Hamiltonian mappings and circle packing phase spaces

A. J. Scott,[1,*] C. A. Holmes,[1] and G. J. Milburn[2]

[1]*Department of Mathematics,*
[2]*Department of Physics,*
The University of Queensland, QLD 4072 Australia.

We introduce three area preserving maps with phase space structures which resemble circle packings. Each mapping is derived from a kicked Hamiltonian system with one of three different phase space geometries (planar, hyperbolic or spherical) and exhibits an infinite number of coexisting stable periodic orbits which appear to 'pack' the phase space with circular resonances.

I. INTRODUCTION

Nonlinear dynamical systems offer a rich and seemingly endless variety of behaviour. In this paper we introduce three Hamiltonian mappings with phase space structures which resemble circle packings [1, 2, 3, 4, 5]. A circle packing is a set with empty interior and whose complement is the union of disjoint open circular discs. Circle packings are often constructed using geometrical or group theoretical methods, for example limit sets of Kleinian groups [6], but here we may have uncovered them in the context of Hamiltonian mechanics. Each of the three mappings acts on a two-dimensional manifold with constant curvature, defining one of three different phase space geometries: spherical (positive curvature), planar (zero curvature) or hyperbolic (negative curvature). In each case an infinite number of stable periodic orbits coexist and seem to 'pack' the manifold with circular resonances. Some analytical data has been found for these maps including the location of a large number of periodic orbits, but there remain many open questions, the most critical being whether the circular resonances densely pack the phase space. That is, whether the residual set not covered by the resonances has empty interior. If this is the case then we have indeed found new examples of circle packings. In sections II, III, and IV we introduce, respectively, the planar, hyperbolic and spherical mappings, and include all analytical findings. Finally, in Section V we discuss some open questions concerning these maps.

II. PLANAR MAP

The Hamiltonian which generates our mapping on the Euclidean plane is

$$H(x, p, t) = \frac{1}{2}\omega(x^2 + p^2) + \mu|x| \sum_{n=-\infty}^{\infty} \delta(t - n),$$

*Electronic address: ajs@maths.uq.edu.au

where x, p and t are position, momentum and time respectively, and, $\omega \in [0, 2\pi)$ and $\mu \geq 0$ are parameters. In the planar case μ may be set to unity by rescaling x and p. The mapping which takes (x, p) from just before a kick to one period later is

$$\begin{bmatrix} x^{n+1} \\ p^{n+1} \end{bmatrix} = F \begin{bmatrix} x^n \\ p^n \end{bmatrix} = \begin{bmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{bmatrix} \begin{bmatrix} x^n \\ p^n - \mu s^n \end{bmatrix} \tag{1}$$

where $s^n \equiv \operatorname{sgn} x^n$ ($\operatorname{sgn} x$ is the signum function with the convention $\operatorname{sgn} 0 = 0$). Between kicks the Hamiltonian is that of a simple harmonic oscillator and all phase space points rotate clockwise about the origin through an angle of ω. The effect of the kick is to add a position dependent shift in the momentum of $-\mu \operatorname{sgn} x$. If $x \neq 0$ the tangent mapping, $\partial F(x, p)/\partial(x, p)$, is simply a linear rotation with eigenvalues $e^{\pm i\omega}$. Thus if an orbit is to have nonzero Lyapunov exponents, such as an unstable periodic orbit, we need at least one iterate of the orbit on the line $x = 0$. However, in this instance the tangent mapping is undefined. We overcome this problem simply by defining an unstable orbit to be one with a point on $x = 0$, and all other orbits stable.

If we let $z = x + ip$ then we can rewrite the mapping as

$$\begin{aligned} z^{n+1} &= F z^n \\ &= e^{-i\omega}(z^n - i\mu s^n). \end{aligned}$$

Suppose that z^0 is a periodic point with period n. That is,

$$\begin{aligned} z^0 &= z^n \\ &= e^{-in\omega} z^0 - i\mu \sum_{k=0}^{n-1} e^{i(k-n)\omega} s^k. \end{aligned}$$

Solving for z^0 we obtain

$$z^0 = \frac{i\mu}{1 - e^{in\omega}} \sum_{k=0}^{n-1} e^{ik\omega} s^k. \tag{2}$$

If z^0 is a stable periodic point then $s^k = \pm 1$, which simply states whether the iterate z^k is on the right or left side of the complex plane. The unstable periodic orbits must have at least one point on $x = 0$. That is, $s^k = 0$ for some $k = 0 \ldots n - 1$. Note that if $n\omega$ is an integer multiple of 2π then equation (2) is invalid. In these cases one finds that there are an infinite number of period n orbits. But when this is not the case, each periodic orbit is uniquely determined by the sequences $\{s^k = 0, \pm 1\}_{k=0 \ldots n-1}$, with at most two s^k being zero (see the Lemma below). By cycling a particular sequence we obtain the n iterates of the periodic orbit. Although each periodic orbit is uniquely represented by a sequence, not every sequence represents a periodic orbit. Hence, we still need to find which sequences are legitimate. This task can be simplified by noting that F can be decomposed into the two involutions I_1 and I_2, where

$$\begin{aligned} I_1 z &= -z^* - i\mu \operatorname{sgn} x \\ I_2 z &= -e^{-i\omega} z^* \\ I_1^2 &= I_2^2 = 1 \\ I_1 F I_1 &= I_2 F I_2 = F^{-1} \end{aligned}$$

and

$$F = I_2 \circ I_1.$$

It is well-known that whenever a mapping admits such a decomposition there exist symmetric periodic orbits having points which lie on the fixed lines of each involution [7]. The fixed line of an involution I is the set

$$\text{Fix}\,(I) \equiv \{z | I z = z\}.$$

Theorem [8]: Let F be an invertible mapping and I an involution ($I^2 = 1$) satisfying $IFI = F^{-1}$. Let $z \in \text{Fix}\,(I)$ and suppose that z is a periodic point of F with least period n. Then $\text{Fix}\,(I)$ contains no other points of the periodic orbit if n is odd, or exactly one other, $F^{n/2}z$, if n is even.

Consider the unstable periodic orbits. From our definition, an unstable orbit must have at least one point lying on $x = 0$. Noting that $\text{Fix}\,(I_1) = \{x + ip | x = 0\}$ and putting $I = I_1$ in the above theorem we obtain the following Lemma:

Lemma : Let $\{s^k\}_{k=0\ldots n-1}$ be the sequence defining an unstable periodic orbit of least period n. If n is odd then the sequence contains exactly one zero, $s^0 = 0$ say, and if n is even the sequence contains exactly two zeros, $s^0 = 0$ and $s^{n/2} = 0$.

We now know that all unstable periodic orbits have one or two points on the fixed line of the involution I_1 ($x = 0$). Many of the stable periodic orbits have points on the fixed line of the second involution I_2, which is given by the equation $x = p \tan \omega/2$. In this case one can also show that the orbits with odd period have one point on this line while the orbits with even period have two. One cannot show, however, that the stable periodic orbits must have point on this line, indeed, some do not. There is a second decomposition of the mapping, $F = \tilde{I}_2 \circ \tilde{I}_1$, where $\tilde{I}_1 = -I_1$ has the fixed line $p = \mu/2 \,\text{sgn}\, x$ and $\tilde{I}_2 = -I_2$ has the fixed line $p = -x \tan \omega/2$. We are unsure as to whether every stable periodic orbit has a point on one of the above fixed lines. However, numerical investigations seem to suggest that they do. For the case $\omega = \pi(\sqrt{5} - 1)$ we have checked all $2^{41} - 2$ of the possible sequences for a stable periodic orbit of period ≤ 40 and found only those with points on these lines. Hence we tentatively conjecture that all stable periodic orbits have one or two points on at least one of the fixed lines of I_1, $\tilde{I}_1$ and $\tilde{I}_2$.

