

# The Statistical Performance of Some Instantaneous Frequency Estimators

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**Abstract**—We examine the class of smoothed central finite difference (SCFD) instantaneous frequency (IF) estimators which are based on finite differencing of the phase of the analytic signal. These estimators are of particular interest since they are closely related to IF estimation via the (periodic) first moment, with respect to frequency, of discrete time-frequency representations (TFR's) in Cohen's class (TFR moment IF estimators). Cohen's class includes representations such as the spectrogram and Wigner-Ville distribution. Indeed, in the case of a monocomponent signal, the variance of a TFR moment IF estimator is bounded from below by the variance of the corresponding SCFD estimator. We determine the distribution of this class of estimators and establish a framework which allows the comparison of several other estimators such as the zero-crossing estimator and a recently proposed estimator based on linear regression on the signal phase.

We find the regression IF estimator is biased and exhibits a large threshold for much of the frequency range because it does not account for the circular nature of discrete-time frequency estimates. By replacing the linear convolution operation in the regression estimator with the appropriate convolution operation for circular data we obtain the parabolic SCFD (PSCFD) estimator. This estimator is unbiased and has a frequency independent variance and yet still retains the optimal performance and simplicity of the original estimator. The PSCFD estimator would be suitable for use as a real-time line or bearing tracker.

In this paper, we propose a number of mathematical operations suitable for circular data which should be used in preference to the conventional linear operations.

## I. INTRODUCTION

ALTHOUGH there are many ways to estimate the instantaneous frequency (IF) of a frequency modulated tone in noise, we will examine a particular class which we call the smoothed central finite difference (SCFD) estimators [4], [5], [21], [22] which are based on finite differencing of the signal phase. Interest in this class arose while we were investigating IF estimation via the first moment, with respect to frequency, of discrete time-frequency representations (TFR's) in Cohen's class [7] (TFR moment IF estimators). Cohen's class includes representations such as the popular spectrogram, or magnitude-squared short-time Fourier transform, and the Wigner-Ville distribution.

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The unsmoothed central finite difference (CFD) IF estimator is obtained from the central finite difference of the phase of the analytic signal; it is analogous to the continuous-time IF estimate obtained from the derivative of the phase of the analytical signal. SCFD IF estimators are calculated by appropriately convolving CFD estimators with a smoothing window function. In the case of a monocomponent signal, it was shown that each TFR moment IF estimator derived from a member of Cohen's general class corresponds to a particular SCFD estimator and that the corresponding SCFD estimator will always be more efficient and easier to calculate. Hence, there is no point in using TFR moment IF estimators when there is a simpler and better alternative.

In this paper, we derive approximate expressions for the statistical performance of the CFD and SCFD IF estimators and examine the behavior of SCFD estimators using various smoothing window functions. We show that several estimators proposed by other researchers can be considered as special cases of the SCFD class and we discuss the parabolic SCFD (PSCFD) estimator which is optimally efficient since it meets the Cramér-Rao lower bound for moderate signal-to-noise ratios (SNR's). The PSCFD is closely related to an estimator based on linear regression on the signal phase which was recently proposed by Kay [12], [15]. Although the PSCFD estimator is asymptotically optimal, it is computationally far simpler than the maximum likelihood estimator and may be adapted for use as a real-time line or bearing tracker.

## II. DISCRETE-TIME DEFINITIONS

For the reader's convenience, some of the fundamental definitions and conditions from [21] and [19] are restated below.

Let the discrete-time signal  $x$  be formed by sampling the continuous-time real signal  $\mathbf{x}$  at frequency  $f_s = 1/T_s$ . Thus

$$x(n) = \mathbf{x}(nT_s).$$

**Definition 1: Discrete-Time Analytic Signal:** The discrete-time analytic signal  $z$  associated with the real discrete-time signal  $x$  is defined by

$$\begin{aligned} z &= A[x] \\ &= x + jH[x] \end{aligned} \quad (1)$$

where  $A[\ ]$  is the linear operator which forms the analytic

signal and  $H[\cdot]$  is the discrete-time Hilbert transform defined by

$$H[x](n) = \sum_{\substack{m=-\infty \\ m \text{ odd}}}^{+\infty} \frac{2x(n-m)}{m\pi}. \quad (2)$$

Assuming that equivalent approximations to the Hilbert transform are made in both discrete and continuous time domains, we can equate the discrete-time and continuous-time analytic signals at the sample points, i.e.,

$$z(n) = \mathbf{z}(nT_s).$$

**Definition 2: CFD IF Estimator:** Let  $z = A[x]$  where  $x$  is a real discrete-time signal. Then the IF of  $x$  at sample  $n$  is estimated by

$$\hat{f}_i^c(n) = \frac{f_s}{4\pi} ((\arg[z(n+1)] - \arg[z(n-1)])_{2\pi}) \quad (3)$$

where  $((\cdot))_{2\pi}$  represents reduction modulo  $2\pi$  onto the domain  $[0, 2\pi)$ .

The CFD estimator will exhibit negligible bias if the discrete-time signal meets the following bandwidth conditions.

**Definition 3: Discrete-Time Bandwidth Conditions:** Let the continuous-time signal  $\mathbf{x}$  be an FM signal with instantaneous frequency  $\mathbf{f}_i$  given by

$$\mathbf{f}_i(t) = f_c + f_\Delta s(t)$$

where  $f_c$  is the carrier frequency,  $f_\Delta$  is the maximum frequency deviation, and  $s(t)$  is the normalized (unity peak amplitude) zero-mean modulating signal with bandwidth  $W$ . Let the deviation ratio  $D = f_\Delta/W$  and let the discrete-time signal  $x$  be obtained by sampling  $\mathbf{x}$  at frequency  $f_s$ . Then the CFD IF estimator will be a negligibly biased estimator of  $\mathbf{f}_i$  at the sample points if the following conditions are met:

$$1) f_c - (D+1)W > 0$$

$$2) f_c + (D+1)W < \frac{f_s}{2}$$

$$3) f_c \gg W.$$

To form a smoothed CFD (SCFD) IF estimator, it does not make sense to simply apply a low-pass filter function using linear convolution; the modulo- $\lambda$  convolution operation should be used.

**Definition 4: Modulo- $\lambda$  Convolution:** Let the sequence  $\tilde{f}$  be of the form  $\tilde{f}(n) = ((f(n)))_\lambda$ ,  $f: \mathbb{Z} \mapsto \mathbb{R}$  and  $\lambda \in \mathbb{R}$ . If we convolve  $\tilde{f}$  with a smoothing function  $h$  of odd length  $P = 2Q + 1$ ,  $h: \mathbb{Z} \mapsto \mathbb{R}$ , then we must use the modulo- $\lambda$  convolution operation defined by

$$\begin{aligned} & \tilde{f}(n) ((*)_\lambda h(n) \\ &= \frac{\lambda}{2\pi} \left( \left( \arg \left[ \sum_{p=-Q}^Q h(n-p) \right. \right. \right. \\ & \quad \cdot \left. \left. \exp[j2\pi\tilde{f}(p)/\lambda] \right] \right)_{2\pi} \right). \end{aligned} \quad (4)$$

Thus the class of SCFD IF estimators is obtained by applying a smoothing window to the raw CFD estimates using the modulo- $\lambda$  convolution.

**Definition 5: SCFD IF Estimator:** Let  $\hat{f}_i^c$  be the CFD IF estimator calculated from the real signal  $x$  and let  $h: \mathbb{Z} \mapsto \mathbb{R}$ , be a smoothing function of length  $P$ , presumed odd. Then the SCFD IF estimator is defined by

$$\hat{f}_i^s(n) = h(n) ((*)_{f_s/2} \hat{f}_i^c(n)), \quad \forall n. \quad (5)$$

### III. STATISTICAL ANALYSIS OF THE CFD IF ESTIMATOR

#### A. Problem Statement and Assumptions

We now consider IF estimators of a monocomponent noise-corrupted signal of the form

$$x(n) = a_c(n) \cos \phi(n) + \epsilon(n) \quad (6)$$

where  $a_c \in \mathbb{R}$  is the envelope function of the amplitude of the signal,  $\phi \in \mathbb{R}$  is the cumulative phase of the signal and  $\epsilon \in \mathbb{R}$  is a zero-mean white Gaussian noise sequence. The envelope may vary with time but it is always positive and the cumulative phase  $\phi$  is a monotonically increasing function of time. At sample  $n$ , the instantaneous signal-to-noise ratio (SNR) is given by  $s(n) = a_c^2(n)/2\sigma_\epsilon^2$  where  $\sigma_\epsilon^2$  is the variance of the noise sequence.

To simplify the analysis, we assume that both  $a_c$  and  $s$  are constant in the vicinity of the sample where we wish to estimate IF and that the SNR is moderate ( $s \geq 5$ ). These assumptions allow us to use a simple approximation to the true distribution.

Sums and differences of angular quantities are always performed modulo  $2\pi$ . Angular quantities are assumed to have domain  $[0, 2\pi)$  or  $[-\pi, +\pi)$  as indicated in the derivation; the latter domain is more convenient for expressing random angular variables with zero mean.

We assume that the monocomponent continuous-time signal  $\mathbf{x}$  satisfies the bandwidth conditions of the definition so that the CFD IF estimator  $\hat{f}_i^c$  derived from the discrete-time signal  $x$  equals the true IF  $\mathbf{f}_i$  of the continuous-time signal  $\mathbf{x}$  at the sample points. So

$$\hat{f}_i^c(n) = f_i(n) = \mathbf{f}_i(nT_s)$$

where  $T_s = 1/f_s$  is the sampling interval.

Using (3) we obtain

$$\hat{f}_i(n) = \frac{f_s}{4\pi} ((\phi(n+1) - \phi(n-1)))_{2\pi} \quad (7)$$

where  $\phi(n) = \arg[z(n)] \in [0, 2\pi)$  and  $z(n) = A[x](n)$ . Let  $\hat{z}(n) = A[\hat{x}](n)$  where  $\hat{x}$  is given by  $\hat{x}(n) = x(n) + \epsilon(n)$  and  $\epsilon$  is a zero-mean white Gaussian noise sequence of variance  $\sigma_\epsilon^2$  corrupting  $x$ . The  $\hat{\cdot}$  symbol indicates estimators which are calculated from noise-corrupted data. An unbiased estimate of  $f_i$  may be calculated from  $\hat{z}$  using the CFD estimator. Thus

$$\hat{f}_i(n) = \frac{f_s}{4\pi} ((\hat{\phi}(n+1) - \hat{\phi}(n-1)))_{2\pi} \quad (8)$$

where  $\hat{\phi}(n) = \arg[\hat{z}(n)] \in [0, 2\pi)$ .

