TIME-FREQUENCY SIGNAL ANALYSIS & INSTANTANEOUS FREQUENCY ESTIMATION
Methodology, Relationships & Implementations

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ABSTRACT: This paper describes a procedure for the time-frequency analysis of signals, based on Time-Frequency Distributions (TFDs) and Instantaneous Frequency (IF) estimation. First, we use a suitable TFD to determine the number of signal components. Then, if the signal is monocomponent, the IF law can be estimated directly. For multicomponent signals, two-dimensional windowing in the time-frequency (t-f) domain (a form of time-varying filtering) is used to isolate each component; IF estimation is then applied to the individual components. The periodic first moment of a TFD is used to estimate the IF. A suitable definition of the periodic first moment is proposed, and the relationship of these estimators to others based on the central finite difference of the phase of the analytic signal is given. A TFD such as the Wigner-Ville Distribution may be used to represent both IF and amplitude variations in the individual signal components at each stage of the analysis.

0. INTRODUCTION

The representation of time-varying signals is a major problem in many signal processing applications. The Short time Fourier Transform (STFT) is often used in such cases. Model based approaches such as time-varying ARMA models have also been used. Time-frequency Distributions (TFDs) have been introduced in an attempt to provide a general solution to this problem and can be considered as an extension to classical Fourier analysis. The latter is primarily designed to deal with stationary or quasi-stationary signals, while TFDs deal with non-stationary ones. Although the STFT is a general member of Cohen's class of Time-Frequency Distributions (TFDs), $\mathcal{P}$, partial integration over a sufficiently large region of the time-frequency (t-f) plane will not give the signal local energy contribution. We say that TFDs such as the Wigner-Ville, Born-Jordan-Cohen and Choi-Williams which possess this property belong to the sub-class, $\mathcal{P}'$.

A concept central to the selection of TFDs for practical analysis is that of instantaneous frequency (IF). The IF is a parameter which corresponds to the frequency of a sine wave which locally matches the signal under analysis. Physically, it makes sense only for monocomponent signals, i.e. where there is only one frequency or a narrow range of frequencies varying as a function of time [1]. For multicomponent signals, the notion of a single valued IF becomes meaningless - (see [2] for a good discussion on multicomponent signals).

It is clear from the above that the *first step* of any general time-frequency analysis procedure is to determine whether the signal under analysis is monocomponent or multicomponent, and whether the signal is stationary or non-stationary. The analysis tool (chosen from Cohen's class of TFDs [2]) must therefore possess the following three properties:

P_1: The tool discriminates between stationary and non-stationary signals.

P_2: The tool discriminates between monocomponent and multicomponent signals.

P_3: The tool allows a break-up of the multicomponent signal into its components (also time-varying).

Many TFDs possess P_1 and P_2. However, interference terms prohibit most TFDs from possessing P_3. Recent work [3] has shown that there are many ways of generating TFDs with reduced magnitude artifacts which can therefore be used for this purpose. However, there is a trade-off between TFD main lobe width and the magnitude of the artifacts as described in section 1.

The *second step* of the analysis procedure is to break down the multicomponent signal into its sub-components. The method used is based on a masking operation in the time-frequency plane. If required, the equivalent time domain signal can be obtained by TFD inversion techniques [1] [6].

The *third step* is to analyze the components. Here, we are back to the problem of analyzing a non-stationary monocomponent signal. Which analysis method should we use? What is desired from the analysis method is

D_1) to track as accurately as possible the spectral variation of the component as given by the IF $f_i(t)$ and

D_2) to indicate at each time the measure of the local spectral spread or instantaneous bandwidth (IB) $B_i(t)$ around the IF.

The ideal tool would be a TFD which has $f_i t)$ as its first moment and $B_i(t)$ as its second moment, and can display these two parameters in a way which is easily readable. However, the intuitive concept of IB needs to be defined clearly in terms of TFDs.

