Quantum slow motion

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Abstract

We simulate the center of mass motion of cold atoms in a standing, amplitude modulated, laser field as an example of a system that has a classical mixed phase-space. We show a simple model to explain the momentum distribution of the atoms taken after any distinct number of modulation cycles. The peaks corresponding to a classical resonance move towards smaller velocities in comparison to the velocities of the classical resonances. We explain this by showing that, for a wave packet on the classical resonances, we can replace the complicated dynamics in the quantum Liouville equation in phase-space by the classical dynamics in a modified potential. Therefore we can describe the quantum mechanical motion of a wave packet on a classical resonance by a purely classical motion.

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To have an intuitive picture of the quantum mechanical dynamics of a wave packet we are usually confined to the semi-classical regime, that is, to orbits with action large compared to Planck's constant [1,2], or to special systems like the harmonic oscillator, where the quantum evolution equations in phase-space are identical to the classical ones [3]. In this letter we propose a scheme which enables us to describe a wave packet, localized near a resonance of a classical mixed phase-space, by classical dynamics in a modified potential. Hereby we replace the potential in the high order quantum Liouville equation by an effective potential in such a way that we obtain a classical Liouville equation. Then we describe the quantum motion as classical motion in this modified potential. We are then able to characterize the quantum effect by comparing the modified dynamics with the dynamics in the original potential. This method is applicable well beyond the semi-classical regime for many different potentials.

Usually quantum effects on wave packets express themselves in the revival and fractional revival properties [4] or in the occurrence of tunneling phenomena [5]. Both take place on a comparatively long time scale so that we intuitively don't expect quantum effects to be visible on a short time scale. We disprove this intuitive assumption in our model where we use the center of mass motion of cold atoms in a standing amplitude modulated laser field. Here we demonstrate that the momentum distribution after each cycle of the modulation is peaked at smaller momenta than we would expect classically. This shows that the atoms are traveling slower than we would expect from classical simulations and we can give a very simple explanation of this "quantum slow motion" phenomenon. Since we here include stimulated and spontaneous transitions we expect that this quantum mechanical effect is of realistic order of magnitude to observe experimentally.

We investigate a cloud of two level atoms situated in a standing laser field, with a periodic modulated amplitude. In this system the Hamiltonian of the center-of-mass motion in the limit of large detuning is [6]

$$H(t) = \frac{p^2}{2} - \kappa(1 - 2\epsilon\cos t)\cos q, \tag{1}$$

where p and q denote scaled dimensionless momentum and position, t time, and κ and ϵ are the parameters defining the depth of the standing wave and the strength of the amplitude modulation, respectively. Note that p and q fulfill the commutator relation $[p, q] = \mathrm{i}\bar{\hbar}$, where $\bar{\hbar}$ is a scaled Planck's constant that is in some sense a measure for the "quantum mechanicality" of the problem since it defines the size of a minimum uncertainty wave packet in relationship to the resonances [7].

In Fig. 1(left) we show as an example the classical stroboscopic phase-space portrait [8] for $\epsilon = 0.2$ with $\kappa = 1.2$. This choice of parameters is capable to show classical stable period-one resonances after each modulation period symmetrically situated along the momentum axes. This specific phase-space structure allows a quantum mechanical wave packet, situated initially near one of these resonances, to coherently tunnel to the other resonance. This takes place on a long time scale in terms of cycles of the modulation. Note that this tunneling cannot be understood in terms of the presence of a potential barrier as it is present in several publications [5] regarding tunneling in mixed systems.

We simulate the tunneling dynamics by starting each realization with a minimum uncertainty wave packet that may be squeezed [], centered on the classical resonance. We then simulate the full quantum mechanical dynamics by applying a split operator algorithm with adapted time step size [9] in the context of a standard quantum Monte Carlo integration scheme to include stimulated and spontaneous transitions [6]. We calculate the mean momenta and the corresponding variance from the Poincare section of the momentum distribution taken after each cycle of the modulation at $t = 2n\pi$. In Fig. 2 (full line) we show the result of this simulation for $\epsilon = 0.2$, $\kappa = 1.2$, and $\bar{\hbar} = 0.25$. Related to recent experiments [10] we used the parameters for Rubidium to obtain a realistic scenario. We plot the mean momentum after each cycle of the modulation of the standing wave against the number of cycles. As expected and clearly indicated by the drop down of the variance, we observe coherent tunneling of the mean momentum from the location of the resonance at approximately $p = 1$ to the corresponding resonance at $p = -1$.