We determine the distribution of  $\hat{f}_i$  in four steps.

1. Determine the distribution of the phase error  $\Theta(n) = ((\hat{\phi}(n) - \phi(n)))_{2\pi}$ .
2. Prove that the phase errors of  $\hat{\phi}(n+1)$  and  $\hat{\phi}(n-1)$  are uncorrelated.
3. Determine the distribution of the phase difference  $\hat{\beta}(n) = \hat{\phi}(n+1) - \hat{\phi}(n-1)$ .
4. Determine the distribution of the CFD IF estimator  $\hat{f}_i$ .

### B. Step 1: Determining the Distribution of the Phase Error $\theta$

We can express  $x$  as  $x(n) = a_c \cos \phi(n)$  where  $a_c$  represents the signal amplitude in the vicinity of the sample where we wish to estimate IF. Since we assume that  $x$  meets the bandwidth conditions, we can express  $z$  as a phasor given by

$$z(n) = A[x](n) = a_c e^{j\phi(n)}. \quad (9)$$

Hence

$$\begin{aligned} \hat{z}(n) &= A[\hat{x}](n) \\ &= A[a_c \cos \phi](n) + A[\epsilon](n) \\ &= a_c e^{j\hat{\phi}(n)} + z_\epsilon(n) \end{aligned} \quad (10)$$

where

$$z_\epsilon(n) = A[\epsilon](n) \quad (11)$$

is an analytic noise sequence.<sup>1</sup> It is useful to express the analytic noise in terms of the in-phase and quadrature components. An equivalent model of the noise in the vicinity of sample  $n_0$  is given by

$$z_\epsilon(n) = e^{j\phi(n_0)} z_\epsilon^i(n) + e^{j\phi(n_0)} j z_\epsilon^q(n). \quad (12)$$

Since  $z_\epsilon$  is analytic

$$\begin{aligned} z_\epsilon^q(n) &= H[z_\epsilon^i(n)] \\ &= \sum_{k \text{ odd}}^{\infty} \frac{2z_\epsilon^i(n-k)}{k\pi}. \end{aligned} \quad (13)$$

Now  $z_\epsilon^i$  is a zero-mean white Gaussian noise sequence with variance  $\sigma_\epsilon^2$  representing the in-phase component of the noise at sample  $n_0$ . So the quadrature component  $z_\epsilon^q$  is also white and identically distributed because  $z_\epsilon^q = H[z_\epsilon^i]$  and  $H[\cdot]$  is an all-pass filter. Thus  $\hat{z}$  can be expressed as the sum of the analytic deterministic signal  $z$  and the in-phase and quadrature components of the analytic noise as shown in Fig. 1.

1) *Linear and Circular Quantities:* A linear quantity has domain  $(-\infty, +\infty)$  and can be visualized by a point on a line; angular quantities are restricted to the domain  $[0, 2\pi)$  and can be visualized by points on a circle. We will use the terms circular, periodic or modulo to describe quantities which possess this angular nature. In digital

<sup>1</sup>Although  $\epsilon$  is a white noise sequence, the analytic noise is colored because the negative frequency components are zero.

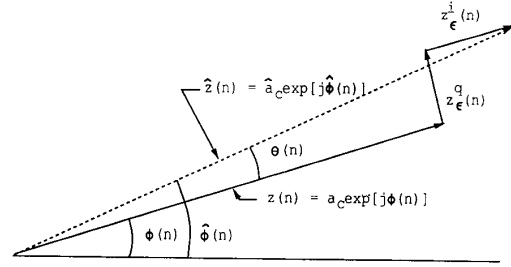


Fig. 1. A phasor diagram of the signal and noise.

signal processing, circular quantities are very common since they arise whenever we perform modulo reduction of linear quantities or a scaling of angular quantities. Indeed, discrete-time frequency estimates are always inherently circular.

2) *Phase Error Analysis Using Distributions on the Circle:* We determine the distribution of the phase error using distributions on the circle and show that the wrapped normal distribution is actually a very good approximation to the true distribution for moderate SNR's.

3) *The Exact Distribution:* In the case of a phasor of magnitude  $a_c$  in complex noise of the form  $\epsilon_i + j\epsilon_q$  where  $\epsilon_i$  and  $\epsilon_q$  are both zero-mean Gaussian random variables of variance  $\sigma^2$ , Blachman [2] and Bennett [1] have shown that the pdf of the phase  $\Theta$  is given by

$$\begin{aligned} p_\Theta(\theta) &= \frac{\exp(-a_c^2/2\sigma^2)}{2\pi} + \frac{a_c \cos \theta}{\sigma\sqrt{2\pi}} \operatorname{erfc} \left[ -\frac{a_c \cos \theta}{\sigma} \right] \\ &\quad \cdot \exp(-a_c^2 \sin^2 \theta / 2\sigma^2) \end{aligned} \quad (14)$$

where the complementary error function is defined by

$$\operatorname{erfc}(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-y^2/2} dy.$$

Equation (14) is the exact pdf of the phase error which we require. Due to the complex nature of this expression, we decided to use an approximation to the true distribution which greatly simplifies later calculations.

4) *Relationship with the Wrapped Normal Distribution:* The wrapped normal distribution can be visualized as a normal distribution which has been wrapped around a circle of unit radius.

*Definition 6:* If the random variable  $X$  is distributed as  $N(0, \sigma)$ . Then  $\Theta = ((X))_{2\pi}$  is wrapped normal  $\tilde{N}(0, \sigma)$  with pdf

$$f_\Theta(\theta) = \frac{1}{\sigma\sqrt{2\pi}} \sum_{k=-\infty}^{+\infty} \exp \left[ -\frac{(\theta + 2\pi k)^2}{\sigma^2} \right] \quad (15)$$

where  $0 < \theta \leq 2\pi$ .

This distribution arises in the study of diffusion processes and, in particular, Brownian motion on the circle; for this reason Stephens [30] and Watson [33] call it the Brownian distribution. Like the normal distribution on the line, the wrapped normal distribution possesses the reproductive property since the sum (or difference), reduced

modulo  $2\pi$ , of two independent wrapped normal angular variables with dispersion parameters  $\sigma_1$  and  $\sigma_2$  yields a wrapped normal variable with dispersion parameter  $\sigma_3$  given by

$$\sigma_3^2 = \sigma_1^2 + \sigma_2^2. \quad (16)$$

Note that for circular random variables such as angles, the sum of the random variables must be reduced modulo  $2\pi$ ; the pdf of this sum is found by circular convolution of the pdf's of the individual random variables.

5) *Wrapped Normal Approximation to the Phase Error Distribution*: The true distribution of the phase error can be closely approximated by the wrapped normal distribution for moderate SNR. That is, for  $s \geq 5$

$$\Theta \sim \tilde{N}(0, \sigma_\Theta) \quad (17)$$

where  $\sigma_\Theta = 1/\sqrt{2s}$ . Figs. 2 and 3 compare the true pdf of  $\Theta$  with the corresponding wrapped normal approximation for  $s = 5$  and  $s = 10$  to indicate the level of the approximation.

### C. Step 2: Proof that the Phase Errors are Uncorrelated

We show that the phase errors of  $\hat{\phi}(n+1)$  and  $\hat{\phi}(n-1)$  are uncorrelated by examining the autocovariance sequence of analytic wide-sense stationary noise.

#### 1) The Autocovariance Sequence of Analytic Noise:

*Lemma 1: Let  $\epsilon$  be a wide-sense stationary noise sequence and let  $z_\epsilon(n) = A[\epsilon](n)$ . Then the autocovariance sequence of  $z_\epsilon$  is given by*

$$R_{z_\epsilon z_\epsilon}(m) = 2A[R_{\epsilon\epsilon}](m) \quad (18)$$

where  $R_{\epsilon\epsilon}$  is the autocovariance sequence of  $\epsilon$ .

*Proof:* Follows from linear system theory [27, p. 272].  $\square$

Hence, if  $\epsilon$  is a zero-mean white noise sequence of variance  $\sigma_\epsilon^2$ , the autocovariance sequence of the corresponding analytic noise is given by

$$R_{z_\epsilon z_\epsilon}(m) = 2\sigma_\epsilon^2[\delta(m) + j\xi(m)] \quad (19)$$

where

$$\xi(m) = \begin{cases} \frac{2}{\pi m} & \text{for } m \text{ odd} \\ 0 & \text{otherwise.} \end{cases}$$

Since  $R_{z_\epsilon z_\epsilon}(m)$  is zero for all even  $m \neq 0$ , any two samples  $z_\epsilon(n)$  and  $z_\epsilon(n+m)$  of discrete, analytic noise are uncorrelated for  $m$  even. A similar result was observed by Tufts and Jackson [11] and Kay [16] where they noted that the complex noise sequence comprising the even samples of a discrete-time, analytic noise sequence is white. Consequently, since

$$\hat{\phi}(n) = \arg[z(n) + z_\epsilon(n)]$$

it can be shown [19] that  $\hat{\phi}(n+1)$  and  $\hat{\phi}(n-1)$  are uncorrelated. Moreover, in the case of white Gaussian noise,  $\hat{\phi}(n+1)$  and  $\hat{\phi}(n-1)$  are also independent.

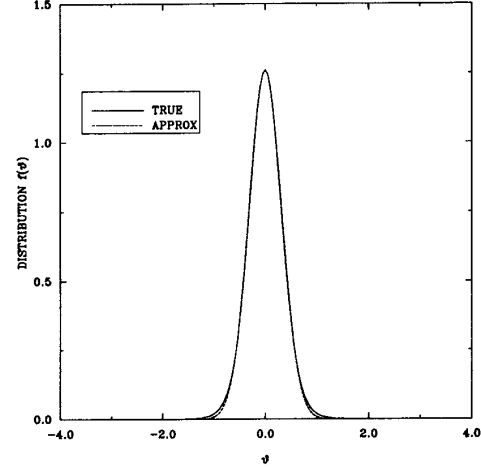


Fig. 2. True pdf of phase error  $\Theta$  and corresponding wrapped normal approximation for  $s = 5$ .