It is known that any particular TFD can be produced by appropriately smoothing another TFD and that many TFDs give the IF by their first moment. However, with sampled signals we must use the periodic first moment because of the periodic nature of the frequency representation. The relationship between TFDs and IF estimation is

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shown in section 2.

The *fourth step* is the reconstruction and modelling part of the analysis, where a mathematical model can be given which accurately represents this signal.

An example of a multicomponent non-stationary signal is the sum of a chirp signal, a narrow-band time-varying signal and noise as shown in fig.1. An appropriate model is:

$$s(t) = \sum_{k=1}^{N} s_k(t) + n(t) \tag{1}$$

where n(t) is a random noise process and the $s_k(t)$ are monocomponent time-varying signals described by amplitude envelope, $a_k(t)$, and IF, $f_i^k(t)$, such that:

$$s_k(t) = a_k(t)\, e^{j2\pi \int_{-\infty}^{t} f_i^k(\tau)d\tau} \tag{2}$$

Then the analysis problem is to find a_k and f_i^k for k=1,N.

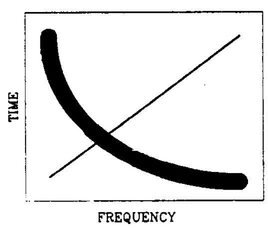

Figure 1. Time-Frequency Plot of an Example Signal

1. SELECTION OF A DISCRETE TFD.

This section examines discrete implementation of Cohen's Class of TFDs. The definition is given in the time-lag domain because of ease of interpretation and implementation. The concept of Bowtie Functions (BFs) is introduced to generate TFDs with the desired properties and to predict their behaviour.

1.1 Cohen's Class of TFDs

Cohen's Class of TFDs [1] is described by

$$\rho_z(t,f) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{j2\pi\nu(u-t)} g(\nu,\tau) z(u+\tau/2) z^*(u-\tau/2) e^{-j2\pi f\tau}\, d\nu\, du\, d\tau \tag{3}$$

where z(t) is the analytic signal associated with the real signal x(t) [4], and $g(\nu,\tau)$ is the arbitrary ambiguity kernel function which characterizes each individual member of the class.

By taking the Fourier transform (FT) of $g(\nu,\tau)$, we obtain:

$$B(t,\tau) = \mathcal{F}_{\nu\to t}\,[g(\nu,\tau)] \tag{4}$$

where $\mathcal{F}$ represents the FT. The expression of $\rho_z(t,f)$ reduces to:

$$\rho_z(t,f) = \mathcal{F}_{\tau\to f}\left[B(t,\tau) \underset{(t)}{*} k_z(t,\tau)\right] \tag{5}$$

where $$k_z(t,\tau) = z(t+\tau/2)z^*(t-\tau/2). \tag{6}$$

$B(t,\tau)$ will be referred to as the kernel function, where t and τ are the time and lag variables. The time-lag domain is assumed unless otherwise noted.

In (3) and (5), the functions $g(\nu,\tau)$ and $B(t,\tau)$ characterize the observation mode chosen by the analyst; they determine how the signal energy is distributed in the time-frequency domain. The $g(\nu,\tau)$ are analogous to the windows used in spectral analysis. Correct choice of B in (5) will yield all TFDs that have been proposed.

The discrete time definition equivalent to the time-lag definition leads to the easiest implementation of a TFD and is expressed as

$$\rho_z(n,k) = \mathcal{F}_{m\to k}\left[B(n,m) \underset{(n)}{*} k_z(n,m)\right] \tag{7}$$

$$\rho_z(n,k) = \sum_{m=-M}^{M} \sum_{p=-M}^{M} B(p-n,m)\, z(p+m) z^*(p-m)\, e^{-j2\pi mk/N} \tag{8}$$

where the discrete real signal, $x(n\Delta t)$, is formed by sampling x(t) at frequency $f_s = 1/\Delta t$ such that $t = n.\Delta t$ and $f = k.\Delta f = k.f_s/N$, N = signal length, and M = (N-1)/2; and where B(n,m) represents the sampled $B(t,\tau)$. For simplicity, we assume $\Delta t = f_s = 1$.

