorrelation estimator

$$R_{aa}(n, 2m) = \frac{1}{2Q+1} \sum_{p=|m|}^{2Q-|m|} \cdot a(n+p+m)a^*(n+p-m) \quad (26)$$

where $a(n)$ is the discrete-time signal under analysis. If we let

$$a(n) = e^{j2\pi f_1 n} + e^{j2\pi f_2 n}$$

and substitute this into (26) we obtain

$$\begin{aligned} R_{aa}(n, 2m) &= \frac{1}{2Q+1} \sum_{p=|m|}^{2Q-|m|} [\exp[j2\pi f_1(n+p+m)] + \exp[j2\pi f_2(n+p+m)]] \\ &\quad \cdot [\exp[-j2\pi f_1(n+p-m)] + \exp[-j2\pi f_2(n+p-m)]] \\ &= \overbrace{\frac{2Q-2|m|+1}{2Q+1}[e^{j4\pi f_1 m} + e^{j4\pi f_2 m}]}^{\text{Autoterms}} \\ &\quad + \overbrace{\frac{2}{2Q+1} \sum_{p=|m|}^{2Q-|m|} \cos[2\pi \Delta f(n+p)] e^{j4\pi \bar{f} m}}^{\text{Cross-terms}} \end{aligned}$$

where $\Delta f = f_1 = f_2$ and $\bar{f} = (f_1 + f_2)/2$. Clearly, if $f_1 M$ and $f_2 M$ are integers (thus avoiding the noninteger delay problem), taking the length $M$ DFT of the autoterms in the above expression will result in Kronecker delta functions at frequency bins $2f_1 M$ and $2f_2 M$ convolved (circularly) with the DFT of the window $w(m) = (2Q - 2|m| + 1)/(2Q + 1)$.

The cross-terms in the estimator $R_{aa}(n, 2m)$ are then given by

$$\begin{aligned} R_X(n, 2m) &= \frac{2}{2Q+1} e^{j4\pi \bar{f} m} \sum_{p=|m|}^{2Q-|m|} \cos(2\pi \Delta f(n+p)) \\ &= \frac{2}{2Q+1} e^{j4\pi \bar{f} m} \frac{1}{2} \sum_{p=|m|}^{2Q-|m|} [e^{j2\pi \Delta f(n+p)} + e^{-j2\pi \Delta f(n+p)}] \\ &= \frac{1}{2Q+1} e^{j4\pi \bar{f} m} \left[ e^{j2\pi \Delta f n} \sum_{p=|m|}^{2Q-|m|} e^{j2\pi \Delta f p} + e^{-j2\pi \Delta f n} \sum_{p=|m|}^{2q-|m|} e^{-j2\pi \Delta f p} \right] \\ &= \frac{1}{2Q+1} e^{j4\pi \bar{f} m} \left[ e^{j2\pi \Delta f n} e^{j2\pi \Delta f |m|} \sum_{p=0}^{2Q-2|m|} e^{j2\pi \Delta f p} \right. \\ &\quad \left. + e^{-j2\pi \Delta f n} e^{-j2\pi \Delta f |m|} \sum_{p=0}^{2Q-2|m|} e^{-j2\pi \Delta f p} \right]. \end{aligned} \quad (27)$$

But

$$\sum_{n=0}^{M} e^{j2\pi knf} = e^{jM\pi kf} \frac{\sin[(M+1)\pi kf]}{\sin[\tau kf]}$$

so (27) becomes

$$\begin{aligned} R_X(n, 2m) &= \frac{1}{2Q+1} e^{j4\pi\bar{f}m}[e^{j2\pi\Delta f(n+|m|)} e^{j(2Q-2|m|)\pi\Delta f} \\ &\quad + e^{-j2\pi\Delta f(n+|m|)} e^{-j(2Q-2|m|)\pi\Delta f}] \frac{\sin[\pi\Delta f(2Q-2|m|+1)]}{\sin[\pi\Delta f]} \\ &= \frac{1}{2Q+1} e^{j4\pi\bar{f}m}[e^{j2\pi\Delta f(n+Q)} + e^{-j2\pi\Delta f(n+Q)}] \frac{\sin[\pi\Delta f(2Q-2|m|+1)]}{\sin[\pi\Delta f]} \\ &= \frac{2\cos[2\pi\Delta f(n+Q)]}{(2Q+1)\sin[\pi\Delta f]} \sin\left[2\pi\Delta f\left(Q-|m|+\frac{1}{2}\right)\right] e^{j4\pi\bar{f}m} \\ &= \overbrace{\frac{2\cos[2\pi\Delta f(n+Q)]}{(2Q+1)\sin[\pi\Delta f]}}^{g_1(n)} \Bigg\{ \overbrace{\sin\left[2\pi\Delta f\left(Q+\frac{1}{2}\right)\right]\cos[2\pi\Delta f|m|]}^{\alpha(m)} \\ &\quad - \overbrace{\cos\left[2\pi\Delta f\left(Q+\frac{1}{2}\right)\right]\sin[2\pi\Delta f|m|]}^{\beta(m)} \Bigg\} \overbrace{e^{j4\pi\bar{f}m}}^{g_2(m)}. \end{aligned}$$