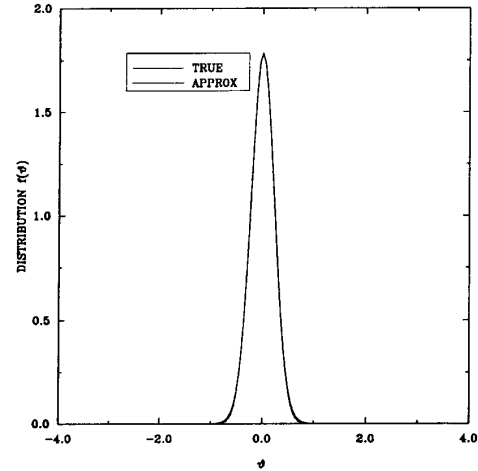


Fig. 3. True pdf of phase error  $\Theta$  and corresponding wrapped normal approximation for  $s = 10$ .

### D. Step 3: Determining the Approximate Distribution of the Phase Difference $\beta$

Let

$$\hat{\beta}(n) = ((\hat{\phi}(n+1) - \hat{\phi}(n-1)))_{2\pi}$$

where  $\hat{\beta} \in [0, 2\pi)$ . The angles  $\hat{\phi}(n+1)$  and  $\hat{\phi}(n-1)$  are uncorrelated with approximate distributions given by the relations

$$\hat{\phi}(n+1) \sim \tilde{N}(\phi(n+1), \sigma_\Theta)$$

and

$$\hat{\phi}(n-1) \sim \tilde{N}(\phi(n-1), \sigma_\Theta).$$

From the reproductive property of the wrapped normal distribution

$$\hat{\beta}(n) \sim \tilde{N}(\beta(n), \sigma_\beta) \quad (20)$$

where

$$\beta(n) = ((\phi(n+1) - \phi(n-1)))_{2\pi}$$

and

$$\sigma_\beta = \sigma_\Theta \sqrt{2} = 1/\sqrt{s} \quad (21)$$

from (16).

#### E. Step 4: Distribution of the CFD IF Estimator $\hat{f}_i$

1) *The Approximate Distribution:* We write the result in terms of the CFD-IF estimator itself to obtain the following theorem:

*Theorem 1: Approximate distribution of  $\hat{f}_i$ :* The CFD IF estimator  $\hat{f}_i$  of a monocomponent, noise-corrupted signal of the form

$$x(n) = a_c \cos \phi(n) + \epsilon(n)$$

where  $\epsilon$  is a zero-mean white Gaussian noise sequence of variance  $\sigma_\epsilon^2$ , is approximately distributed as wrapped normal  $\tilde{N}(f_i(n), \sigma_c)$  where

$$f_i(n) = \frac{f_s}{4\pi} ((\arg [z(n+1)] - \arg [z(n-1)]))_{2\pi}$$

and

$$\sigma_c = f_s/4\pi\sqrt{s} \quad (22)$$

where  $f_s$  is the sampling frequency and  $s = a_c^2/2\sigma_\epsilon^2$  is the SNR.

2) *The Exact Distribution:* An exact, although somewhat intractable, expression for the distribution of the CFD IF estimator may be obtained by circularly convolving the exact phase error distribution of (14) with itself and then scaling the result to convert from the angular domain to the frequency domain.

*Theorem 2: Exact distribution of  $\hat{f}_i$ :* The exact pdf  $p_{\text{CFD}}(f)$  is given by

$$p_{\text{CFD}}(f) = p_B \left( \frac{\beta f_s}{4\pi} \right) \quad (23)$$

where

$$p_B(\beta) = p_\Theta(\theta) \circledast p_\Theta(\theta). \quad (24)$$

Here  $\circledast$  denotes circular convolution and  $p_\Theta(\theta)$  is obtained from (14).

#### F. Validation of These Results

To demonstrate the validity of the wrapped normal approximations to the phase errors, we calculated the exact distribution of the CFD IF estimator by circularly convolving the exact pdf given by (14) as described above. This convolution was performed by:

- 1) sampling the exact pdf at 128 points,
- 2) forming the 128 point discrete Fourier transform,
- 3) calculating the squared magnitude of the transform, and
- 4) finally taking the inverse discrete Fourier transform.

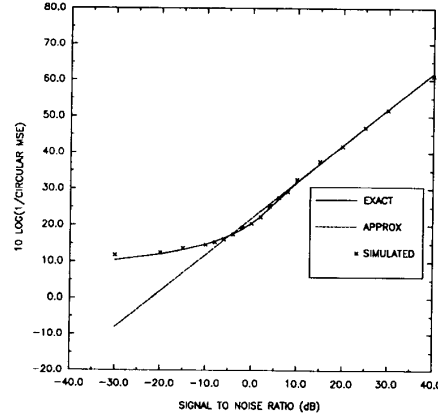


Fig. 4. Comparison of estimates of the CFD IF estimator dispersion parameter from simulations with the exact value from the theory, and the approximation of  $f_s/(4\pi\sqrt{s})$ . For the simulated results, 100 simulations were used above 10 dB, and 500 simulations at lower signal-to-noise ratios. The dispersion of all estimates was based on the circular mean square error of (36).

This series of operations gave us a sampled description of the exact pdf of the CFD IF estimator. From the exact pdf we calculated the standard deviation using the circular mean square error defined later in this paper as (36). This yielded the solid curve in Fig. 4. The dashed curve corresponds to the wrapped normal approximation and the crosses show the measured dispersion from simulations. This plot shows excellent agreement between the exact dispersion derived from the foregoing theory and the simulated results. There is also excellent agreement with the wrapped normal approximation at signal-to-noise ratios above 5 dB.

*Remarks:* It is interesting to note that the CFD IF estimator is identical to the TFR moment estimator calculated from the Wigner-Ville distribution [4], [21]. Later we will show that the CFD estimator is also asymptotically optimal since it meets the Cramér-Rao lower bound for moderate SNR.

#### IV. ANALYSIS OF SCFD IF ESTIMATORS

The class of SCFD estimators is closely related to several other estimators which have been proposed in the literature. In particular, they have a close relationship with TFR moment IF estimators and estimators based on linear regression of the signal phase. They can be used with fixed window length to perform IF estimation at low computational cost and are particularly attractive for computationally efficient adaptive IF estimation algorithms [21].

The analysis of these estimators from [19] is included as the Appendix. It assumes that the smoothing window function is nonnegative, the SNR is constant and moderate ( $s \geq 5$ ), and the IF variation within the window is bounded. The analysis yields the following theorem [21]:

*Theorem 3: Distribution of  $\hat{f}_i^s$ :*

Let  $\hat{f}_i^s$  be the SCFD IF estimator calculated from the CFD estimator  $\hat{f}_i$  using the modulo convolution operator

with a nonnegative smoothing window  $h$  of odd length  $P = 2Q + 1$ . Then  $\hat{f}_i(n)$  is distributed approximately as wrapped normal  $\tilde{N}(f_i(n), \sigma_c)$  and  $\hat{f}_i^s(n)$  is distributed approximately as  $\tilde{N}(f_i^s(n), \sigma_s)$  where

$$f_i^s(n) = f_i(n) ((*)_{f_s/2} h(n) \quad (25)$$

and

$$\sigma_s^2 = \frac{\mathbf{h}^T \mathbf{p} \mathbf{h}}{(\mathbf{h}^T \mathbf{1})^2} \sigma_c^2 \quad (26)$$

and  $\sigma_c$  is the dispersion of  $\hat{f}_i^c$  given by  $\sigma_c = f_s / 4\pi\sqrt{s}$ ; the  $P$ -dimensional smoothing function vector and unit vector are defined by

$$\mathbf{h} = [h_Q \cdots h_{-Q}]^T \quad (27)$$

and

$$\mathbf{1} = [1 \cdots 1]^T \quad (28)$$

respectively; the  $P \times P$  dimensional correlation coefficient matrix  $\mathbf{p}$  has elements

$$\rho_{ij} = \begin{cases} 1 & \text{for } m = 0 \\ -\frac{\sin[(m-2)\beta^s(n)/2]}{(m-2)\pi} + \frac{2\sin[m\beta^s(n)/2]}{m\pi} & \text{for } m \text{ odd} \\ -\frac{\sin[(m+2)\beta^s(n)/2]}{(m+2)\pi} & \text{for } m = 2 \\ -\frac{1}{2} & \text{otherwise} \\ 0 & \text{otherwise} \end{cases}$$

where  $m = |i - j|$  and  $\beta^s(n) = 4\pi f_i^s(n)/f_s$ .

Fig. 5 compares the dispersion improvement ratio,  $\sigma_c/\sigma_s$ , obtained from simulations with the predicted value from (26) for the case of an SCFD IF estimator with a 5 point rectangular smoothing window. This figure confirms (26) and clearly shows the frequency dependence of this SCFD estimator which is caused by dependencies between phase estimates introduced by the Hilbert transform used to form the analytic signal. The dispersion of the SCFD estimator with a 5-point rectangular smoothing window is up to 3.5 times lower than the unsmoothed CFD estimator.

## V. STATISTICAL PERFORMANCE OF PARTICULAR SCFD ESTIMATORS

### A. SCFD Estimators Corresponding to TFR's

We have already stated that the dispersion parameter for the TFR moment IF estimator calculated with the Wigner-Ville distribution is  $\sigma_c$  from (22). Figs. 6 and 7 show the predicted dispersion improvement ratios for the

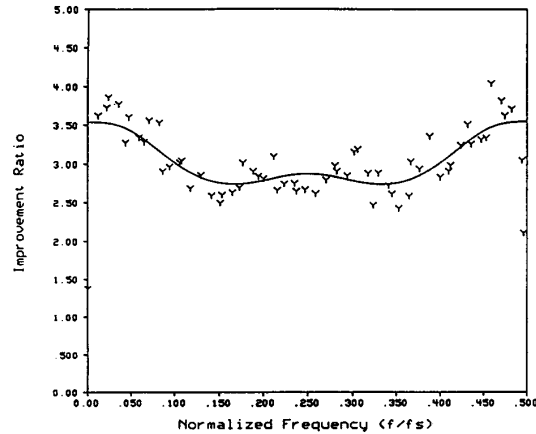


Fig. 5. Comparison of measured and calculated values of dispersion improvement,  $\sigma_c/\sigma_s$ , against frequency for an SCFD IF estimator using a 5 point rectangular smoothing window. The simulations were performed by calculating the SCFD estimator 100 times at each of the 64 different frequencies with a SNR of 20 dB.