Eq. (7) indicates that the implementation of a TFD requires three steps:

i) formation of the bilinear product $k_z(n,m) = z(n+m)z^*(n-m)$.
ii) discrete convolution in the n (time) direction
iii) discrete FT with respect to m.

1.2 Desirable Properties of TFDs

To be useful as a tool for time-frequency signal analysis [1], a TFD must have the following properties.

P_1: The TFD must be real-valued; this imposes:

$$B(n,m) = B^*(n,-m) \tag{9}$$

P_2: The marginals of the distribution should be equal to the spectrum and instantaneous power:

$$\sum_{n=0}^{N-1} \rho_z(n,k) = |Z(k)|^2 \qquad (10)$$

$$\sum_{k=0}^{N-1} \rho_z(n,k) = |z(n)|^2 \qquad (11)$$

This is obtained if

$$B(n,0) = \delta(n) \quad \text{and} \quad \sum_{n=0}^{N-1} B(n,m) = 1 \qquad (12)$$

where $\delta(n)$ is the Kronecker delta.

P_3: The representation should be zero outside the time and frequency regions where the signal is present:

$$\rho_z(n,k) = 0 \text{ when} \qquad (13)$$
$$z(n) = 0 \text{ for } n < n_1 \text{ and } n > n_2;$$
$$Z(k) = 0 \text{ for } k < k_1 \text{ and } k > k_2.$$

This is obtained if

$$B(n,m) = 0 \text{ for } |n| > |m|. \qquad (14)$$

P_4: The normalized periodic first moment in frequency (section 3) should yield the instantaneous frequency (Central Finite Difference definition). This imposes the conditions:

$$B(n,0) = \delta(n) \quad \text{and} \quad B(n,1) = \delta(n) \qquad (15)$$

TFDs which possess properties P_1 through P_4 belong to the class, $\mathcal{P}'$.

1.3 Bowtie Functions

Accounting for all the constraints on the shape and magnitude of the kernel, the shaded section of Figure 2 indicates the region in the time-lag plane where the kernel function must be zero. Due to the shape of this region, the B(n,m) functions describing the TFDs which are members of $\mathcal{P}'$ will be called Bowtie Functions (BFs).

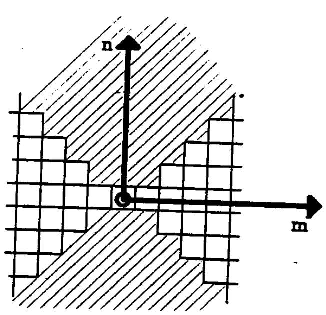

Figure 2 - Bowtie Functions - Kernel shape of TFDs satisfying P_1 through P_4

Table 1 lists B for the most common TFDs. Each B is plotted in Figures 3A-3F.

The BF concept is useful because many qualitative characteristics of TFD behaviour can be inferred directly:

A) Energy Concentration: For the distribution to exhibit a high energy concentration about the IF law of a signal, the kernel function must be concentrated about the n=0 axis. The kernel function of the discrete WVD is only non-zero along this axis, so it will exhibit this property. The Choi-Williams Distribution will also give high concentration, for large values of the σ parameter.

B) Artifacts: The shape of the artifacts of TFDs is given by the cross-section of B(n,m) in the n direction (with a constant value of m). This occurs only when the ambiguity kernel function is a product kernel; that is when g(l,m) is a function of the product of l and m.

The discrete WVD, Born-Jordan-Cohen and Choi-Williams Distributions artifact shapes will all be given by the cross-section of the kernel function. The artifacts of the STFT, for which the kernel function $g(v,\tau)$ shown in (3) is not a product kernel, must be calculated differently [2]. The STFT artifacts are superimposed on the true spectral components and cause them to oscillate or jitter slightly; they are much smaller in magnitude than the true components.

A) and B) mean that there is a trade-off between energy concentration (mainlobe width) and artifact magnitude (see fig.3).