Proposition : If $\{s^k\}_{k=0\ldots n-1}$ is the sequence defining a periodic orbit of least period n with z^0 being the first point, then

1. $s^k = -s^{n-k}$ if $z^0 \in \text{Fix}\,(I_1)$,

2. $s^k = -s^{n-k-1}$ if $z^0 \in \text{Fix}\,(I_2)$,

3. $s^k = s^{n-k}$ if $z^0 \in \text{Fix}\,(\tilde{I}_1)$,

4. $s^k = s^{n-k-1}$ if $z^0 \in \text{Fix}\,(\tilde{I}_2)$,

for $k = 1 \ldots n - 1$.

Proof : We will only prove the first case. The others are similar. We know that $F^n z^0 = z^0$ and $I_1 z^0 = z^0$, since $z^0 \in \text{Fix}\,(I_1)$. The iterate z^k, $1 \leq k < n$, is another point of the

periodic orbit and

$$
\begin{aligned}
I_1 z^k &= I_1 F^k z^0 \\
&= I_1 I_1 F^{-k} I_1 z^0 \\
&= F^{-k} z^0 \\
&= F^{n-k} z^0 \\
&= z^{n-k}.
\end{aligned}
$$

Thus $I_1 z^k$ is also a point of the periodic orbit, and since I_1 changes the sign of x the above equation gives the formula $s^k = -s^{n-k}$. $\square$

The phase space portrait for $\omega = \pi(\sqrt{5}-1)$ and $\mu = 1$ is shown in Fig. 1. The unstable orbits are in black and consist of unstable periodic and aperiodic motion. They were generated by plotting all images and preimages of the vertical line $x = 0$. The circular discs are resonances and are packed entirely with stable orbits. At the center of each of these lies a stable periodic orbit. The positions of a large number of periodic orbits have been found analytically. For want of a better term, we will call these 'first order' periodic orbits. All of the resonances visible in Fig. 1 are of first order. Only on greater levels of magnification does one encounter resonances of higher order.

The planar mapping has a stable first order periodic orbit of period n if there exists an integer m such that $1 \leq m \leq n$, $\gcd(n, m) = 1$, and

$$
\frac{m-1}{n} < \frac{\omega}{2\pi} < \frac{m}{n} \quad \text{if } n \text{ is even,}
$$

or

$$
\frac{m-1/2}{n} < \frac{\omega}{2\pi} < \frac{m}{n} \quad \text{if } n \text{ is odd.}
$$

These orbits are born near the origin for the smaller value of ω and then are destroyed at infinity when $\omega/2\pi = m/n$. The sequences defining these orbits are

$$
s^k = \text{sgn}\left(\sin 2\pi \frac{km + 1/4}{n}\right) \tag{3}
$$

$k = 0 \ldots n - 1$. The n iterates of each periodic orbit are found by inserting the n cycles of the corresponding sequence into (2). The orbits with odd period come in pairs. The second of each pair is found by negating every term of the sequence. All these orbits, apart from those of period 1 and 2, are created with a corresponding unstable periodic orbit. These unstable orbits exist for the same ω as the stable and have sequences

$$
s^k = \text{sgn}\left(\sin 2\pi \frac{km}{n}\right). \tag{4}
$$

There is also an unstable period 1 orbit at the origin which exists for all ω.

The first order periodic orbits were found by observing that at infinity the kick has no effect and the map is simply a linear rotation through an angle of ω. Consequently, when $\omega/2\pi = m/n$ there will be period n orbits at infinity and these orbits must have sequences given by (3) or (4), depending on their stability. Further investigation reveals that they were born near the origin at a smaller value of ω when points of the stable periodic orbit

intersected the line $x = 0$. Hence one can use (2) to find out when this occurred and obtain the above results.

All these orbits revolve around the origin under iteration, and in general, the orbits for which m/n is larger are closer to the origin since they are destroyed at infinity for a greater value of ω. In Fig. 2 and Fig. 3 we have plotted, respectively, the positions of all stable and unstable first order periodic orbits. The values of ω and μ are the same as in Fig. 1. The radii of the circular resonances enclosing each of the stable first order periodic orbits can also been found. This may be done by noting that every stable periodic orbit must have an iterate with its circular resonance tangent to the line $x = 0$. The radii is then given by the x position of that iterate. These reduce to

$$r_n = \frac{1}{2} \tan \frac{\omega n}{4} \quad \text{if } n \text{ is even,}$$

or

$$r_n = \frac{1}{2} \tan \left(\frac{\omega n}{2} + \frac{\pi}{2} \right) \quad \text{if } n \text{ is odd.}$$

All iterates of an orbit inside a resonance can be written as

$$z^k = z_p^{k \bmod n} + (z^0 - z_p^0)e^{-ik\omega} \tag{5}$$

where z_p^0 is the period n point at the center of the resonance containing z^0, and

$$|z^0 - z_p^0| < r_n.$$

If $\omega/2\pi$ is irrational then the orbit iterates in a quasiperiodic circular motion about the central periodic orbit. However, when $\omega/2\pi$ is rational, that is, $\omega/2\pi = p/q$ where p and q are integers with $\gcd(p, q) = 1$, then $z^{nq} = z^0$. Hence the orbit is periodic. In this case each image of the vertical line $x = 0$ is a collection of parallel line segments at one of only q different possible angles. Consequently, the resonances are all polygons which tile the phase plane. The phase space portrait for $\omega/2\pi = 5/8$ and $\mu = 1$ is shown in Fig. 4. Here the phase plane is tiled with octagons of decreasing size. In Fig. 5 where $\omega/2\pi = 3/5$ the phase plane is tiled with pentagons and decagons.

III.   HYPERBOLIC MAP

The hyperbolic map is derived from the Hamiltonian

$$H(\mathbf{K}, t) = \omega K_3 + \mu |K_1| \sum_{n=-\infty}^{\infty} \delta(t - n),$$

where $\mathbf{K} = (K_1, K_2, K_3) = (-x_2 p_3 - x_3 p_2, x_3 p_1 + x_1 p_3, x_1 p_2 - x_2 p_1)$ is the Minkowski 3-vector for a particle confined to a pseudosphere[9], normalized such that $\mathbf{K}$ lies on the hyperboloid

$$1 + K_1{}^2 + K_2{}^2 = K_3{}^2, \quad K_3 > 0. \tag{6}$$

The evolution of $\mathbf{K}$ under the above Hamiltonian is governed by the equations

$$\dot{K}_i = \{K_i, H\}, \quad \{K_1, K_2\} = -K_3, \quad \{K_2, K_3\} = K_1, \quad \{K_3, K_1\} = K_2,$$

where $\{\cdot,\cdot\}$ are the Poisson brackets. The mapping which takes $\mathbf{K}$ from just before a kick to one period later is

$$
\begin{bmatrix} K_1^{n+1} \\ K_2^{n+1} \\ K_3^{n+1} \end{bmatrix} = F \begin{bmatrix} K_1^n \\ K_2^n \\ K_3^n \end{bmatrix} = \begin{bmatrix} \cos\omega & -\sin\omega & 0 \\ \sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh\mu s^n & \sinh\mu s^n \\ 0 & \sinh\mu s^n & \cosh\mu s^n \end{bmatrix} \begin{bmatrix} K_1^n \\ K_2^n \\ K_3^n \end{bmatrix}
$$
$$
\equiv \mathrm{F}(s^n)\mathbf{K}^n
$$

where $s^n \equiv \mathrm{sgn}\, K_1^n$.