SCFD estimators corresponding to the spectrogram using a rectangular data window and the Choi-Williams exponential [6] TFR.

In the case of the Margenau-Hill-Rihaczek (MHR) [29] TFR, the corresponding SCFD has a  $P = 3$  point smoothing window function  $h$  of the form:  $h(-1) = 1$ ,  $h(0) = 0$  and  $h(1) = 1$ . This estimator has a dispersion improvement ratio of 2 for all frequencies. In the next section we will see why the dispersion is frequency independent in this case.

### B. Estimators with Frequency Independent Dispersion

Consider SCFD estimators which use a *comb* smoothing function  $h$  which has  $h(n) = 0$  for all odd  $n$ . All of the frequency dependent terms in (26) are multiplied by 0 and the dispersion  $\sigma_s$  becomes frequency independent. Such estimators only require phase estimates from every second sample of the signal; these estimators effectively estimate the IF of the resampled signal comprising every second sample of the original analytic signal using a finite differencing operation between adjacent samples. Since Lemma 1 shows that the even (or odd) samples of analytic noise are uncorrelated, we see that  $\sigma_s$  is frequency independent because the resampled noise sequence is white.

In particular, consider an estimator with a rectangular comb smoothing window of the form

$$h(n) = \begin{cases} 1 & \text{for even } n \leq 2q \\ 0 & \text{otherwise} \end{cases} \quad (29)$$

where the window length  $P = 4q + 1$ ,  $q \in \mathbb{Z}^+$ . Using (26), the improvement ratio is given by

$$\frac{\sigma_c}{\sigma_s} = \frac{P + 1}{2}. \quad (30)$$

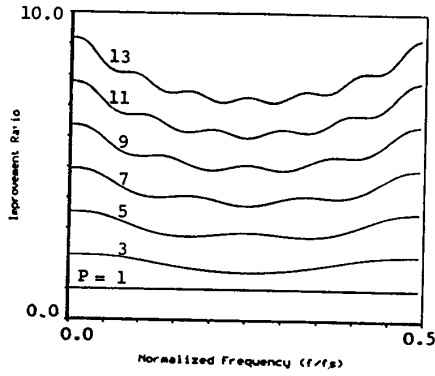


Fig. 6. Calculated values of dispersion improvement ratio,  $\sigma_c/\sigma_s$ , against frequency for an SCFD IF estimator using with rectangular smoothing windows of length  $P = 1, 3, 5, 7, 9, 11$ , and  $13$ . Each of these estimators corresponds to a TFR moment IF estimator derived from a spectrogram calculated with a rectangular data window of length  $P + 2$ .

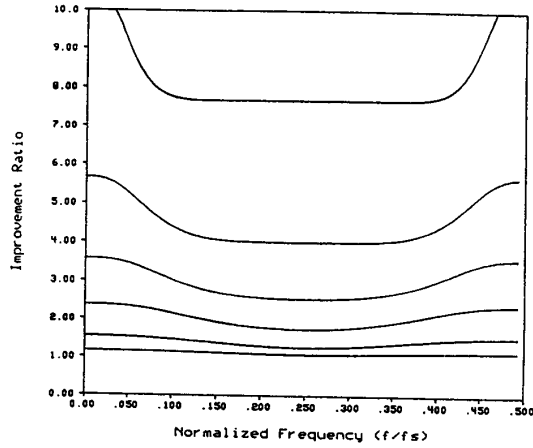


Fig. 7. Calculated values of dispersion improvement ratio,  $\sigma_c/\sigma_s$ , against frequency for an SCFD IF estimator using exponential smoothing windows of the form  $h(n) = \exp[-dn^2/4]\sqrt{d\pi}/2$  where the smoothing parameter  $d = 10, 5, 2, 1, 0.5$ , and  $0.2$ . These estimators correspond to TFR moment IF estimators derived from the Choi-Williams exponential TFR with smoothing parameter  $d$ .

In this case the dispersion is solely due to the phase estimation errors at the endpoints of the smoothing function i.e.,  $\Theta(n - 2q - 1)$  and  $\Theta(n + 2q + 1)$ ; all phase estimation errors at internal samples cancel. This estimator is similar to estimators based on zero crossing detection.

Compare this estimator to an SCFD estimator with a rectangular window of the form

$$h(n) = \begin{cases} 1 & \text{for } n \leq 2q \\ 0 & \text{otherwise.} \end{cases} \quad (31)$$

The dispersion improvement ratio of this class of SCFD estimators has already been presented in Fig. 6. As  $f_i$  approaches 0 Hz (or equivalently  $f_s/2$ ), the improvement ratio of estimators with rectangular windows approaches

a maximum value of

$$\lim_{f_i \rightarrow 0} \frac{\sigma_c}{\sigma_s} = \frac{P}{\sqrt{2}}. \quad (32)$$

A minimum value of

$$\min \left[ \frac{\sigma_c}{\sigma_s} \right] \approx \frac{P}{1.8} \quad (33)$$

is attained near  $f_i = f_s/4$  for large  $P$  (using numerical evaluation of (26)). These values seem to indicate that we can obtain slightly more efficient estimators in some sections of the frequency range by using window functions which use all samples of the original analytic signal rather than just the even or odd samples.

It is interesting to investigate this point further. An analytic signal is completely specified by its even (or odd) samples because it requires only half the sampling rate of the corresponding real signal to prevent aliasing. Thus the odd samples of an analytic signal can be recovered from the even by using the sampling theorem [27, p. 337]. At first glance it seems strange that an improved SCFD estimate can be obtained by using all samples instead of just the even samples, when the odd samples are linearly dependent on the even. Upon reflection we see that the odd samples near the ends of the window function are significantly dependent on even samples outside the window; their inclusion has a similar effect to slightly increasing the length of the comb smoothing window in a frequency dependent manner.

In IF estimation we minimize mean-square estimation error at each instant by ensuring that we choose the largest possible window consistent with the constraint that IF variation within the window is kept below a given level. Thus there is no point in investigating the performance of SCFD estimators which use every sample because they effectively violate the window length constraint by depending on samples outside the window. We therefore focus our attention on SCFD estimators with comb smoothing windows.

Now we must answer the questions, "Which comb window function will yield the most efficient SCFD estimator?" and "What is the most efficient estimator we can hope for?"

## VI. OPTIMAL IF ESTIMATION

### A. Cramer-Rao Lower Bound

An optimal (efficient) estimator of a quantity is unbiased, maximum likelihood, normally distributed and possesses the lowest possible variance for a given data set. Rife and Boorstyn [28] have determined the Cramér-Rao (CR) lower bound for the variance of any unbiased frequency estimator in the case of a complex sinusoid with unknown phase and amplitude in white, Gaussian noise. We have adapted their result so it can be applied to real signals.

**Theorem 4: Cramér-Rao (CR) Lower Bound for a Real Signal:** Let  $\hat{x}(n) = x(n) + \epsilon(n)$  where  $x$  is a real sinusoid of the form  $x(n) = a_c \cos [2\pi f_0 n]$  and  $\epsilon$  is a zero-mean white Gaussian noise sequence with variance  $\sigma_\epsilon^2$ . Then the CR lower bound on the variance of an unbiased estimator of the frequency  $f_0$  (using only the odd or even samples of the analytic signal) is given by

$$\text{var} [\hat{f}_o] \geq \frac{f_s^2}{(4\pi)^2} \frac{6}{sN_i(N_i^2 - 1)} \quad (34)$$

where  $N_i = (M + 1)/2$  and  $M$  is the number of samples in the data window, presumed odd. The signal-to-noise ratio is given by  $s = a_c^2/2\sigma_\epsilon^2$ .

The length  $M$  represents the length of the window used to form the periodogram in the case of the maximum likelihood estimator (see Section VI-C). For a SCFD estimator with a smoothing window of odd length  $P$  we use  $M = P + 2$ . We can produce estimators which have a variance which is slightly lower than (34) by using all samples of the analytic signal, but this is effectively increasing our window length  $M$  as shown previously. Theorem 4 can be used to give a lower bound on the variance of IF estimators if we assume that the windows are selected to ensure that IF is nearly constant within the window.

Thus attaining the Cramér-Rao lower bound is the best that we can hope to achieve with SCFD estimators. Now we wish to determine the comb window function which will yield the most efficient SCFD estimator. Fortunately, this window function was already discovered by other researchers performing parallel research which we will soon examine.

### B. Circular Sample Estimators

The above formula for the CR lower bound assumes that the frequency estimates can take any value on the real line, yet discrete-time frequency estimates must always lie in the interval  $[0, f_s/2)$ . Therefore, although (34) indicates that CR lower bound approaches infinity as the SNR approaches 0, the discrete-time frequency estimates will actually approach a uniform distribution over  $[0, f_s/2)$  with a (linear) mean of  $f_s/4$ , and a (linear) variance of

$$\begin{aligned} \text{var} [\hat{f}_0] &= E[\hat{f}_0^2] = \frac{2}{f_s} \int_{-f_s/4}^{+f_s/4} \hat{f}_o^2 d\hat{f}_o \\ &= \frac{f_s^2}{48} \end{aligned} \quad (35)$$

which is less than the CR lower bound! Indeed, the value of the (linear) mean is also quite unsatisfactory since intuitively there is no reason why we should have  $\hat{f}_0 = f_s/4$  at low SNR regardless of the signal.

These problems highlight the dangers of using linear operators on circular random variables. As SNR approaches 0, it is better say that the distribution of  $\hat{f}_0$  approaches a uniform circular distribution which has an undefined circular mean.

One method of resolving this difficulty would be to reformulate the CR bounds using circular random variables as suggested by Mardia [23, p. 118]. Instead of using this approach, we have found it convenient to linearize the circular variance to obtain a quantity which is equivalent to the linear variance at high SNR but avoids wrapping point ambiguities. The linearizing transformation assumes that the statistic is wrapped normally distributed [23, p. 74] and may be invalid if this condition no longer holds (e.g., at small SNR).