Time-Frequency Representation	B(n,m)
Discrete WVD	$\delta(n)$, $m \in [-(M-1)/2,(M-1)/2]$ 0 otherwise.
Smoothed WVD using a Rectangular window of odd length P	$\frac{1}{P}$, $n \in [-(P-1)/2,(P-1)/2]$ 0 otherwise.
Rihaczek-Margenau	$\frac{1}{2}[\delta(n+m)+\delta(n-m)]$
STFT using a Rectangular Window of odd length P.	$\frac{1}{P}$, $\lvert m+n\rvert <= (P-1)/2$ 0 otherwise.
Born-Jordan-Cohen	$\frac{1}{\lvert m\rvert+1}$, $\lvert m\rvert <= \lvert n\rvert$ 0 otherwise.
Choi- Williams (with parameter σ)	$\frac{\sqrt{\sigma/\pi}}{2m} e^{-\sigma n^2/4m^2}$

Table 1 - Time-lag kernels of some TFDs.

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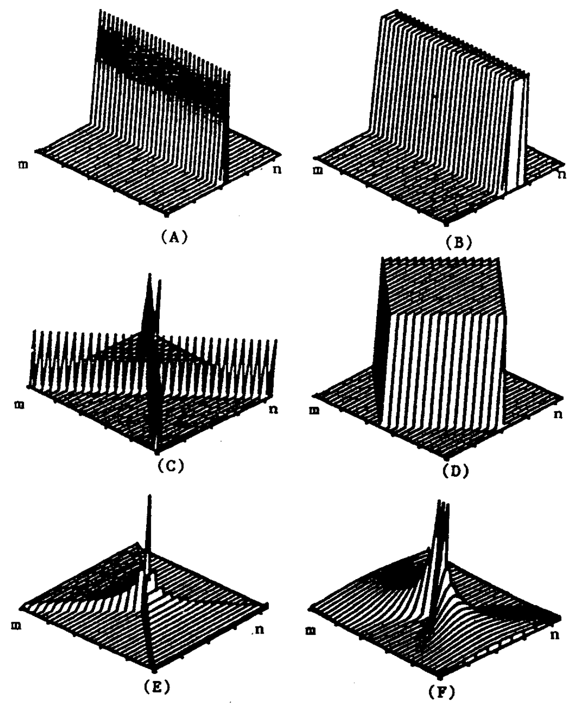

Figure 3 - B(n,m) for various TFDs
A= windowed discrete WVD; B= smoothed windowed discrete WVD; C=Rihaczek; D=STFT; E=Born-Jordan-Cohen; F= Choi-Williams

1.4 The choice of a TFD for t-f signal analysis

The shape of B allows the properties of a TFD to be predicted. The properties P_1-P_4 are verified if B is a Bowtie Function. Other properties, such as energy concentration and artifact shape (for product ambiguity kernel functions) may also be found from B.

The best TFD for a particular application must be selected according to the signal under analysis. The only discrete TFDs which satisfy all of P_1, P_2, P_3 and P_4 are the windowed discrete WVD and the Choi-Williams Distribution (for large σ). Although the Born-Jordan-Cohen Distribution gives the IF law exactly in the continuous case, the discrete Born-Jordan-Cohen Distribution does not satisfy P_4 exactly. The periodic first moment in frequency of the Born-Jordan-Cohen Distribution will give a smoothed (3-point MA filtered) version of the true IF law. In the case of a monocomponent signal, the WVD gives the greatest energy concentration about the frequency law. However, the Choi-Williams Distribution or Born-Jordan-Cohen Distribution is probably the better choice when analyzing multicomponent signals due to reduced artifacts.

All TFDs suffer from artifact terms. If the signals under analysis are multicomponent, then the time-frequency display becomes confused with artifacts appearing where no component exists or interfering with true spectral components. Another method of reducing artifacts is to use windowed versions of TFDs. The analysis window limits the extent to which a TFD is affected by far future and distant past occurences in the signal, and thus reduces or eliminates the artifacts which occur between components at different times.

