On Händel's Analysis of Kay's Weighted Linear Predictor Frequency Estimator

B.G. Quinn, V. Clarkson and P.J. Kootsookos

Abstract— **Händel, in a recent correspondence, provides an alternate analysis to that of Clarkson, Kootsookos and Quinn, of Kay's linear predictor frequency estimator, claiming that his proof is more concrete. We show, in fact, that his proof is in error, and emphasise the care needed when carrying out such analyses.**

I. Introduction

The analysis of Kay's estimator in [1] showed that the estimator was unbiased, and evaluated bounds on the variance of the estimator. The authors believe that this is the first occasion on which it has been able to do this for any frequency estimator. Often it is possible to derive a central limit theorem for an estimator of frequency, and to identify the variance in this limiting distribution as the 'asymptotic variance' of the estimator. More often, and erroneously, some type of Taylor series expansion is carried out, approximations are made, and a 'proof' of 'asymptotic variance' results. There is no theoretical basis for such proofs. Händel claims that what his 'approach lacks in elegance it gains in concreteness'. On the contrary, we shall demonstrate that our approach is the only one that leads to precise results, and that Händel's approach is invalid.

II. Händel's Analysis

Starting out with the model

$$s_n = Ae^{i\omega n} + z_n \quad n = 0, \ldots, N-1 \tag{1}$$

where the complex amplitude A and the frequency ω are unknown, and $\{z_n\}$ is zero mean complex Gaussian with variance σ^2, the weighted linear predictor is given by [1], [2], [3]

$$\begin{aligned} \widehat{\omega} &= \Im\left[\widehat{R}\right] \\ \widehat{R} &= \sum_{n=1}^{N-1} w_n s_n s_{n-1}^* \\ w_n &= \frac{6n\,(N-n)}{N\,(N^2-1)} \end{aligned} \tag{2}$$

B.G. Quinn is with the Department of Mathematical Sciences, Goldsmiths College, University of London, New Cross, London, U.K.
V. Clarkson is with the Electronic Warfare Division, Defence Science and Technology Organisation, Salisbury, Australia.
P.J. Kootsookos is with the Cooperative Research Centre for Robust and Adaptive Systems, Canberra, Australia.
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The essence of Händel's proof is to expand $\widehat{R}$, and subsequently $\widehat{\omega}$ about the parameters they estimate, considering only first order terms. The problems with his analysis are due to the fact that the z_n are iid complex Gaussian random variables, and, as such, take on *all* values in the complex plane, regardless of the value of σ^2.

Thus $\widehat{R}$ takes on all values in the complex plane. In fact, although it is possible to obtain a Gaussian central limit theorem for $\widehat{R}$, with asymptotic variance small for high SNR, the fact of the matter is that Händel's equation (4)

$$\delta\omega = \Im \log\left[1 + \frac{\delta R}{R}\right] \simeq \Im\left[\frac{\delta R}{R}\right],$$

is valid only when $\left|\frac{\delta R}{R}\right| < 1$, which is not true, with nonzero probability, for any SNR. This error is compounded by the fact that Händel, along with many others, believe that writing an estimator $\widehat{\theta}$ of θ in the form

$$\widehat{\theta} = \theta + \delta\theta + \varepsilon,$$

where the variance of $\delta\theta$ is easily obtained and ε is of lower order in magnitude, allows bounds to be placed on the variance of $\widehat{\theta}$. Here is a simple counter-example: Let $X_N = 1 + Z_N + \frac{1}{N}Y$, where the second moment of Z_N converges to zero, but Y is distributed as standard Cauchy (Y therefore does not have a mean or variance). Then, although $\{X_N\}$ converges to 1 in probability, as convergence in mean square implies convergence in probability, the variance of X_N does not exist for any N, and therefore does not exist in the limit. If, however, it can be shown that the distribution of $\sqrt{N}Z_N$ converges to that of a zero mean random variable with variance, say, ν^2, it is true to say that the 'asymptotic variance' of X_N is $\frac{\nu^2}{N}$, as $\sqrt{N}\,(X_N - 1)$ has a distribution which converges to that of a zero mean random variable with variance ν^2. This argument may be applied in the above context, with care, and is valid only in the limit as $N \to \infty$. Firstly, one must show, from Händel's equation (6), that $\delta\omega$ admits a Gaussian central limit theorem (has an asymptotic Gaussian distribution as $N \to \infty$). What was done in [1] was quite different. The authors there, through careful and 'concrete' analysis, placed bounds on the exact variance of Kay's estimator for fixed SNR and N.

III. Conclusion

The authors have demonstrated the need for care in obtaining bounds to the variances of estimators. There is quite a difference between asymptotic bounds on the variances of estimators and bounds on asymptotic variances. It is not enough to do first order Taylor series expansions and hope for the best. The authors believe that if statistical analysis is carried out, it should be done rigorously, as there is no guarantee that anything less will yield the correct results.

References

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