The unstable orbits are defined to be those with at least one point with $K_1 = 0$. In a similar fashion to the planar case we can label all periodic orbits with sequences $\{s^k = 0, \pm 1\}_{k=0\ldots n-1}$. Then the position of the periodic point must be the solution of

$$
\mathbf{K} = \mathrm{F}(s^{n-1})\mathrm{F}(s^{n-2})\ldots\mathrm{F}(s^0)\mathbf{K}, \tag{7}
$$

which lies on the hyperboloid (6). By using the previous theorem and decomposing F into the two involutions,

$$
I_1\mathbf{K} = (-K_1, K_2\cosh(\mu\,\mathrm{sgn}\,K_1) + K_3\sinh(\mu\,\mathrm{sgn}\,K_1), K_2\sinh(\mu\,\mathrm{sgn}\,K_1) + K_3\cosh(\mu\,\mathrm{sgn}\,K_1)),
$$
$$
I_2\mathbf{K} = (-K_1\cos\omega - K_2\sin\omega, -K_1\sin\omega + K_2\cos\omega, K_3),
$$

where $F = I_2 \circ I_1$, we can show that the lemma and proposition in the previous section are also true in the hyperbolic case. The second pair of involutions for the hyperbolic map are

$$
\tilde{I}_1\mathbf{K} = (K_1, -K_2\cosh(\mu\,\mathrm{sgn}\,K_1) - K_3\sinh(\mu\,\mathrm{sgn}\,K_1), K_2\sinh(\mu\,\mathrm{sgn}\,K_1) + K_3\cosh(\mu\,\mathrm{sgn}\,K_1)),
$$
$$
\tilde{I}_2\mathbf{K} = (K_1\cos\omega + K_2\sin\omega, K_1\sin\omega - K_2\cos\omega, K_3),
$$

with $F = \tilde{I}_2 \circ \tilde{I}_1$. Hence the mapping has symmetry lines on the hyperboloid (6) of $K_1 = 0$, $K_1 = -K_2\tan\omega/2$, $K_2 = -K_3\tanh(\mu/2\,\mathrm{sgn}\,K_1)$, and $K_2 = K_1\tan\omega/2$ which are the fixed lines of the involutions I_1, I_2, $\tilde{I}_1$, and $\tilde{I}_2$, respectively.

To display the hyperbolic phase space we will use the conformal disk model [10] where all angles are Euclidean. Hence all circles will look like Euclidean circles, but some will appear larger than others of the same periodic orbit because the model distorts distances. The mapping

$$
x = \frac{K_1}{1 + K_3}, \quad y = \frac{K_2}{1 + K_3},
$$

takes all points on the hyperboloid (6) into the disk $x^2 + y^2 < 1$ in the Euclidean plane. The phase space portraits for $\omega = \pi(\sqrt{5} - 1)$, $\mu = 0.04$ and $\mu = 0.2$ are shown in Fig. 6 and Fig. 7, respectively. On comparing Fig. 6 with Fig. 1 we can see that the hyperbolic map resembles the planar map when μ is small. In fact, under a suitable transformation of variables the hyperbolic map can be approximated by the planar map when close to the origin and μ is small enough. Consequently, the first order periodic orbits of the hyperbolic map exist for the same ω as in the planar map with the exception that they are destroyed at infinity ($x^2 + y^2 = 1$) at a smaller value of ω dependent on μ. The hyperbolic mapping has a stable first order periodic orbit of period n if there exists an integer m such that $1 \le m \le n$, $\gcd(n, m) = 1$, and

$$
\frac{m-1}{n} < \frac{\omega}{2\pi} < f_{mn}(\mu) \le \frac{m}{n} \quad \text{if } n \text{ is even,}
$$

or

$$\frac{m-1/2}{n} < \frac{\omega}{2\pi} < f_{mn}(\mu) \leq \frac{m}{n} \quad \text{if } n \text{ is odd.}$$

The sequences defining these orbits are given by (3), and the k-th iterate is found by inserting the k-th cycle of the sequence into (7) and solving on the hyperboloid (6). Again, the orbits with odd period come in pairs. The second of each pair is found by negating the sequence. The corresponding unstable orbits have sequences given by (4). The functions $f_{mn}(\mu)$ could not be found analytically except in the special cases of $(m, n) = (1, 1)$ where

$$f_{11}(\mu) = 1 - \frac{1}{\pi} \arcsin\left(\tanh\frac{\mu}{2}\right),$$

and $(m, n) = (1, 2), (1, 4), (3, 4)$ where

$$f_{mn}(\mu) = \frac{m}{n} - \frac{1}{\pi} \arcsin\left(\sin\frac{\pi}{n}\tanh\frac{\mu}{2}\right).$$

These can be found by explicitly solving (7) for the position of each periodic orbit. A much simpler way to find these orbits of low period is to make use of the fact that some iterates will lie on the symmetry lines. Although the functions $f_{mn}(\mu)$ are generally unknown, the task of finding all first order periodic orbits is still quite simple. One simply replaces the inequality $\omega/2\pi < f_{mn}(\mu)$ with the weaker inequality $\omega/2\pi < m/n$ together with the condition that the nontrivial solutions of (7) (with (3) or (4)) have $K_3{}^2 - K_1{}^2 - K_2{}^2 = t^2 > 0$ (the case of $f_{mn}(\mu) < \omega/2\pi < m/n$ corresponds to $K_3{}^2 - K_1{}^2 - K_2{}^2 = -t^2$).

In the previous section we found that when $\omega/2\pi$ is rational the phase plane of the planar map is entirely filled with polygons. This does not occur in the hyperbolic case. The tangent mapping to the planar map is the same linear rotation matrix for both the left and right half of the phase plane. Consequently, the local rotation about every stable periodic orbit is $-\omega$ (see Eq. (5)), and if $\omega/2\pi$ is rational then this rotation is itself periodic. The hyperbolic map, however, has different tangent mappings for the left and right half of the hyperbolic plane. Hence the local rotation about each stable periodic orbit will be different. However we can still choose ω and μ such that the rotation about one particular orbit is periodic. When this occurs we find that each resonance of the periodic orbit forms a polygon. For example, by choosing $\mu = 0.5$ and setting $\omega = \pi + \arccos\tanh^2\mu/2 = 4.652..$ we find that the resonances enclosing each of the two period 1 orbits are hyperbolic squares (see Fig. 8). Each point inside these squares is a period 4 orbit, rotating clockwise exactly $\pi/2$ radians at each iteration.