Using this knowledge of the effect of the kernel shape on the behaviour of the TFD, it may be possible to design a kernel that will yield a TFD similar to the STFT, but with improved resolution.

(After completing this manuscript, we became aware of a similar approach by M. Amin for designing two-dimensional window functions used for smoothing the Autocorrelation function of random signals to produce a smoothed Wigner-Ville spectrum [8].)

2. INSTANTANEOUS FREQUENCY ESTIMATION

Intuitively and from (2), it is clear that the IF of a signal is only a function of its phase, and that its IB (spread about IF) should be a function of its envelope a(t). Motivated by the increasing use of time-frequency signal analysis as a tool for analyzing the time-varying spectral characteristics of signals, and in particular their IF, the performance of the IF estimate based on the direct definition was compared with the first moment of the discrete WVD as a function of signal to noise ratio (SNR) and the statistical behaviour of the WVD based definition which was studied in [7].

Many authors thought that by using the first moment of the WVD to estimate the IF, noise contributions would be smoothed out and this would result in improved IF estimation over the direct use of the definition. Analytical results and simulations have shown that in fact both methods are identical if we use the periodic first moment [5]. If we use the linear first moment, the discrete WVD based estimate exhibits more variance. However, the influence of noise on the WVD IF estimate will decrease if we smooth the WVD along the t-axis before taking the first moment, thus effectively producing a different TFD, or alternatively perform the integration over a selectively reduced range of frequencies (this uses the full potential of time-frequency signal analysis). Next we discuss the performance of IF estimators based on TFDs.

2.1. The Analytic Signal

The continuous time analytic signal, z(t), associated with the continuous time real signal, x(t), is given by

$$z(t) = \mathcal{A}[x(t)] = x(t) + j\mathcal{H}[x(t)] \qquad (16)$$

where $\mathcal{H}[\ ]$ is the continuous time Hilbert transform defined by

$$\mathcal{H}[x(t)] = \frac{1}{\pi}\ \text{p.v.}\left[\int_{-\infty}^{+\infty}\frac{x(t-\xi)}{\xi}d\xi\right]. \quad (17)$$

where p.v. denotes the Cauchy principle value.

The discrete time analytic signal, z(n), associated with the real discrete time signal, x(n), is given by

$$z(n) = \mathcal{A}[x(n)] = x(n) + j\mathcal{H}[x(n)] \quad (18)$$

where $\mathcal{H}[\ ]$ is the discrete time Hilbert Transform defined by

$$\mathcal{H}[x(n)] = \sum_{-\infty}^{+\infty} \frac{2x(n-m)}{m\pi}\ ,\ (m\ \text{odd})\ . \quad (19)$$

This definition of $\mathcal{H}[\ ]$ describes the ideal case, but the corresponding filter is not realizable. However, realizable approximations are easily obtainable.

2.2. *IF and the CFD Estimator*

Let $z(t) = \mathcal{Q}[x(t)]$ where x(t) is a continuous time real signal. Then the **continuous time instantaneous frequency** of x(t) is defined by the derivative of the phase of z(t).

$$f_i(t) = \lim_{\delta t \to 0} \frac{1}{4\pi\delta t}\ ((\ \arg[z(t+\delta t)] - \arg[z(t-\delta t)]\ ))_{2\pi}\ . \quad (20)$$

where the notation $((\ ))_{2\pi}$ represents a modulo 2π operation. The limiting and modulo operations are required because arg[z(t)] is only defined on $[-\pi,+\pi)$.

The concept of instantaneous frequency is extended to discrete time signals by using the central finite difference (CFD) of the phase of a discrete time analytic signal.