However there are additional oscillations that might lead to the conclusion that the wave

packet is not sitting precisely on the classical period-one fixed point but is indeed circulating around an alternative stable point in phase-space. It seems like the wave packet, centered on the classical resonance is not appropriately centered on the "true" resonance but sitting beside it. Therefore the mean momentum at each kick strongly oscillates around its mean motion.

This lead us to the conclusion that if we moved the initial wave packet onto this alternative stable point and started the simulation of the dynamics from there, we expect the oscillations to vanish. This is exactly what we see in Fig. 2 (dashed line). The oscillations are strongly compressed and we face essentially the situation of a well localized wave packet which undergoes coherent tunneling on the longer time scale. For the dynamics of the wave packet the classical resonance is obviously not important but a modified resonance, shifted towards slower momentum. How can we explain this effect?

To give an explanation we first recall that a wave packet localized near a classical resonance has been shown [8] to remain localized without changing its shape, at least for a long time. Therefore we may assume that a minimum uncertainty wave packet sitting near a classical resonance will remain unchanged in shape for several cycles. This is the main assumption we need to apply a theory of Henriksen et. al [11] where the effect of quantum mechanics on a wave packet is described as classical motion, that is as motion following the classical Liouville equations in phase-space, but in a modified potential.

The convenient quantum mechanical phase-space representation is the Wigner function $W(q,p,t)$,because it has the correct quantum mechanical marginal distributions. Since in the experiments we are seeking to describe the momentum distribution and the position distribution of the center of mass motion, this property of the Wigner function allows us to compare the marginals directly with the measured distributions.

The phase-space dynamics of the Wigner function is given by [12,11]

$$\frac{\partial W}{\partial t} = -p\frac{\partial W}{\partial q} + \frac{\mathrm{i}}{\hbar}\left(\sum_{\nu=0}^{\infty}\frac{1}{\nu!}\left(\frac{\hbar}{2\mathrm{i}}\right)^{\nu}\frac{\partial^{\nu}V(q,t)}{\partial q^{\nu}}\frac{\partial^{\nu}W}{\partial p^{\nu}} - \sum_{\nu=0}^{\infty}\frac{1}{\nu!}\left(-\frac{\hbar}{2\mathrm{i}}\right)^{\nu}\frac{\partial^{\nu}V(q,t)}{\partial q^{\nu}}\frac{\partial^{\nu}W}{\partial p^{\nu}}\right) \qquad (2)$$

where $V(q,t) = \kappa(1-\epsilon\cos t)\cos q$ denotes the potential. This, for the following convenient

representation, corresponds to the well known one given by Wigner where only one sum over odd derivatives occurs.

Due to the special spatial dependence of our cosine potential, where the odd derivatives reproduce themselves, we can replace the infinite sum by defining an effective potential V_{eff} by

$$\frac{\partial V_{eff}}{\partial q} = \frac{\mathrm{i}}{\hbar}\left(\sum_{\nu=0}^{\infty}\frac{1}{\nu!}\left(\frac{\hbar}{2\mathrm{i}}\right)^{\nu}\frac{\partial^{\nu}V(q,t)}{\partial q^{\nu}}\frac{\partial^{\nu}W}{\partial p^{\nu}} - \sum_{\nu=0}^{\infty}\frac{1}{\nu!}\left(-\frac{\hbar}{2\mathrm{i}}\right)^{\nu}\frac{\partial^{\nu}V(q,t)}{\partial q^{\nu}}\frac{\partial^{\nu}W}{\partial p^{\nu}}\right)/\frac{\partial W}{\partial p}. \tag{3}$$

Then Eq. 2 is replaced by the first order equation

$$\frac{\partial W}{\partial t} = -p\frac{\partial W}{\partial q} + \frac{\partial V_{eff}}{\partial q}\frac{\partial W}{\partial p} \tag{4}$$

which is identical to the classical Liouville equation describing the classical dynamics in the modified potential V_{eff}. In this sense the action of quantum mechanics can be described by the classical motion in a modified potential.