IV. SPHERICAL MAP

The spherical map is given by the Hamiltonian

$$H(\mathbf{J}, t) = \omega J_3 + \mu|J_1| \sum_{n=-\infty}^{\infty} \delta(t - n),$$

where $(\mathbf{J})_i = J_i = \epsilon_{ijk}x_j p_k$ $(i = 1, 2, 3)$ are the three components of angular momentum for a particle confined to a sphere, normalized such that

$$J_1{}^2 + J_2{}^2 + J_3{}^2 = 1. \tag{8}$$

The evolution of **J** under the above Hamiltonian is governed by the equations

$$\dot{J}_i = \{J_i, H\}, \quad \{J_i, J_j\} = \epsilon_{ijk} J_k.$$

The mapping which takes **J** from just before a kick to one period later is

$$\mathbf{J}^{n+1} = \begin{bmatrix} J_1^{n+1} \\ J_2^{n+1} \\ J_3^{n+1} \end{bmatrix} = \begin{bmatrix} \cos\omega & -\sin\omega & 0 \\ \sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\mu s^n & -\sin\mu s^n \\ 0 & \sin\mu s^n & \cos\mu s^n \end{bmatrix} \begin{bmatrix} J_1^n \\ J_2^n \\ J_3^n \end{bmatrix} \tag{9}$$
$$\equiv \mathrm{F}(s^n)\mathbf{J}^n$$

where $s^n \equiv \mathrm{sgn}\, J_1^n$.

The unstable orbits are defined to be those with at least one point for which $J_1 = 0$. Again, we can label all periodic orbits with sequences $\{s^k = 0, \pm 1\}_{k=0\ldots n-1}$. Then the position of the periodic point must be the solution of

$$\mathbf{J} = \mathrm{F}(s^{n-1})\mathrm{F}(s^{n-2})\ldots \mathrm{F}(s^0)\mathbf{J}, \tag{10}$$

on the unit sphere (8) with $\mathrm{sgn}\, J_1 = s^0$. By decomposing F into the two involutions,

$$I_1\mathbf{J} = (-J_1, J_2\cos(\mu\,\mathrm{sgn}\, J_1) - J_3\sin(\mu\,\mathrm{sgn}\, J_1), J_2\sin(\mu\,\mathrm{sgn}\, J_1) + J_3\cos(\mu\,\mathrm{sgn}\, J_1)),$$
$$I_2\mathbf{J} = (-J_1\cos\omega - J_2\sin\omega, -J_1\sin\omega + J_2\cos\omega, J_3),$$

where $F = I_2 \circ I_1$, we can show that the lemma and proposition of Section II hold in this case. The second pair of involutions for the spherical map are

$$\tilde{I}_1\mathbf{J} = (J_1, -J_2\cos(\mu\,\mathrm{sgn}\, J_1) + J_3\sin(\mu\,\mathrm{sgn}\, J_1), J_2\sin(\mu\,\mathrm{sgn}\, J_1) + J_3\cos(\mu\,\mathrm{sgn}\, J_1)),$$
$$\tilde{I}_2\mathbf{J} = (J_1\cos\omega + J_2\sin\omega, J_1\sin\omega - J_2\cos\omega, J_3),$$

with $F = \tilde{I}_2 \circ \tilde{I}_1$. Hence the mapping has symmetry lines on the unit sphere (8) of $J_1 = 0$, $J_1 = -J_2\tan\omega/2$, $J_2 = J_3\tan(\mu/2\,\mathrm{sgn}\, J_1)$, and $J_2 = J_1\tan\omega/2$ which are the fixed lines of the involutions I_1, I_2, $\tilde{I}_1$, and $\tilde{I}_2$, respectively.

We will only display the eastern hemisphere ($J_1 > 0$) of the spherical phase space. Under the transformation $(J_1, J_2, J_3) \to (-J_1, -J_2, J_3)$ the mapping (9) remains invariant, hence the phase space structure in the the western hemisphere will be symmetrical to the eastern under this reflection. The mapping

$$x = \frac{J_2}{1 + J_1}, \quad y = \frac{J_3}{1 + J_1},$$

takes all points on the eastern hemisphere into the disk $x^2 + y^2 < 1$ in the Euclidean plane. All circles in the spherical phase space are transformed to Euclidean circles, but some will appear larger than others of the same periodic orbit because the mapping distorts distances. The phase space portraits for $\omega = \pi(\sqrt{5} - 1)$, $\mu = 0.02$ and $\mu = \pi(\sqrt{5} - 1)$ are shown in Fig. 9 and Fig. 10, respectively. For small μ the southern hemisphere can be approximated by the planar map, while the northern hemisphere can be approximated by the planar map under the transformation $\omega \to 2\pi - \omega$ (compare Fig. 9 with Fig. 1). Hence we would expect there to be a stable first order periodic orbit of period n if there exists an integer m such that $1 \le m \le n$, $\gcd(n, m) = 1$, and

$$\frac{m-1}{n} < \frac{\omega}{2\pi} < \frac{m}{n} \quad \text{or} \quad \frac{m-1}{n} < 1 - \frac{\omega}{2\pi} < \frac{m}{n} \quad \text{if } n \text{ is even,}$$

or

$$\frac{m-1/2}{n} < \frac{\omega}{2\pi} < \frac{m}{n} \quad \text{or} \quad \frac{m-1/2}{n} < 1 - \frac{\omega}{2\pi} < \frac{m}{n} \quad \text{if } n \text{ is odd,}$$

together with some condition on $\mu \in [0, 2\pi)$. Again, the sequences defining these orbits are given by (3), while the corresponding unstable orbits have sequences given by (4). The position of each periodic orbit is found by solving (10). The condition on μ for the existence of each periodic orbit could not be found analytically except in the special cases of $n = 1, 2, 3, 4, 6$. The orbits of period $n = 1, 2, 4$ exist for all μ (and ω). The period 3 orbits exist for

$$\cos \mu > \frac{\cos \omega}{1 - \cos \omega}$$

while the period 6 orbits exist for

$$\cos \mu > \frac{- \cos \omega}{1 + \cos \omega}.$$

For orbits of higher period one can simply just ignore any condition imposed on μ. After solving (10) we then need to check whether we have actually found a periodic orbit. All first order periodic orbits can be found in this manner. As for the hyperbolic mapping, some choices of ω and μ will produce polygonal resonances.

V.   DISCUSSION AND CONCLUSION

We are left with the question as to whether the resonances of the stable periodic orbits densely pack the phase space. Recall that if $\omega/2\pi$ was rational in the planar map then the phase plane was found to be tiled with polygons. There exist trivial examples of the tiling when $\omega/2\pi = 0, 1/2, 1/4, 3/4, 1/3, 2/3, 1/6, 5/6$. In these cases there are no periodic orbits of higher order. When $\omega/2\pi = 1/4, 3/4$ the phase plane is tiled with a grid of squares. A similar situation occurs when $\omega/2\pi = 1/3, 2/3, 1/6, 5/6$. Then the phase plane is tiled with triangles and hexagons. In all of these cases the unstable set (closure of the set of all images and preimages of the vertical line $x = 0$) has zero Lebesgue measure and is one dimensional. For other rationals something different occurs. When $\omega/2\pi = 1/8, 3/8, 5/8, 7/8$ the phase plane is tiled with octagons of decreasing size in a selfsimilar fashion (see Fig. 4). Using selfsimilarity one can show that the unstable set has zero measure with the Hausdorff dimension [11] of $\log 3 / \log(1 + \sqrt{2}) = 1.246...$ When $\omega/2\pi = 1/5, 2/5, 3/5, 4/5, 1/10, 3/10, 7/10, 9/10$ the phase plane is tiled with pentagons and decagons (see Fig. 5) and the unstable set has the Hausdorff dimension of $\log 6 / \log(2 + \sqrt{5}) = 1.241...$ For other rationals the phase plane is tiled with many different types of polygons in patterns of great complexity (see Fig. 11). We were unable to find the dimension of the unstable set for other rationals, but it would be safe to conjecture that they also have zero measure. Hence one could assume that whenever $\omega/2\pi$ is rational the unstable set has empty interior and the phase plane is densely packed with polygonal resonances. However when $\omega/2\pi$ is irrational we are uncertain as to whether the phase plane will be densely packed with circles. But this seems likely since any irrational can be approximated by a sequence of rationals. Hence a phase plane portrait for $\omega/2\pi$ irrational can be approximated by a sequence of dense polygonal packings. As we get closer to the irrational new smaller polygons are created and the number of sides of each existing polygon increases until they approximate circles. This argument is quite naive however and a mathematical proof is needed.