In practice we use the linearized circular mean-square error (CMSE) rather than the variance to highlight the bias inherent in some estimators. We have chosen the following definition:

**Definition 7: Circular mean-square error (CMSE):** Let  $\{\hat{f}(k)\}$  be a set of  $K$  discrete-time frequency estimates of a sinusoid with frequency  $f_0$  in noise sampled at  $f_s$  Hz. Then the sample estimator of the (linearized) CMSE is defined by

$$\begin{aligned} e_p^2 &= -\frac{f_s^2}{8\pi^2} \ln \left[ \frac{1}{K} \left| \sum_{k=0}^{K-1} \exp [j4\pi\hat{f}(k)/f_s] \right| \right] \\ &+ \left[ \frac{f_s}{4\pi} \left\{ \left( \left( 4\pi \frac{(\hat{f}_0 - f_0)}{f_s} + \pi \right) \right)_{2\pi} - \pi \right\} \right]^2 \end{aligned} \quad (36)$$

where  $\hat{f}_0$  is the circular sample mean given by

$$\hat{f}_0 = \frac{f_s}{4\pi} \left( \left( \arg \left[ \sum_{k=0}^{K-1} \exp (j4\pi\hat{f}/f_s) \right] \right)_{2\pi} \right) \quad (37)$$

The circular sample mean [21], [22], [23, p. 20] is the natural measure of location for a set of circular random variables such as discrete-time frequency estimates. It may be visualized as the argument of a unit phasor sum; the argument of each unit phasor would be proportional to the corresponding frequency estimate. The magnitude of the phasor sum divided by the number of phasors is used to derive a measure of circular sample variance.

For sample bias less than  $f_s/4$ , the CMSE increases with increasing bias just like the linear MSE. However, the bias can never exceed  $f_s/4$  because of the circular nature of these estimators. In general, the CMSE can be used to replace the linear MSE in the evaluation of estimators of circular quantities. The linear MSE will give about the same results as the CMSE provided the variance of the estimators is small and the origin is chosen to avoid circular wraparound.

It is interesting to note that Rife and Boorstyn [28] were aware of the difficulties associated with linear estimators of MSE and commented that the bias was greatest when  $f_0$  was near 0 or  $f_s/2$  Hz. For this reason they restricted their signal frequencies to be near  $f_s/4$ . The CMSE provides an elegant answer to this common problem.

### C. Maximum Likelihood Estimation

The optimal maximum likelihood (ML) estimator of a complex sinusoid in complex white Gaussian noise is given by the location of the peak of the periodogram [28].

A coarse estimate can be made directly from the DFT of the signal as was done by Palmer [26].

Frequency estimators generally suffer from threshold effects. There is usually an SNR value, called the threshold, below which the dispersion of the estimator rises very rapidly at the SNR decreases. This threshold generally decreases with increasing data window length  $M$  [28]. Although several estimators are optimal at high SNR, they all exhibit higher thresholds than the ML estimator for a given data set.

If the ML estimator is used to estimate IF at successive samples, we can replace the FFT calculation with a recursive DFT [10] to reduce the computational load. Nevertheless, in many instances, computation of the true ML estimator may be prohibitive.

#### D. The Kay Estimator

Kay [12], [15] recently proposed an estimator based on finite differencing of the phase of even or odd samples of the analytic signal; an operation which involves far less computation than the ML estimator and can be easily formulated for real-time IF tracking. This estimator is based upon linear regression on the phase as proposed by Tretter [31]. It was claimed that this phase regression estimator is unbiased and that its variance attains the CR lower bound for moderate SNR (albeit at a somewhat higher threshold than the ML estimator). We find that this is not the case for much of the frequency range because the estimator does not handle the circular nature of discrete-time frequency estimators correctly.

We have reformulated the Kay estimator using our notational conventions so that it resembles an SCFD estimator. Kay's estimator is given by

$$\begin{aligned}\hat{f}_i^k(n) &= \frac{f_s}{4\pi} \sum_{p=-Q}^Q h_p(p) \\ &\quad \cdot \arg [\hat{z}(n-p+1)\hat{z}^*(n-p-1)] \\ &= \sum_{p=-Q}^Q h_p(p) \hat{f}_i^c(n-p) \\ &= h_p(n) * \hat{f}_i^c(n)\end{aligned}\quad (38)$$

where

$$h_p(p) = \begin{cases} \frac{3N_i}{2(N_i^2-1)} \left(1 - \left[\frac{p}{N_i}\right]\right)^2 & \text{for even } p \leq Q \\ 0 & \text{otherwise} \end{cases} \quad (39)$$

is the comb smoothing window with length  $P = 2Q + 1$  containing  $N_i = (P + 3)/2$  independent samples (as in Theorem 4). The  $*$  operator denotes linear convolution and the CFD IF estimator of the noisy signal  $\hat{f}_i^c$  is given by

$$\hat{f}_i^c(n) = \frac{f_s}{4\pi} ((\arg [\hat{z}(n+1)] - \arg [\hat{z}(n-1)]))_{2\pi}. \quad (40)$$

Kay uses a parabolic window function  $h_p$  which minimizes the variance of the estimator according to his analysis. The parabolic shape arises because of the dependency between successive CFD estimators. Kay has compared the dispersion of this estimator with a rectangularly windowed estimator which he calls the unwindowed estimator given by

$$\hat{f}_i^u = \frac{f_s}{4\pi} \sum_{p=-Q}^Q h_r(p) \arg [z(n-p+1)z^*(n-p-1)] \quad (41)$$

where

$$h_r(n) = \begin{cases} \frac{1}{N_i-1} & \text{for even } n \leq Q \\ 0 & \text{otherwise.} \end{cases} \quad (42)$$

He has shown that the ratio of variances of the unwindowed estimator to the parabolically windowed Kay estimator approaches  $N_i/6$  for large  $N_i$ .

Contrary to Kay's claims we find that these estimators are biased and exhibit a high threshold when the IF approaches<sup>2</sup> 0 or  $f_s/2$  Hz. This effect occurs because some of the  $\hat{f}_i^c$  in the summation in (38) wrap around the circular domain so that estimates that should be near 0 Hz sometimes appear near  $f_s/2$  and greatly increase the variance of the sum. The poor performance of the estimator is a direct result of using the linear convolution operation on circular data.

#### E. The Parabolic SCFD (PSCFD) Estimator

These problems may be overcome by replacing the linear convolution operation in (38) with the modulo- $\lambda$  convolution operation from definition 4 to obtain a new estimator which we call the parabolic SCFD (PSCFD) estimator given by

$$\hat{f}_i^o(n) = h_p(p) ((*)_{f_s/2} \hat{f}_i^c(p). \quad (43)$$

It is easier to compare the two estimators when we expand (43) using (40) and the definition of the modulo- $\lambda$  convolution to obtain

$$\begin{aligned}\hat{f}_i^o(n) &= \frac{f_s}{4\pi} \left( \left( \arg \left[ \sum_{p=-Q}^Q h_p(p) \exp \{j \arg [\hat{z}(n-p \right. \right. \right. \\ &\quad \left. \left. \left. + 1)\hat{z}^*(n-p-1)]\} \right] \right) \right)_{2\pi}. \end{aligned} \quad (44)$$

Substituting (39) into (26) and performing some algebraic manipulation we find that the dispersion of  $\hat{f}_i^o$  for moderate SNR is given by

$$\sigma_o^2 = \frac{f_s^2}{(4\pi)^2} \frac{6}{sN_i(N_i^2-1)}. \quad (45)$$

Now (45) is just the CR lower bound given by Theorem 4. It follows that  $\sigma_o^2$  is the minimum value that the SCFD

<sup>2</sup>In [12] estimator bias was minimized near 0 Hz because the phases were expressed in  $[-\pi, +\pi)$  instead of  $[0, 2\pi)$ .

dispersion  $\sigma_s^2$  from (26) can attain using comb smoothing window functions. Although we can attain a lower variance by using arbitrary window functions, these estimators violate the window length constraint discussed in Section V-B. When  $N_i = 2$  the PSCFD estimator becomes the CFD estimator and so we see that the CFD estimator is also optimal at moderate SNR. Kay's unwindowed estimator corresponds to the SCFD estimator with a rectangular comb smoothing window which has a dispersion of

$$\sigma_r^2 = \frac{f_s^2}{(4\pi)^2} \frac{1}{s(N_i - 1)^2} \quad (46)$$

from (30). Dividing (46) by (45) we obtain

$$\frac{\sigma_r^2}{\sigma_o^2} = \frac{N_i(N_i + 1)}{6(N_i - 1)} \approx \frac{N_i}{6} \quad (47)$$

a result which is identical to Kay's. We see that parabolic windowing offers substantial dispersion improvement for large window lengths. As mentioned previously, the SCFD estimator with a rectangular comb smoothing window corresponds to a form of zero-crossing frequency estimator.

## VII. COMPARISON OF SIMULATION RESULTS

Since we get spurious large values for (linear) MSE when frequency estimates wrap around near 0 or  $f_s/2$  Hz, we compare these estimators using the CMSE of definition 7. Figs. 8 and 9 show the performance of the ML, PSCFD, and Kay estimator for normalized frequencies of  $f_o/f_s = 0.05$  and  $0.20$ , respectively. These figures show that the Kay estimator has a frequency dependent threshold. The Kay estimator is also biased towards a normalized frequency of  $0.25$  at low SNR. Both the PSCFD and ML estimators are unbiased and have a constant threshold.