The length $M$ DFT of $\alpha(m)$ (assuming $M\Delta f$ is an integer) is just

$$\underset{m\to k}{F}[\alpha(m)] = \sin[2\pi\Delta fQ][\delta(k - M\Delta f) + \delta(k + M\Delta f)]$$

where $\delta(n)$ is the Kronecker delta. Modulation of this by the $g_2(m)$ modulation term involves only a frequency shift of $M\bar{f}$ (yet again we assume $M\bar{f}$ falls exactly in a frequency bin to avoid the noninteger delay complication). Thus

$$\begin{aligned} \underset{m\to k}{F}[\alpha(m)g_2(m)] &= \sin[2\pi\Delta f(Q+\tfrac{1}{2})][\delta(k - M\Delta f - 2M\bar{f}) + \delta(k + M\Delta f - 2M\bar{f})] \\ &= \sin[2\pi\Delta f(Q+\tfrac{1}{2})][\delta(k - M(f_1 - f_2 + f_1 + f_2)) + \delta(k + M(f_1 - f_2 - f_1 - f_2))] \\ &= \sin[2\pi\Delta f(Q+\tfrac{1}{2})][\delta(k - 2Mf_1) + \delta(k - 2Mf_2)]. \end{aligned}$$

*Remark:* Part of the effect of the cross-terms of the STFT is to add small amplitude Kronecker delta functions which overlay the autoterms exactly.

The length $M = 2L + 1$ DFT of $\beta(m)$ (assuming $M\Delta f$ is an integer) is given by

$$\begin{aligned} \underset{m\to k}{F}[\beta(m)] &= \cos[2\pi\Delta f(Q+\tfrac{1}{2})] \sum_{m=-L}^{+L} \sin[2\pi\Delta f|m|] e^{-j2\pi mk/M} \\ &= \cos[2\pi\Delta f(Q+\tfrac{1}{2})] \sum_{m=1}^{+L} \sin[2\pi\Delta fm][e^{j2\pi mk/M} + e^{-j2\pi mk/M}] \\ &= 2\cos[2\pi\Delta f(Q+\tfrac{1}{2})] \sum_{m=1}^{+L} \sin[2\pi\Delta fm]\cos[2\pi mk/M] \\ &= \cos[2\pi\Delta f(Q+\tfrac{1}{2})] \sum_{m=1}^{+L} [\sin[2\pi m(\Delta f - k/M)] + \sin[2\pi m(\Delta f + k/M)]]. \end{aligned}$$

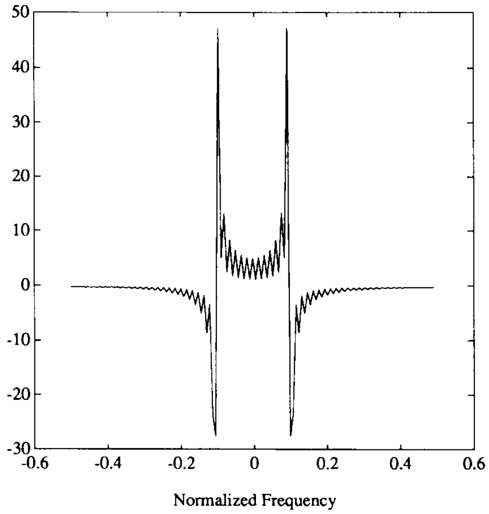


Fig. 8. $F_{m \to k}[\beta(m)]$ evaluated for $L = 63$ and $\Delta f = 0.1$.

Using

$$\sum_{m=1}^{L} \sin[m\psi] = \frac{\sin[(L+1)\psi/2] \sin[L\psi/2]}{\sin[\psi/2]} \tag{28}$$

we have

$$\underset{m \to k}{F}[\beta(m)] = \cos\left[2\pi\Delta f\left(Q + \frac{1}{2}\right)\right]\left[\begin{array}{c}\dfrac{\sin[(L+1)\pi(\delta f - k/M)] \sin[L\pi(\Delta f - k/M)]}{\sin[\pi(\Delta f - k/M)]} \\ + \dfrac{\sin[(L+1)\pi(\Delta f + k/M)] \sin[L\pi(\Delta f + k/M)]}{\sin[\pi(\Delta f + k/M)]}\end{array}\right].$$

This function is plotted in Fig. 8 ($L = 63$, $\Delta f = 0.1$). Note the energy is concentrated about $f = -0.1, +0.1$.

If $\beta(m)$ is now modulated by $g_2(m)$, this will induce a frequency sift of $M\bar{f}$ in $F_{m \to k}[\beta(m) g_2(m)]$. Now the energy will be concentrated about $2Mf_1$ and $2Mf_2$ instead of $+\Delta f$ and $-\Delta f$.

*Remark:* The other effect of the cross-terms of the STFT is to exactly overlay the autoterms with the function of (28).

Hence the cross-terms of the STFT will have peaks where the autoterms occur.

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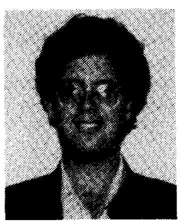

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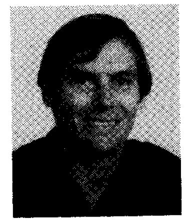

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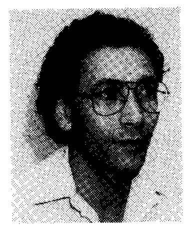

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