In the hyperbolic and spherical cases there are no polygonal packings and the above argument does not apply. Whether these mappings produce circle packings is also an open question. Consider the case when $\omega = \mu = \pi(\sqrt{5} - 1)$ in the spherical map (Fig. 10). Only the first order periodic orbits of periods 1, 2 and 4 exist. The largest circle is a resonance of a period 2 orbit, the second largest is of period 1, while the two smaller circles are of period 4. All smaller resonances are of higher order. Part of the phase portrait is shown in detail in Fig. 12. At this level of magnification one can see that the layout of the high order resonances is highly irregular and there seems to be no selfsimilarity. This is a great departure from the simple polygonal packings in Fig. 4 and Fig. 5 where the selfsimilarity is immediately apparent. One can also see that if the unstable set has zero measure then its fractal dimension would be extremely close to two. Indeed, in Fig. 12 the unstable set looks more likely to be a set of positive measure.

It is worthwhile to point out that phase space structures similar to that in the planar mapping can be produced in the sawtooth standard map

$$
\begin{aligned}
x^{n+1} &= x^n + k \operatorname{saw}(y^n) \\
y^{n+1} &= y^n + x^{n+1}
\end{aligned}
$$

if $k = -4\sin^2(\theta/2)$, where $(x, y) \in [0, 1)^2$ and $\operatorname{saw}(y) = y - \lfloor y \rfloor - 1/2$. Numerical investigations have led Ashwin [12] to conjecture that the closure of the set of images of the discontinuity ($y = 0$) has positive Lebesgue measure whenever θ/π is irrational. If this conjecture proves true we also expect it to hold for our planar map (when $\omega/2\pi$ is irrational) as the two mappings seem closely related. We should also point out that the polygonal tilings of the planar map also arise in 'polygonal dual billiards' [13, 14, 15].

Regardless of whether we have discovered new examples of circle packings, these deceptively simple mappings demonstrate that phase space structure of stunning complexity can arise in Hamiltonian dynamics. As we have shown, a large class of periodic orbits can be found. Perhaps some form of periodic orbit quantization might be realizable. If nothing else, their aesthetic beauty is an admirable quality.

[1] Bullet S and Mantica G 1992 Group theory of hyperbolic circle packings *Nonlinearity* **5** 1085-109.

[2] Bullet S and Penrose C 1999 Perturbing circle-packing Kleinian groups as correspondences *Nonlinearity* **12** 635-72.

[3] Herrmann H J, Mantica G and Bessis D 1990 Space-filling bearings *Phys. Rev. Lett.* **65** 3223-6.

[4] Parker J R 1995 Kleinian circle packings *Topology* **34** 489-96.

[5] Keen L, Maskit B and Series C 1993 Geometric finiteness and uniqueness for Kleinian groups with circle packing limit sets *J. Reine Angew. Math.* **436** 209-19.

[6] Maskit B 1988 *Kleinian Groups* (Berlin:Springer).

[7] Piña E and Jiménez Lara L 1987 On the symmetry lines of the standard mapping *Physica* **26D** 369-78.

[8] Lamb J S W and Roberts J A G 1998 Time-reversal symmetry in dynamical systems: A survey *Physica* **112D** 1-39.

[9] Balazs N L and Voros A 1986 Chaos on the pseudosphere *Physics Reports* **143** 109-240.

[10] Ratcliffe J G 1994 *Foundations of Hyperbolic Manifolds* (New York:Springer).

[11] Hausdorff 1918 Dimension und Außeres Maß *Math. Annalen.* **79** 157.

[12] Ashwin P 1997 Elliptic behaviour in the sawtooth standard map *Phys. Lett. A* **232** 409-16.

[13] Vivaldi F and Shaidenko A V 1987 Global stability of a class of discontinuous dual billiards *Commun. Math. Phys.* **110** 625-40.

[14] Tabachnikov S 1993 Dual billiards *Russ. Math. Surv.* **48** 75-102.

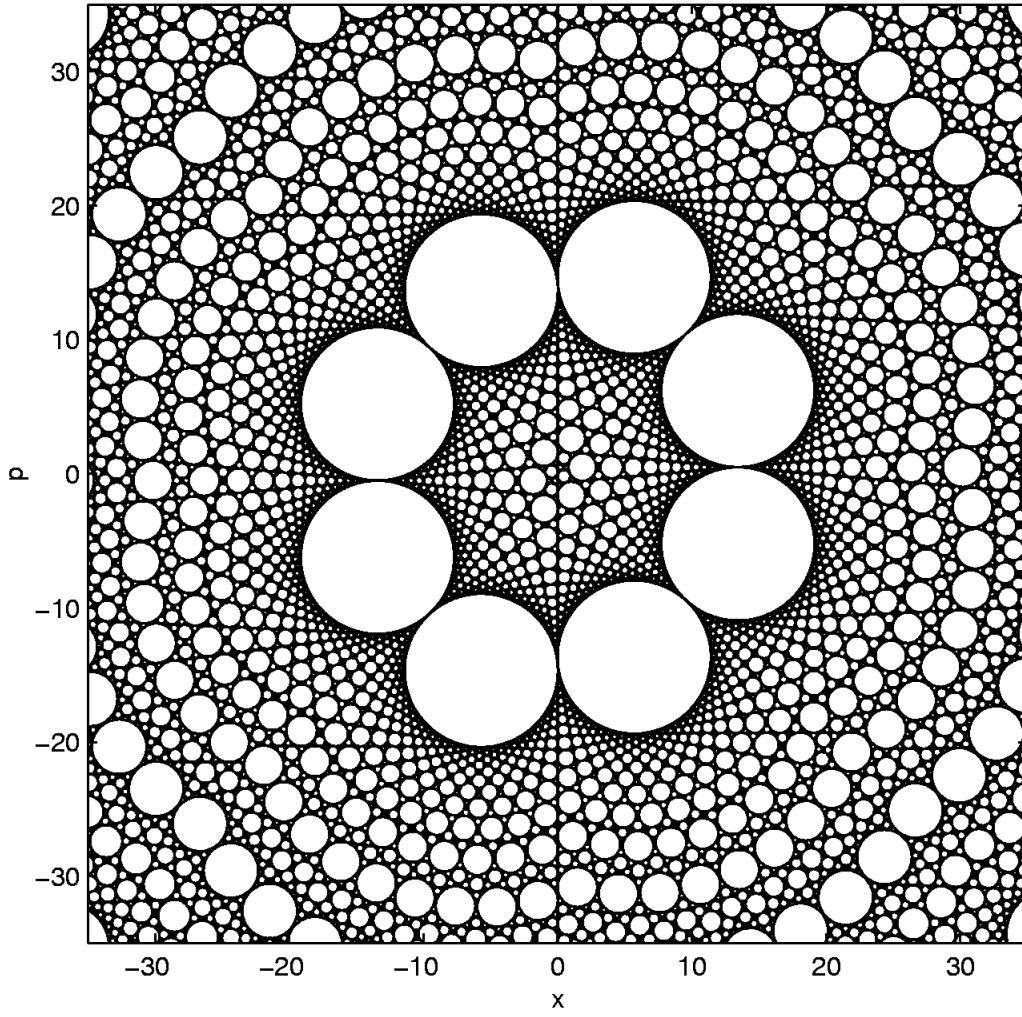[15] Tabachnikov S 1995 On the dual billiard problem *Adv. Math.* **115** 221-49.