### A. TFR Moment IF Estimators Revisited

Kay mentions two additional estimators which are equivalent to the Kay estimator (38) and the unwindowed estimator (41) at high SNR. These are, respectively,

$$\hat{f}_i^a = \frac{f_s}{4\pi} \arg \left( \left( \sum_{p=-Q}^Q h_p(p) z(n-p) + 1 \right) z^*(n-p-1) \right) \Bigg|_{2\pi} \quad (48)$$

and

$$\hat{f}_i^b = \frac{f_s}{4\pi} \arg \left( \left( \sum_{p=-Q}^Q h_r(p) z(n-p) + 1 \right) z^*(n-p-1) \right) \Bigg|_{2\pi}. \quad (49)$$

These estimators are formed by interchanging the operations of argument and summation. Lank *et al.* [17] pro-

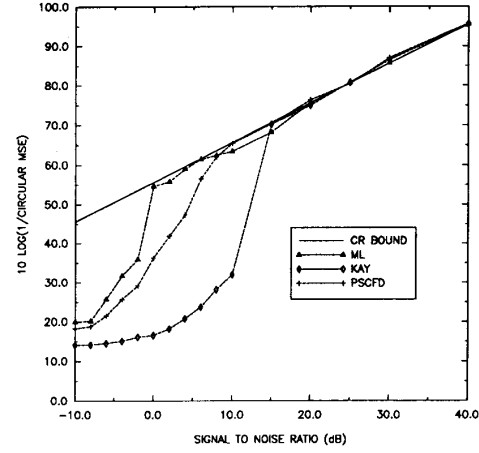


Fig. 8. Comparison of the dispersion of the ML, PSCFD, and Kay estimators against SNR for a normalized frequency of  $0.05$ . The window length  $M = 47$  for the ML estimator and the smoothing window length  $P = 45$  for the PSCFD and Kay estimators. The dispersions were calculated using the frequency estimates from 100 simulations at each SNR value.

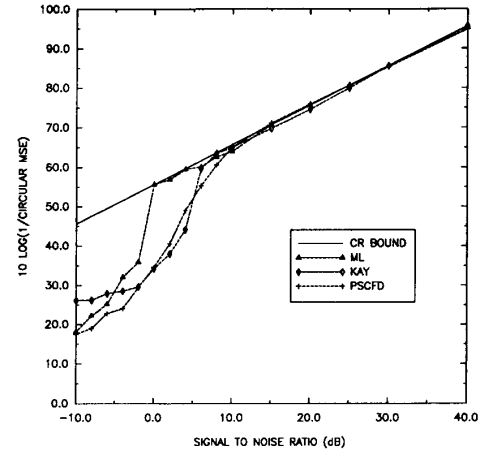


Fig. 9. Comparison of the dispersion of the ML, PSCFD, and Kay estimators against SNR for a normalized frequency of  $0.20$ . The window length  $M = 47$  for the ML estimator and the smoothing window length  $P = 45$  for the PSCFD and Kay estimators. The dispersions were calculated using the frequency estimates from 100 simulations at each SNR value.

posed the estimator  $\hat{f}_i^b$  of (49) and it was later studied by Jackson and Tufts [11] and Kay [14]. The variance of (49) is given in [17] and is identical to our result for the “unwinded” estimator (46) at high SNR although Kay shows that  $\hat{f}_i^b$  has a higher threshold than  $\hat{f}_i^a$ . It can be shown [22] that (49) is identical to a TFR moment estimator; in fact, if  $h_r$  were constant for all samples within the window rather than just the even samples, (49) would be identical to the TFR IF moment estimator derived from the spectrogram with a rectangular data window of length  $M = P + 2$ . Similarly (48) is the TFR moment IF estimator corresponding to the PSCFD. This does not exactly correspond to any well-known TFR although an infinite number of TFR's could be designed which yield (48) by

their first moment. However, (48) approximately corresponds to the TFR moment IF estimator from a spectrogram formed with a low-sidelobe window such as the Hamming or Blackman–Harris window [9] which are very similar to the parabolic window. It is interesting to note that such windows may be used to reduce the variance of DFT based spectral estimates [13, p. 80] just as we use the parabolic window to minimize the variance of the SCFD estimator.

### VIII. CONCLUSIONS

We have determined the distribution of CFD and SCFD IF estimator which are closely related to IF estimation via TFR moment estimators. These results show that there is no point in considering IF estimators which use every sample of the analytic signal because the variance of these estimators changes with frequency and is influenced by signal behavior outside the analysis window. We only need to consider estimators based on every second sample of the analytic signal. This paper gives a framework which allows the comparison of several other estimators such as the zero-crossing estimator and the estimator proposed by Kay. The Kay estimator was found to be biased and exhibit a large threshold for certain frequencies because it does not account for the circular nature of discrete-time frequency estimates. We replaced the linear convolution operation in the Kay estimator with the appropriate convolution operation for circular data to arrive at the PSCFD estimator. This estimator is unbiased and has frequency independent variance but it retains the asymptotically optimal performance and simplicity of the Kay estimator.

It has become clear that many problems encountered in discrete-time signal processing can be traced to poor understanding of the circular nature of some operations such as frequency estimation. We have highlighted some of these problems and proposed a number of mathematical operations suitable for circular data which should be used in preference to the conventional linear operations.

### APPENDIX

#### ANALYSIS OF SCFD IF ESTIMATORS

##### A. Problem Statement and Assumptions

We assume the following conditions to simplify the derivation.

- 1) The window function  $h$  is nonnegative.
- 2) The SNR is constant and moderate ( $s \geq 5$ ) within the window.
- 3) The IF variation within the window is bounded such that

$$\max_{-Q \leq p \leq Q} |f_i(n-p) - f_i^s(n)| \leq \frac{f_s}{40\pi}$$

where  $f_i^s$  is the SCFD estimate which would be obtained in the absence of noise.

We derive the distribution of  $\hat{\beta}^s(n) = 4\pi \hat{f}_i^s(n)/f_s$  rather than  $\hat{f}_i^s$  to simplify the derivation. The estimator  $\hat{\beta}^s$  may be visualized as the argument of the phasor sum used to

calculate the smoothed IF estimator  $\hat{f}_i^s$ . Then from definition 5 we have

$$\hat{\beta}^s(n) = \left( \left( \arg \left[ \sum_{p=-Q}^Q h(p) e^{j\hat{\beta}(n-p)} \right] \right) \right)_{2\pi} \quad (50)$$

where  $\hat{\beta}(n) = ((\hat{\phi}(n+1) - \hat{\phi}(n-1)))_{2\pi}$  and  $\hat{\beta}^s, \hat{\beta} \in [0, 2\pi)$ .

Similarly we define the noise-free angle

$$\beta^s(n) = \left( \left( \arg \left[ \sum_{p=-Q}^Q h(p) e^{j\beta(n-p)} \right] \right) \right)_{2\pi} \quad (51)$$

where  $\beta(n) = ((\phi(n+1) - \phi(n-1)))_{2\pi}$  and  $\beta^s, \beta \in [0, 2\pi)$ .

We determine the distribution of  $\hat{f}_i^s$  in four steps.

1. Determine an expression for the phase error  $\gamma(n) = ((\hat{\beta}^s(n) - \beta^s(n)))_{2\pi}$ .
2. Show  $\gamma$  is distributed as zero-mean, wrapped normal.
3. Find the dispersion parameter of  $\gamma$ .
4. Determine the distribution of the SCFD estimator  $\hat{f}_i^s$ .

##### B. Step 1: Determining an Expression for the Phase Error $\gamma$

We need to determine the distribution of  $\gamma$ , where  $\gamma$  represents the phase error in the smoothed IF estimator due to noise and is defined by

$$\gamma(n) = ((\hat{\beta}^s(n) - \beta^s(n)))_{2\pi}. \quad (52)$$

Since  $\gamma$  is a zero-mean angular error, it is convenient to express it in the domain  $[-\pi, +\pi)$  rather than  $[0, 2\pi)$ . Let the phase errors of the individual unsmoothed CFD IF estimators be denoted  $\alpha$ , where

$$\alpha(n) = ((\hat{\beta}(n) - \beta(n)))_{2\pi} \quad (53)$$

and  $\alpha \in [-\pi, +\pi)$ . When the SNR ratio is moderate ( $s \geq 5$ ), the  $\alpha(n)$  will be distributed as  $\tilde{N}(0, \alpha_\alpha)$  where  $\sigma_\alpha = \sigma_\beta$  as given by (21). From (52) and (50) we have

$$\begin{aligned} \gamma(n) &= \left( \left( \arg \left[ \sum_{p=-Q}^Q h(p) \exp [j\hat{\beta}(n-p)] \right] - \beta^s(n) \right) \right)_{2\pi} \\ &= \arg \left( \left[ \sum_{p=-Q}^Q h(p) \exp [j\hat{\beta}(n-p)] \right] e^{-j\beta^s(n)} \right) \\ &= \arg \left[ \sum_{p=-Q}^Q h(p) \exp \{ j[\hat{\beta}(n-p) - \beta^s(n)] \} \right] \\ &= \arctan \frac{\sum_{p=-Q}^Q h(p) \sin [\hat{\beta}(n-p) - \beta^s(n)]}{\sum_{p=-Q}^Q h(p) \cos [\hat{\beta}(n-p) - \beta^s(n)]}. \end{aligned} \quad (54)$$

Substituting (53) into (54) we obtain

$\gamma(n)$

$$= \arctan \frac{\sum_{p=-Q}^Q h(p) \sin [\beta(n-p) - \beta^s(n) + \alpha(n-p)]}{\sum_{p=-Q}^Q h(p) \cos [\beta(n-p) - \beta^s(n) + \alpha(n-p)]}. \quad (55)$$

We expand the trigonometric terms to obtain

$$\begin{aligned} \gamma(n) = \arctan & \left[ \left( \sum_{p=-Q}^Q h(p) \sin [\beta(n-p) - \beta^s(n)] \right. \right. \\ & \cdot \cos [\alpha(n-p)] \\ & + \sum_{p=-Q}^Q h(p) \cos [\beta(n-p) - \beta^s(n)] \\ & \cdot \sin [\alpha(n-p)] \Big) \\ & \cdot \left( \sum_{p=-Q}^Q h(p) \cos [\beta(n-p) - \beta^s(n)] \right. \\ & \cdot \cos [\alpha(n-p)] \\ & - \sum_{p=-Q}^Q h(p) \sin [\beta(n-p) - \beta^s(n)] \\ & \cdot \sin [\alpha(n-p)] \Big)^{-1} \Big]. \quad (56) \end{aligned}$$