Let $z(n) = \mathcal{A}[x(n)]$ where x(n) is a discrete sequence formed by sampling the continuous time signal, x(t), at frequency f_s. Then the **discrete instantaneous frequency of x(n)** is defined by

$$f_i(n) = \frac{f_s}{4\pi}\ ((\ \arg[z(n+1)] - \arg[z(n-1)]\ ))_{2\pi} \quad (21)$$

If we have a noisy signal, $\hat{x}(n) = x(n) + \epsilon(n)$, where $\epsilon(n)$ is a random noise sequence corrupting x(n), we may estimate the IF of x(n) by using the **CFD estimator** defined by

$$\hat{f}_i(n) = \frac{f_s}{4\pi}\ ((\ \arg[\hat{z}(n+1)] - \arg[\hat{z}(n-1)]\ ))_{2\pi} \quad (22)$$

where $\hat{z}(n) = \mathcal{A}[\hat{x}(n)]$.

2.3 *IF estimation using the first moment of a TFD*

Let $\rho(n,k)$ be a discrete TFD from Cohen's class which may be represented by an N by M matrix. Then the **periodic first moment** of $\rho(n,k)$ with respect to frequency is defined by

$$i_1(n) = \frac{M}{2\pi}\ ((\ \arg\left[\sum_{k=0}^{M-1} e^{j2\pi k/M}\rho(n,k)\right]))_{2\pi} \quad (24)$$

In the special case where $\rho(n,k)$ is the discrete WVD, the normalized periodic first moment equals the CFD IF estimator,

$$\hat{f}_i(n) = \frac{f_s}{2M} i_1(n) \quad (25)$$

Some authors use a conventional linear definition of the first moment to form an estimator of IF. It can be shown that this estimator is biased and exhibits higher variance than the estimator based on the periodic first moment and in general, the equality in (25) no longer holds [5].

If $\rho(n,k)$ is a *general* member of Cohen's class, the normalized periodic first moment of (25) may yield an IF estimator with excessive bias [5].

2.4. *Relationship between Smoothed CFD and TFD IF Estimators*

In this section we describe IF estimators which are calculated by applying a time-averaging or smoothing window to the CFD estimator. Since the CFD estimator is periodic in value (it is modulo $f_s/2$) and a linear function of time, we cannot simply calculate the smoothed estimator by linear convolution with a smoothing function. Instead, we use modulo convolution defined as follows:

Let the IF sequence, $f(n)$, be of the form $f(n) = ((\ f(n)\ ))_A$ where f and $A \in \mathcal{R}$; if we convolve $f(n)$ with window h(n) of odd length $P = 2Q + 1$, $h \in \mathcal{R}$ then we must use the **modulo A convolution operation** defined by

$$f(n)\ ((\ *\ ))_A\ h(n) = \frac{A}{2\pi}((\ \arg\left[\sum_{p=-Q}^{Q} h(p)e^{j2\pi f(n-p)/A}\right]))_{2\pi}\ . \quad (26)$$

This definition ensures that values of $f(n)$ are averaged sensibly to reflect their periodic nature and is effectively the argument of a phasor sum.

Let f_i be the CFD IF estimator which is modulo $f_s/2$ and let h be a smoothing function of odd length P. Then the **smoothed CFD (SCFD) IF estimator** is defined by

$$\hat{f}_i^s(n) = \hat{f}_i(n)\ ((\ *\ ))_{f_s/2}\ h(n)\ . \quad (27)$$

We determine the functions, h(n), which make $\hat{f}_i^s$ correspond to an IF estimate calculated as the first moment of a TFD from Cohen's class as follows. The normalized periodic first moment of a TFD as given in (24) becomes:

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$$\hat{f}_i^c(n) = \frac{f_s}{4\pi}\left(\left(\arg\left[\sum_{p=-(N-1)/2}^{(N-1)/2} B(p,1)z(n-p+1)z^*(n-p-1)\right]\right)\right)_{2\pi} \quad (28)$$