FIG. 1: The phase space portrait for the planar map when $\omega = \pi(\sqrt{5} - 1)$ and $\mu = 1$.
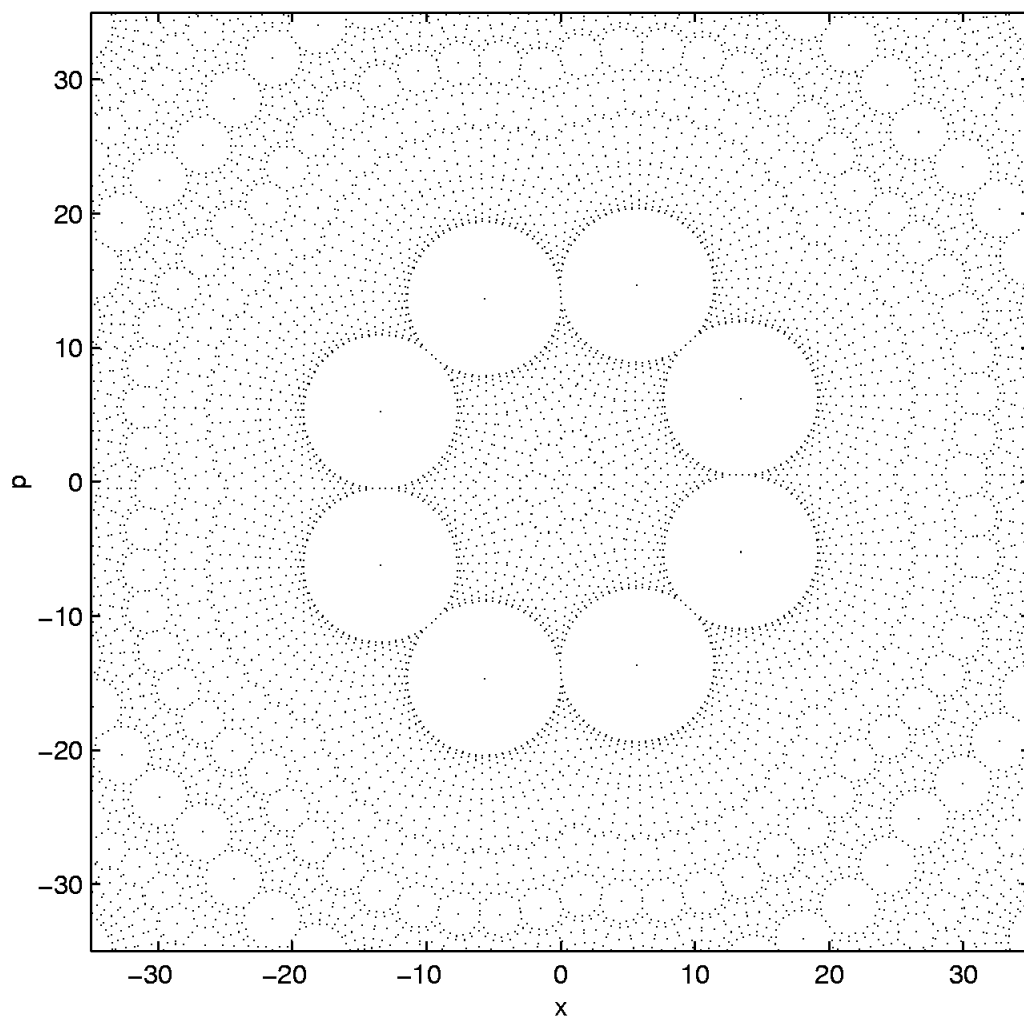
FIG. 2: The stable first order periodic orbits in Fig. 1.