Assuming a Gaussian squeezed minimum uncertainty wave packet with time dependent squeeze parameter $\xi(t)$, we take the Wigner function to be of the form

$$W(q,p,t) = \frac{1}{\pi\hbar}\exp\left(-\frac{\xi}{\hbar}(q-\langle q\rangle)^2 - \frac{1}{\hbar\xi}(p-\langle p\rangle)^2\right) \tag{5}$$

with the mean time dependent momentum and position, $\langle p\rangle(t)$ and $\langle q\rangle(t)$, respectively, chosen in such a way, that the wave packet always stays centered on the resonance in order the assumption of staying unchanged in shape to remain valid. It is not important to know the explicit time dependence of these parameters. Then the effective potential is

$$V_{eff}(q,t) = V(q,t)\exp\left(-\frac{\hbar}{4}\right)\frac{\sinh(p-\langle p\rangle)}{p-\langle p\rangle} \tag{6}$$

That means the motion of the wave packet is locally described by the original potential compressed by a factor of $\exp(-\hbar/4)$ since the sinh-factor can, for the sake of qualitative discussion, locally be approximated by 1.

In Fig. 1 (middle and right) we show for $\hbar = 0.25$ and $\hbar = 0.35$ classical stroboscopic phase-space portraits for the effective potential and compare them to the phase-space portrait of the original potential. Note, that our approximation is only valid in the vicinity

of the period-one resonances. However, since we are interested in exactly these regions of phase space this kind of representation gives an idea of what is going on, although the other phase-space regions are not to be taken as a valid description of the dynamics there. The main conclusion regarding the resonances is that the central resonance at $(q, p) = (0, 0)$ becomes smaller and the second order resonances we are interested in are pushed towards smaller momenta p, which exactly corresponds to the observation made in Fig. 2, where we could simulate the tunneling phenomenon best for initially situating the wave packet at the shifted resonance.

Eq. 6 indicates that the effect scales with $\hbar$ which identifies it as a purely quantum mechanical effect. We can clearly see this property by comparing Fig. 1 (middle) and (right), where we can directly see the relocation of the classical resonance for two values of $\hbar$. In Fig. 3 we simulate wave packets for different values of $\hbar$ for only a few cycles. We start each simulation with a minimum uncertainty wave packet in such a way that the oscillations in the evolution are most suppressed and observe that in correspondence to the modified potential the mean momenta and therefore the wave packets themselves are relocated towards smaller velocities with increasing $\hbar$. Note that the curve corresponding to $\hbar = 0.25$ corresponds to a situation where the conditions for tunneling are fulfilled, therefore the mean momentum starts to decrease.

There is a second important consequence of this phenomenon in the scenario of present experiments [10] of investigating the short time behavior of loading all the resonances from a spatially uniform distributed cloud of atoms. In order to effectively load the resonances we start with a phase shift of $-\pi/2$, that is to say, we now investigate the Hamiltonian

$$H(t) = \frac{p^2}{2} - \kappa(1 - 2\epsilon \sin t) \cos q, \tag{7}$$

and take the snapshots at $t = \pi/2 + 2n\pi$. Then the resonances are initially aligned on the q-axes and are therefore covered best by the cloud of atoms. A classical picture of the dynamics suggests, that as time goes by only those atoms initially sitting close to a resonance remain, whereas all the other atoms perform a nonlinear motion corresponding to

the fact that they are sitting in a chaotic region [13] of phase-space. Therefore we expect to observe after some time only the three peaks of loaded resonances. Since the assumption of a durable wave packet is only valid for a wave packet initially situated on a resonance and not for all the other wave packets this motivates us to believe that the local relocation of the resonance described above only happens to those atoms trapped at the resonance. This should change the overall momentum distribution in comparison to a pure classical simulation.