When we substitute

$$|z(n-p+1)z^*(n-p-1)|e^{j4\pi\hat{f}_i(n-p)/f_s}.$$

for $z(n-p+1)z^*(n-p-1)$ in (28) we find that, apart from the $|z(n-p+1)z^*(n-p-1)|$ multiplier term, (28) is of the form of a modulo convolution. If we assume high SNR and constant signal amplitude within the window defined by $B(n,1)$, then this multiplier term is constant and we may write

$$\hat{f}_i^c(n) \tilde{=} \hat{f}_i(n) \; ((\;*\;))_{f_s/2} \; B(n,1) \; . \quad (29)$$

Thus the IF estimate calculated as the first moment of any TFD is almost exactly equivalent to a smoothed CFD IF estimator with window function, h, equal to $B(n,m)$ evaluated at $m = 1$. In fact, the smoothed CFD estimator will always have a dispersion parameter which is slightly less than the corresponding estimator calculated as the first moment of a TFD due to variations in $|z(n-p+1)z^*(n-p-1)|$ from noise.

Since $h(n) = \delta(n)$ for the discrete WVD, the IF estimator derived from the periodic first moment of the WVD is identical to the unsmoothed CFD estimator as seen previously. The analysis of the distribution of $\hat{f}_i^s$ is given in [5].

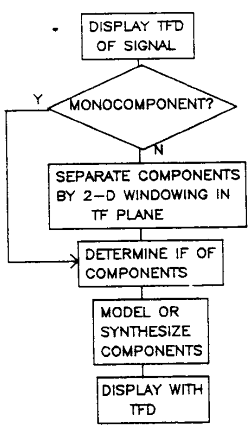

Figure 4. Diagram of the TFD Analysis System

3. CONCLUSION

The previous sections have described the techniques used for separation of the signal into components modelled by $s_i(t)$ as in (2), where the phase is estimated via the instantaneous frequency (IF) and the envelope via the instantaneous bandwidth (IB). Analytic expressions could then be given, using polynomial fitting techniques for the phase and gaussian expansions for the envelope. Results will appear elsewhere.

A comprehensive procedure for time-frequency signal analysis based on TFDs should incorporate TFD properties selection, component separation, IF and IB estimation, TFD inversion (see fig.4) and must be sufficiently general to be applied in all situations where analysts are dealing with time-varying signals.

From the results presented in this paper, it is concluded that there is perhaps no best universal TFD: the best we can do, for a particular class of signals, is to select a TFD which will be optimal for that class. The most useful TFDs are the STFT and the WVD because the STFT has negligible artifacts and the WVD gives the least blurred time-frequency representation in the case of a monocomponent signal.

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REFERENCES

[1] Boashash, B. (1988) "Time-Frequency Signal Analysis and Synthesis - The Choice of a Method and its Application", *Proc. SPIE Conf. on Advanced Algorithms and Architectures for Sig. Proc.*, vol.975, San Diego.

[2] Cohen, L. (1988) "Time-Frequency Distributions and Instantaneous Frequency", *Proc. SPIE Conf. on Advanced Algorithms and Architectures for Sig. Proc.*, vol.975, San Diego.

[3] Choi, H.I. & Williams, W.J. (1989) "Improved Time-Frequency Representation of Multicomponent Signals Using Exponential Kernels", *IEEE trans. on ASSP*, to appear.

[4a] Boashash, B., (1988) "Note on the use of the Wigner Distribution for Time-Frequency Signal Analysis", *IEEE trans. on ASSP*, vol.36, pp.1518-1521, Sept.

[4b] Boashash, B. amd P. Black, (1987) "Efficient Real-Time Implementation of the Wigner-Ville Distribution", *IEEE trans. on ASSP*, vol.35, pp.1611-1618, Nov.

[5] Lovell, B.C., *PhD thesis*, University of Queensland, to appear.

[6] Kootsookos, P.J., *PhD thesis*, University of Queensland, to appear.

[7] White, L.B. (1989), "Some Aspects Of Time-Frequency Analysis Of Non-Stationary Random Signals", *Ph.D thesis*, University of Queensland

[8] Amin, M., "Smoothing and Recursion in time-varying spectrum estimation", preprint, submitted to IEEE Trans. on ASSP