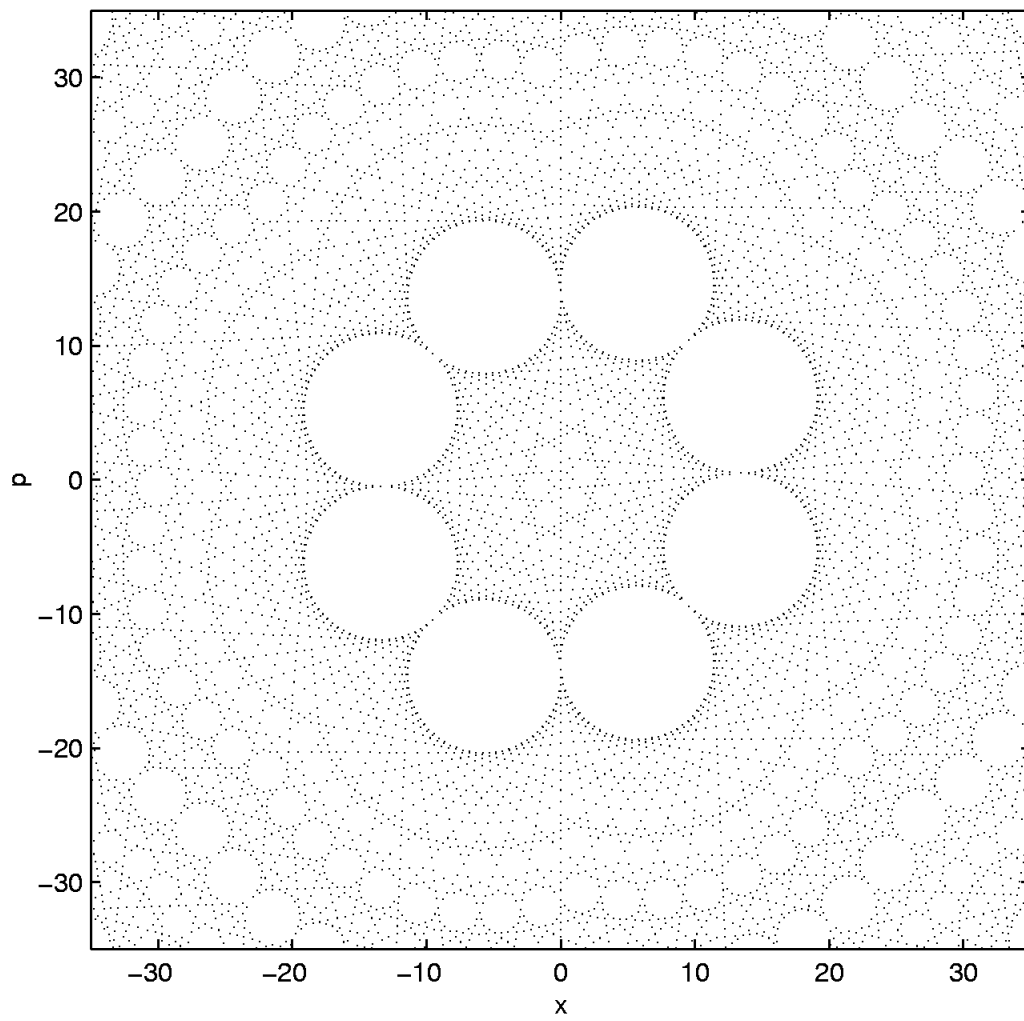
FIG. 3: The unstable first order periodic orbits in Fig. 1.
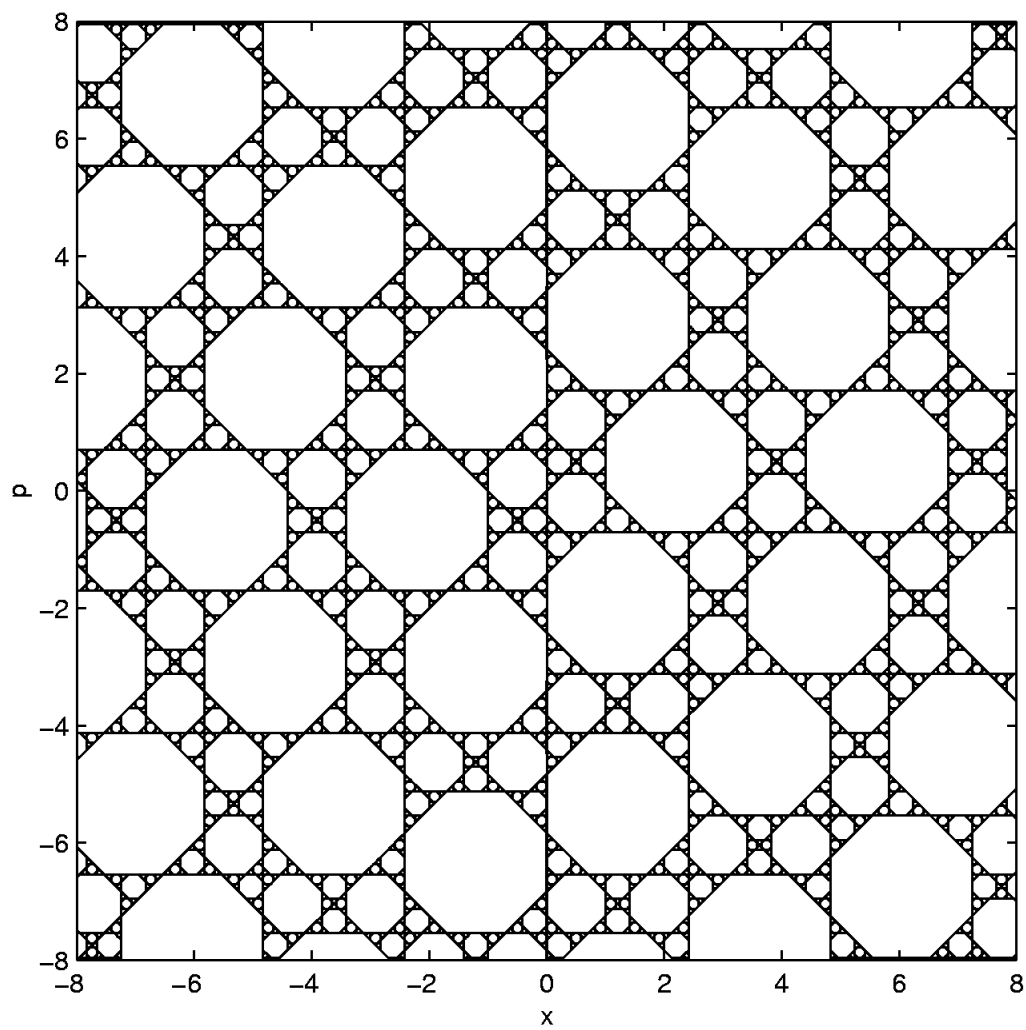
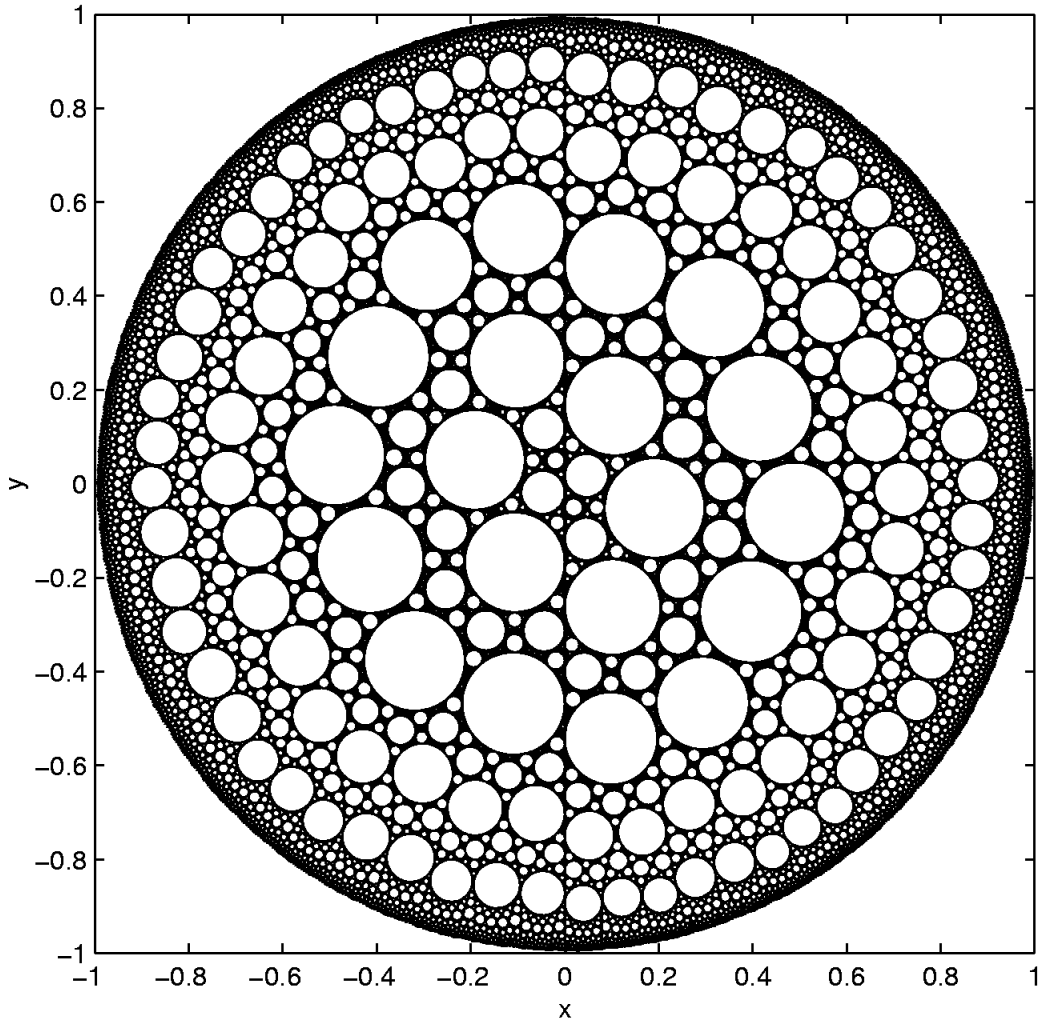