In Fig. 4 (left) we compare the momentum distributions of snapshots at $t = 9\pi/2$, that is after 2.25 modulation cycles only, of three different simulations: a quantum mechanical (top), a modified classical (middle), and a purely classical simulation(bottom). Note that this is a very short time compared to tunneling and revival experiments. For the quantum simulation (top) we start with a large number of wave packets of the width of the distribution in momentum of the atom cloud. The width in position is chosen in order to have a minimum uncertainty wave packet. We distribute them uniformly on the q-axis, apply the Monte Carlo integration scheme and finally add the contribution of each wave packet up to get the whole momentum distribution. In the purely classical simulation (bottom) we simply take a cloud of point particles uniformly distributed in $q-$direction and Gaussian distributed in $p-$direction. Then the individual motion of the atoms is treated classically by letting the atoms evolve following the classical Liouville dynamics, but we still have included stimulated and spontaneous transitions in a Monte-Carlo-Integration scheme.

Note that the quantum peaks are shifted towards smaller momenta. This shift becomes larger with the scaled Planck's constant $\tilde{\hbar}$ which is a further indication, that this effect can be explained by the quantum mechanical effect described above. To show that the occurrence of the effective potential may in principle be sufficient to explain this feature, we simulate this by applying the classical simulation again (middle), where we now change the trajectory according to the effective potential once we start on a resonance. This is a very simple approach which is certainly only useful to show qualitatively that our explanation is suitable to describe the quantum dynamics. But we note that this modified classical

simulation indeed shows the essential features of the pure quantum simulations.

In Fig. 4(right) we show the same simulations but without any modulation. Here the difference corresponding to the quantum mechanical effect vanish and now more or less all three simulations show the same structure. This structure is due to classical transient effects, which appear in the first few cycles and are closely related to the motion in the standing wave, since they are independent of the modulation. This transient is always there and interferes with the quantum mechanical effect investigated in this paper. However the quantum mechanical effect is easily to identify since it vanishes for $\epsilon = 0$. Therefore this effect is clearly related to the modulation and therefore shows a quantum feature of the classical mixed phase space. Note that the effect is vanishing for $\epsilon = 0$ is consistent with our theory since in this case we face classical integrable motion. A wave packet in such a system is not stabilized but spreads and changes its shape and therefore the assumption for applying the theory of Henriksen et al. is no longer valid.

To conclude, we have shown that we can use the property of wave packets staying localized on resonances of a classical mixed phase space to simplify the complicated quantum dynamics in phase space. In this case we can describe the quantum dynamics of the wave packet by the classical motion in a modified potential. This is not only valid for the cosine potential investigated, but also as already mentioned in [11] to polynomial potentials of arbitrary high order and to other systems that has been topic of investigations of the relationship of classical chaotic motion and the corresponding quantum dynamics. First there is the atomic bouncer in an evanescent field [14]

$$V(q,t) = \lambda q + \kappa(1 + \epsilon\cos t)\exp(-q) \tag{8}$$

This setup of evanescent light waves can be modified to get a Morse potential [15] which serves as an atomic trap. In these two cases it is also very straightforward to find the modified potential and to come to similar conclusions to those in this paper.

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FIGURES

FIG. 1. Left: Stroboscopic phase space portrait of the classical motion described by the Hamiltonian Eq. 1 for $\kappa = 1.2$ and $\epsilon = 0.2$. Middle and right: Stroboscopic phase space portraits of the corresponding effective potentials Eq. 6 with $\bar{k} = 0.25$ and $\bar{k} = 0.35$

FIG. 2. Left: Mean momentum $\langle p \rangle$ of the quantum mechanical simulation of the dynamics of two wave packets in dependence of number of cycles s. The first (straight line) is initially sitting on the classical resonance ($p_m = 1.03$), the second (dashed line) on the modified at $p_m = 0.84$. Here the parameters are $\bar{k} = 0.25, \kappa = 1.2$, and $\epsilon = 0.2$ Right: Corresponding variance $V[p]$.

FIG. 3. Mean momentums $\langle p \rangle$ of the quantum mechanical simulation of the dynamics of several wave packets in dependence on number of cycles s with $\kappa = 1.2$ and $\epsilon = 0.2$. Here $\bar{k}$ takes on the values $0.15, 0.2, 0.25, 0.3$ (from top).

FIG. 4. Left: Quantum, modified classical, and purely classical simulation (from top) of momentum distributions $P[p]$ of snapshots after 2.25 modulation cycles of the Hamiltonian Eq.7 with $\kappa = 1.2, \epsilon = 0.2$, and $\bar{k} = 0.35$. Right: The same simulations but for the unmodulated case $\epsilon = 0$.