Since we assume moderate SNR, we use the small angle approximations  $\cos \alpha(n) \approx 1$  and  $\sin \alpha(n) \approx \alpha(n)$ . Dividing both the numerator and denominator by

$$\sum_{p=-Q}^Q h(p) \cos [\beta(n-p) - \beta^s(n)]$$

gives

$$\begin{aligned} \gamma(n) \approx \arctan & \left[ \left( \frac{\sum_{p=-Q}^Q h(p) \sin [\beta(n-p) - \beta^s(n)]}{\sum_{p=-Q}^Q h(p) \cos [\beta(n-p) - \beta^s(n)]} + \frac{\sum_{p=-Q}^Q h(p) \cos [\beta(n-p) - \beta^s(n)] \alpha(n-p)}{\sum_{p=-Q}^Q h(p) \cos [\beta(n-p) - \beta^s(n)]} \right) \right. \\ & \cdot \left( 1 - \frac{\sum_{p=-Q}^Q h(p) \sin [\beta(n-p) - \beta^s(n)] \alpha(n-p)}{\sum_{p=-Q}^Q h(p) \cos [\beta(n-p) - \beta^s(n)]} \right)^{-1} \Big]. \quad (57) \end{aligned}$$

It is easily shown [19] that

$$\begin{aligned} & \Im \left[ \sum_{p=-Q}^Q h(p) \exp \{j[\beta(n-p) - \beta^s(n)]\} \right] \\ & = \sum_{p=-Q}^Q h(p) \sin [\beta(n-p) - \beta^s(n)] = 0 \quad (58) \end{aligned}$$

and this shows that the first term in the numerator of (57) is zero. Substituting (58) into (57) we obtain

$$\gamma(n) \approx \arctan \frac{Y}{1-X} \quad (59)$$

where

$$Y = \frac{\sum_{p=-Q}^Q h(p) \cos [\beta(n-p) - \beta^s(n)] \alpha(n-p)}{\sum_{p=-Q}^Q h(p) \cos [\beta(n-p) - \beta^s(n)]} \quad (60)$$

and

$$X = \frac{\sum_{p=-Q}^Q h(p) \sin [\beta(n-p) - \beta^s(n)] \alpha(n-p)}{\sum_{p=-Q}^Q h(p) \cos [\beta(n-p) - \beta^s(n)]}. \quad (61)$$

1) *Bounding the Variances of X and Y:* By using the assumptions that the window function  $h$  is nonnegative and the IF variation within the window is bounded, we can determine upper bounds for the variances of  $Y$  and  $X$  as follows [19]:

$$\sigma_Y^2 \leq \sigma_\alpha^2 \quad (62)$$

and

$$\sigma_X^2 \leq \frac{\sin^2 \Delta_\beta}{\cos^2 \Delta_\beta} \sigma_\alpha^2. \quad (63)$$

2) *Further Approximations to  $\gamma$ :* Using (21) and the assumption that  $s \geq 5$ , we get

$$\sigma_\alpha \leq \frac{1}{\sqrt{5}} = 0.45.$$

By assuming that  $\Delta_\beta \leq 0.1$ , (63) gives

$$\sigma_X \leq 0.045 \ll 1$$

and so we can neglect the denominator in (59). From (62) we know that

$$\sigma_Y \leq \sigma_\alpha$$

and therefore we can further simplify (59) by using the small angle approximation,  $\arctan Y \approx Y$ . Thus (59) sim-

plifies to

$$(n) \approx Y$$

$$\approx \frac{\sum_{p=-Q}^Q h(p) \cos [\beta(n-p) - \beta^s(n)] \alpha(n-p)}{\sum_{p=-Q}^Q h(p) \cos [\beta(n-p) - \beta^s(n)]}. \quad (64)$$

Although  $\gamma$  is really a circular random variable, at this stage it is safe to treat it as a conventional linear random variable because it has zero mean and a standard deviation which is small with respect to its angular domain  $[-\pi, +\pi]$ . Thus there is no need to express the right side of (64) as a quantity reduced modulo  $2\pi$ .

Since  $\Delta_\beta$  is a small angle

$$\cos [\beta(n-p) - \beta^s(n)] \approx \cos (\Delta_\beta) \approx 1$$

and thus

$$\gamma(n) \approx \frac{\sum_{p=-Q}^Q h(p) \alpha(n-p)}{\sum_{p=-Q}^Q h(p)}. \quad (65)$$

We are now in a position to describe the distribution of  $\gamma$  and hence  $\hat{f}_i^s$ .

### C. Step 2: Wrapped Normal Approximation to the Distribution of $\gamma$

Equation (65) shows us that  $\gamma$  is approximately equal to the weighted sum of the  $\alpha(n)$  which are approximately distributed as  $\tilde{N}(0, \sigma_\alpha)$ . Hence, using the reproductive property of the wrapped normal distribution

$$\gamma \sim \tilde{N}(0, \sigma_\gamma). \quad (66)$$

We must determine the dispersion parameter  $\sigma_\gamma$  allowing for the dependencies between the  $\alpha(n)$  due to the Hilbert transformer used to form the analytic signal.

### D. Step 3: Finding the Dispersion Parameter of the Distribution

First we define the following  $P$ -dimensional vectors (recall that  $P = 2Q + 1$ ).

$$\begin{aligned} \beta &= [\beta(n-Q) \cdots \beta(n+Q)]^T \\ \hat{\beta} &= [\hat{\beta}(n-Q) \cdots \hat{\beta}(n+Q)]^T \\ \phi_n &= [\phi(n-Q) \cdots \phi(n+Q)]^T \\ \hat{\phi}_n &= [\hat{\phi}(n-Q) \cdots \hat{\phi}(n+Q)]^T \\ \alpha &= [\alpha(n-Q) \cdots \alpha(n+Q)]^T \\ h &= [h(-Q) \cdots h(Q)]^T \end{aligned}$$

and

$$\mathbf{1} = [1 \cdots 1]^T.$$

We have from (65)

$$\begin{aligned} \sigma_\gamma^2 &= E[\gamma^2(n)] \\ &= E \left[ \left( \frac{\sum_{p=-Q}^Q h(p) \alpha(n-p)}{\sum_{p=-Q}^Q h(p)} \right)^2 \right] \\ &= E \left[ \left( \frac{\mathbf{h}^T \alpha}{\mathbf{h}^T \mathbf{1}} \right)^2 \right] \\ &= \frac{\mathbf{h}^T E[\alpha \alpha^T] \mathbf{h}}{(\mathbf{h}^T \mathbf{1})^2} \\ &= \frac{\mathbf{h}^T \mathbf{p} \mathbf{h}}{(\mathbf{h}^T \mathbf{1})^2} \sigma_\alpha^2 \end{aligned} \quad (67)$$

where we define the correlation coefficient matrix  $\mathbf{p}$  by

$$\mathbf{p} = \frac{1}{\sigma_\alpha^2} E[\alpha \alpha^T]. \quad (68)$$

Now

$$\begin{aligned} \alpha &= \hat{\beta} - \beta \\ &= \hat{\phi}_{n+1} - \hat{\phi}_{n-1} - \phi_{n+1} + \phi_{n-1} \\ &= \Theta_{n+1} - \Theta_{n-1} \end{aligned} \quad (69)$$

where

$$\Theta_n = [\Theta(n-Q) \cdots \Theta(n+Q)]^T$$

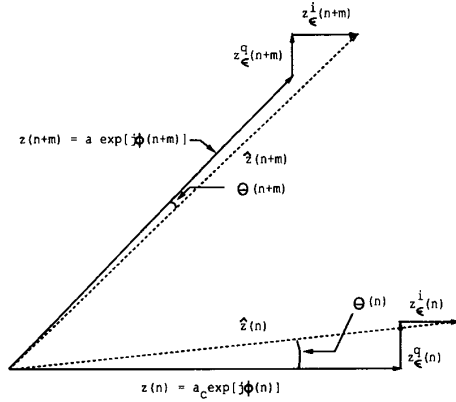
$$\Theta(n) = ((\hat{\phi}(n) - \phi(n)))_{2\pi} = ((\arg[\hat{z}(n)] - \arg[z(n)]))_{2\pi}$$

and  $\Theta \in [-\pi, +\pi]$  represents the phase error in estimating  $\arg[z(n)]$  due to noise. Substituting (69) into (68) we get

$$\begin{aligned} \mathbf{p} &= \frac{1}{\sigma_\alpha^2} E[(\Theta_{n+1} - \Theta_{n-1})(\Theta_{n+1} - \Theta_{n-1})] \\ &= \frac{1}{\sigma_\alpha^2} [E[\Theta_{n+1} \Theta_{n+1}^T] + E[\Theta_{n-1} \Theta_{n-1}^T] \\ &\quad - E[\Theta_{n+1} \Theta_{n-1}^T] - E[\Theta_{n-1} \Theta_{n+1}^T]] \\ &= \frac{1}{\sigma_\alpha^2} [2\mathbf{R} - E[\Theta_{n+1} \Theta_{n-1}^T] - E[\Theta_{n-1} \Theta_{n+1}^T]]. \end{aligned} \quad (70)$$

where  $\mathbf{R} = E[\Theta_n \Theta_n^T] = E[\Theta_{n+1} \Theta_{n+1}^T] = E[\Theta_{n-1} \Theta_{n-1}^T]$  since we assume that the noise is wide-sense stationary.

1) *Determining the Autocovariance Matrix for  $\Theta$* : Fig. 10 shows the noise-free phasors  $z(n)$  and  $z(n+m)$  of magnitude  $a_c$  and the corresponding noise corrupted phasors  $\hat{z}(n)$  and  $\hat{z}(n+m)$ . The noise phasors are represented as the sums of in-phase and quadrature components denoted  $z_\epsilon^i$  and  $z_\epsilon^q$ , respectively. Because of the moderate SNR assumption, we use the small angle approximation

Fig. 10. Phasor diagram of  $\hat{z}$ .

$\tan \Theta \approx \Theta$ . We have

$$\begin{aligned} \Theta(n) &= \arctan \frac{z_c^q(n)}{a_c} \\ &\approx \frac{z_c^q(n)}{a_c} \end{aligned} \quad (71)$$

and similarly

$$\begin{aligned} \Theta(n+m) &= \arctan \frac{z_c^q(n+m) \cos [\phi(n+m) - \phi(n)] - z_c^i(n+m) \sin [\phi(n+m) - \phi(n)]}{a_c} \\ &\approx \frac{z_c^q(n+m) \cos [\phi(n+m) - \phi(n)] - z_c^i(n+m) \sin [\phi(n+m) - \phi(n)]}{a_c}. \end{aligned} \quad (72)$$

We will use the approximation

$$\phi(n+m) - \phi(n) \approx \frac{m\beta^s(n)}{2}. \quad (73)$$

Here we replace the actual difference between angles  $m$  samples apart with  $m$  times the average angle difference between samples. This approximation is good since we assume that IF variation within the window is small. Substituting (73) into (72) yields

$$\begin{aligned} \Theta(n+m) &\approx \frac{1}{a_c} \left[ z_c^q(n+m) \cos \frac{m\beta^s(n)}{2} \right. \\ &\quad \left. - z_c^i(n+m) \sin \frac{m\beta^s(n)}{2} \right]. \end{aligned} \quad (74)$$

So using (74) and (71) we have

$$\begin{aligned} E[\Theta(n)\Theta(n+m)] &\approx \frac{1}{a_c^2} E \left[ z_c^q(n) \left( z_c^q(n+m) \cos \frac{m\beta^s(n)}{2} \right. \right. \\ &\quad \left. \left. - z_c^i(n+m) \sin \frac{m\beta^s(n)}{2} \right) \right]. \end{aligned} \quad (75)$$

Since the noise is analytic, we know from (13) that

$$z_c^q(n) = \sum_{\substack{k=-\infty \\ k \text{ odd}}}^{+\infty} \frac{2z_c^i(n-k)}{k\pi} \quad (76)$$

and both  $z_c^i$  and  $z_c^q$  are zero mean, white Gaussian noise sequences with variance  $\sigma_c^2$ . Since white noise sequences are orthogonal

$$E[z_c^q(n)z_c^q(n+m)] = E[z_c^i(n)z_c^i(n+m)] = \sigma_c^2 \delta(m).$$

Using this result and substituting (76) into (75) yields

$$\begin{aligned} E[\Theta(n)\Theta(n+m)] &\approx \frac{\sigma_c^2}{a_c^2} \delta(m) - \frac{1}{a_c^2} E \left[ z_c^i(n+m) \right. \\ &\quad \left. \cdot \sin \frac{m\beta^s(n)}{2} \sum_{\substack{k=-\infty \\ k \text{ odd}}}^{+\infty} \frac{2z_c^i(n-k)}{k\pi} \right]. \end{aligned} \quad (77)$$

Now from (20),  $\sigma_\Theta^2 = \sigma_\beta^2/2$  and  $\sigma_\alpha = \sigma_\beta$  since  $\alpha$  is just

the phase error associated with  $\hat{\beta}$ . Thus the elements  $r_{ij}$  of  $\mathbf{R}$  are given by

$$\begin{aligned} r_{ij} &= E[\Theta(n)\Theta(n+m)] \\ &= \begin{cases} \frac{\sigma_\alpha^2}{2} & \text{for } m = 0 \\ \frac{\sigma_\alpha^2}{2} \frac{2}{m\pi} \sin [m\beta^s(n)/2] & \text{for } m \text{ odd} \\ 0 & \text{for even } m \neq 0 \end{cases} \end{aligned} \quad (78)$$

where  $m = |i - j|$ .

All terms in (70) can now be determined. The elements  $r'_{ij}$  of  $E[\Theta_{n+1}\Theta_{n-1}^T]$  are given by

$$r'_{ij} = E[\Theta(n+1)\Theta(n-1)] = r_{i,j+2} \quad (79)$$

and the elements  $r''_{ij}$  of  $E[\Theta_{n-1}\Theta_{n+1}^T]$  are given by

$$r''_{ij} = E[\Theta(n-1)\Theta(n+1)] = r_{i,j-2}. \quad (80)$$

Substituting (78)–(80) into (70) we see that the correlation

coefficient matrix  $\mathbf{p}$  is a Toeplitz matrix with elements

$$\rho_{ij} = \frac{1}{\sigma_\alpha^2} [2r_{ij} - r_{i,j+2} - r_{i,j-2}].$$

Thus

$$\rho_{ij} = \begin{cases} 1 & \text{for } m = 0 \\ -\frac{\sin[(m-2)\beta^s(n)/2]}{(m-2)\pi} + \frac{2\sin[m\beta^s(n)/2]}{m\pi} & \\ -\frac{\sin[(m+2)\beta^s(n)/2]}{(m+2)\pi} & \text{for } m \text{ odd} \\ -\frac{1}{2} & \text{for } m = 2 \\ 0 & \text{otherwise} \end{cases}$$

where  $m = |i - j|$ .

#### E. Step 4: Approximate Distribution of the SCFD IF Estimator $\hat{f}_i^s$

Since the IF estimators only differ from the angular quantities by a simple scale factor, these results can be rewritten in terms of the smoothed CFD IF estimator itself to obtain Theorem 3 as required.

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#### REFERENCES

- [1] W. R. Bennett, "Methods of solving noise problems," *Proc. IRE*, vol. 44, pp. 609-639, May 1956.
- [2] N. M. Blachman, "A comparison of the informational capacities of amplitude and phase modulation communication systems," *Proc. IRE*, vol. 41, pp. 748-759, June 1953.
- [3] N. M. Blachman, *Noise and its Effect on Communication* (McGraw-Hill Electronic Sciences Series). New York: McGraw-Hill, 1966.
- [4] B. Boashash, B. C. Lovell, and P. J. Kootsookos, "Time-frequency signal analysis and instantaneous frequency estimation," in *Proc. IEEE Int. Symp. Circuits Syst.* (Portland, OR), 1989, pp. 1237-1241.
- [5] B. Boashash, "Time-frequency signal analysis," in *Advances in Spectrum Estimation and Array Processing*, vol. 1, S. Haykin, Ed. Englewood Cliffs, NJ: Prentice-Hall, 1990, pp. 418-517.
- [6] H. I. Choi and W. J. Williams, "Improved time-frequency representation of multicomponent signals using exponential kernels," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. 37, pp. 862-871, June 1989.
- [7] L. Cohen, "Generalized phase-space distribution functions," *J. Math. Phys.*, vol. 7, pp. 781-786, 1966.
- [8] E. J. Gumbel, J. A. Greenwood, and D. Durand, "The circular normal distribution: Theory and tables," *J. Amer. Stat. Ass.*, vol. 48, pp. 131-152, 1953.
- [9] F. J. Harris, "On the use of windows for harmonic analysis with the discrete Fourier transform," *Proc. IEEE*, vol. 66, pp. 51-83, Jan. 1978.
- [10] G. H. Hostetter, "Recursive discrete Fourier transformation," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-28, pp. 184-190, Apr. 1980.
- [11] L. B. Jackson and D. W. Tufts, "Frequency estimation by linear prediction," in *Proc. IEEE Int. Conf. ASSP* (Tulsa, OK), 1976.
- [12] S. M. Kay, "Statistically/computationally efficient frequency estimation," in *Proc. IEEE Int. Conf. ASSP* (New York), 1988, pp. 2292-2295.
- [13] S. M. Kay, *Modern Spectral Estimation*. Englewood Cliffs, NJ: Prentice-Hall, 1988.
- [14] S. M. Kay, "Comments on frequency estimation by linear prediction," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-27, pp. 198-199, Apr. 1979.
- [15] S. M. Kay, "A fast and accurate single frequency estimator," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. 37, no. 12, pp. 1987-1990, Dec. 1990.
- [16] S. M. Kay, "Maximum entropy spectral estimation using the analytical signal," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-26, pp. 467-469, Oct. 1978.
- [17] G. W. Lank, I. S. Reed, and G. E. Pollon, "A semicoherent detection and Doppler estimation statistic," *IEEE Trans. Aerosp. Electron. Syst.*, vol. AES-9, pp. 151-165, Mar. 1973.
- [18] H. Leib and S. Pasupathy, "The phase of a vector perturbed by Gaussian noise and differentially coherent receivers," *IEEE Trans. Inform. Theory*, vol. IT-34, pp. 1491-1501, Nov. 1988.
- [19] B. C. Lovell, "Techniques for nonstationary spectral analysis," Ph.D. dissertation, Univ. Queensland, Brisbane, Australia, 1990.
- [20] B. C. Lovell, R. C. Williamson, and B. Boashash, "The relationship between instantaneous frequency and time-frequency representations," *IEEE Trans. Signal Processing*, to be published, May 1993.
- [21] B. C. Lovell and B. Boashash, "Efficient estimation of the instantaneous frequency of a rapidly time-varying signal," in *Proc. Australian Symp. Signal Processing Its Appl.* (Adelaide, Australia), Apr. 17-19, 1989, pp. 314-318.
- [22] B. C. Lovell, P. J. Kootsookos, and R. C. Williamson, "Efficient frequency estimation and time-frequency representations," in *Proc. Int. Symp. Signal Processing Its Appl.* (Gold Coast, Australia), Aug. 27-31, 1990, pp. 170-173.
- [23] K. V. Mardia, *Statistics of Directional Data*. London: Academic, 1972.
- [24] R. von Mises, "Über die 'ganzahligkeit' der atomgewichte und verwandte fragen," *Phys. Zeitschrift*, vol. 19, pp. 490-500, 1918.
- [25] A. V. Oppenheim and R. W. Schaffer, *Discrete-Time Signal Processing*. Englewood Cliffs, NJ: Prentice-Hall, 1989.
- [26] L. C. Palmer, "Coarse frequency estimation using the discrete Fourier transform," *IEEE Trans. Inform. Theory*, vol. IT-20, pp. 104-109, Apr. 1967.
- [27] A. Papoulis, *Probability, Random Variables, and Stochastic Processes*, second ed. Singapore: McGraw-Hill, 1984.
- [28] D. C. Rife and R. R. Boorstyn, "Single-tone parameter estimation from discrete-time observations," *IEEE Trans. Inform. Theory*, vol. IT-20, pp. 591-598, Sept. 1974.
- [29] A. W. Rihaczek, "Signal energy distribution in time and frequency," *IEEE Trans. Inform. Theory*, vol. T-14, pp. 369-374, Mar. 1968.
- [30] M. A. Stephens, "Random walk on a circle," *Biometrika*, vol. 50, pp. 385-390, 1963.
- [31] S. A. Tretter, "Estimating the frequency of a noisy sinusoid by linear regression," *IEEE Trans. Inform. Theory*, vol. IT-31, pp. 832-835, Nov. 1985.
- [32] A. J. Viterbi, *Principles of Coherent Communication*. New York: McGraw-Hill, 1966.
- [33] G. S. Watson, "Distributions on the circle and sphere," *Essays Stat. Sci.*, vol. 19A, pp. 265-280, 1982.



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