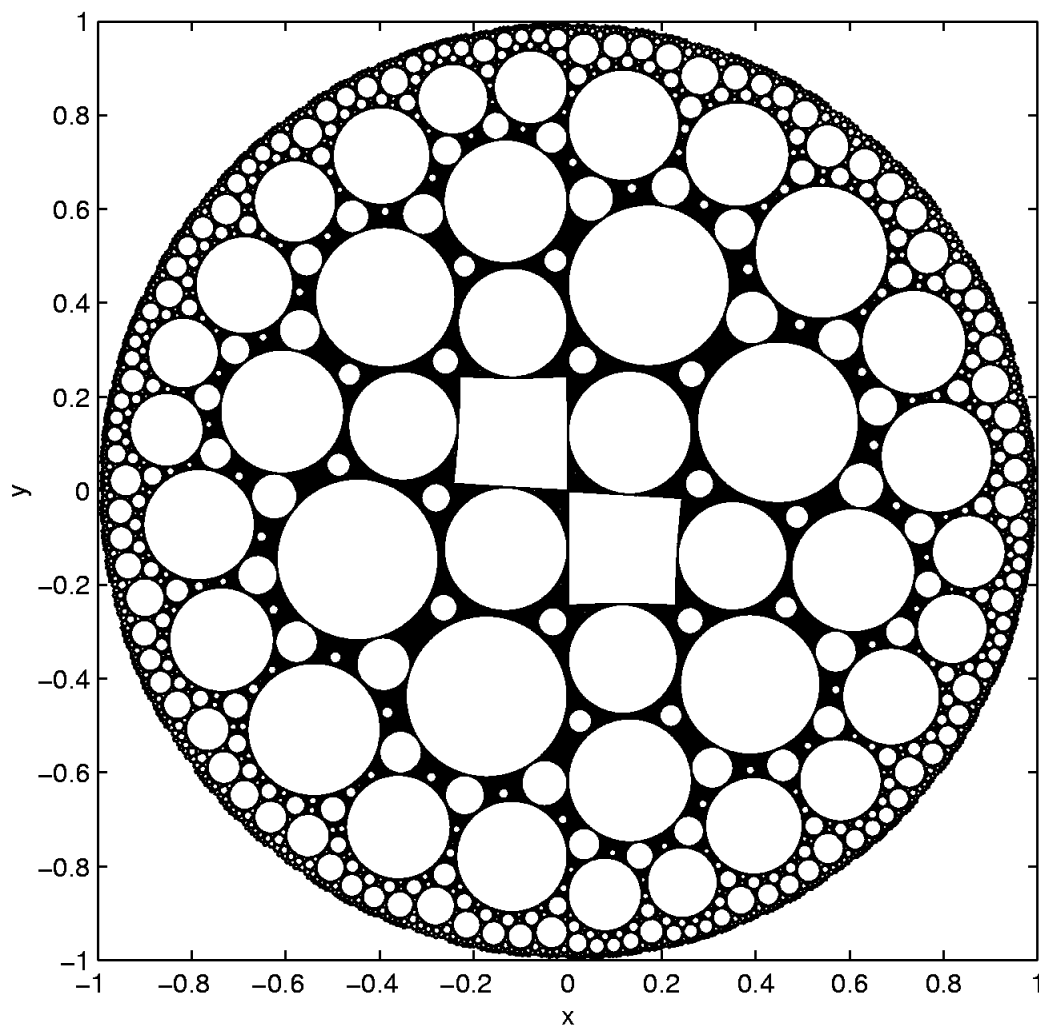FIG. 4: The phase space portrait for the planar map when $\omega/2\pi = 5/8$ and $\mu = 1$.
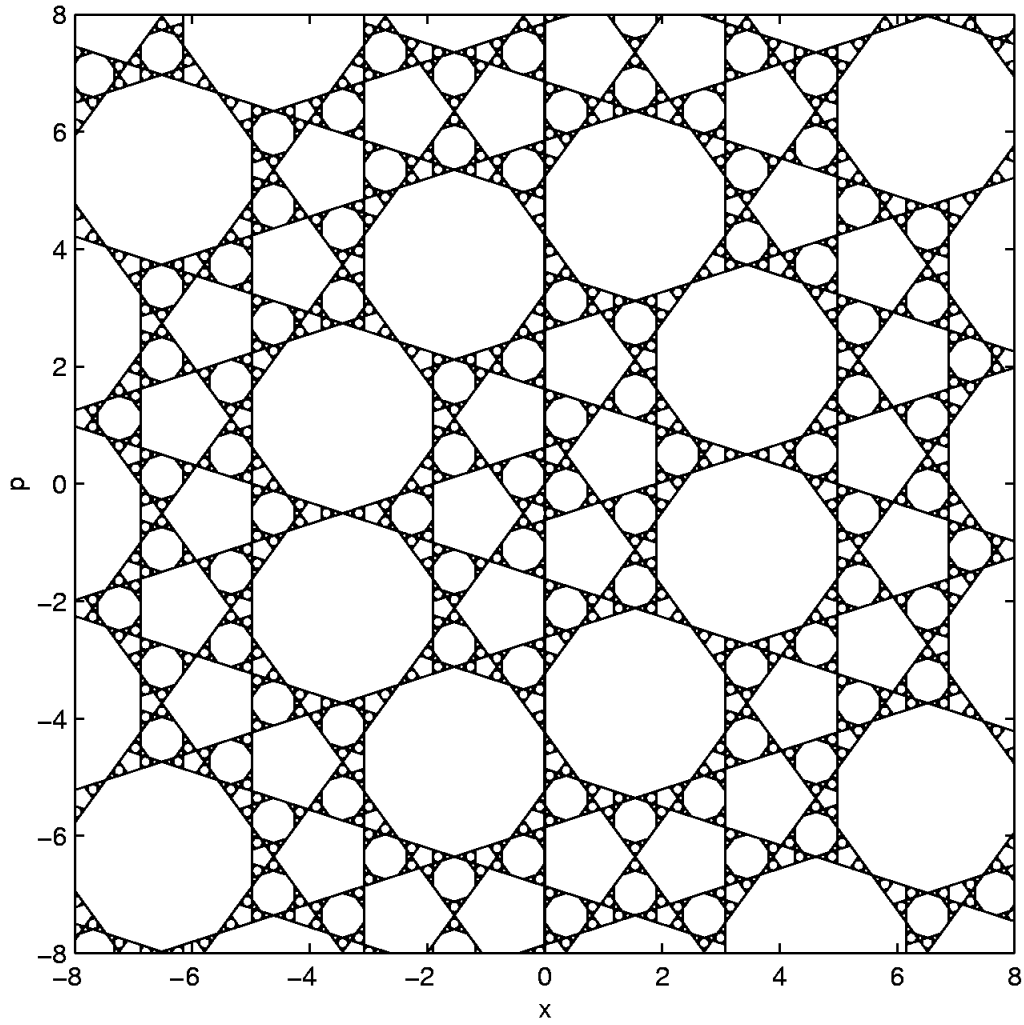
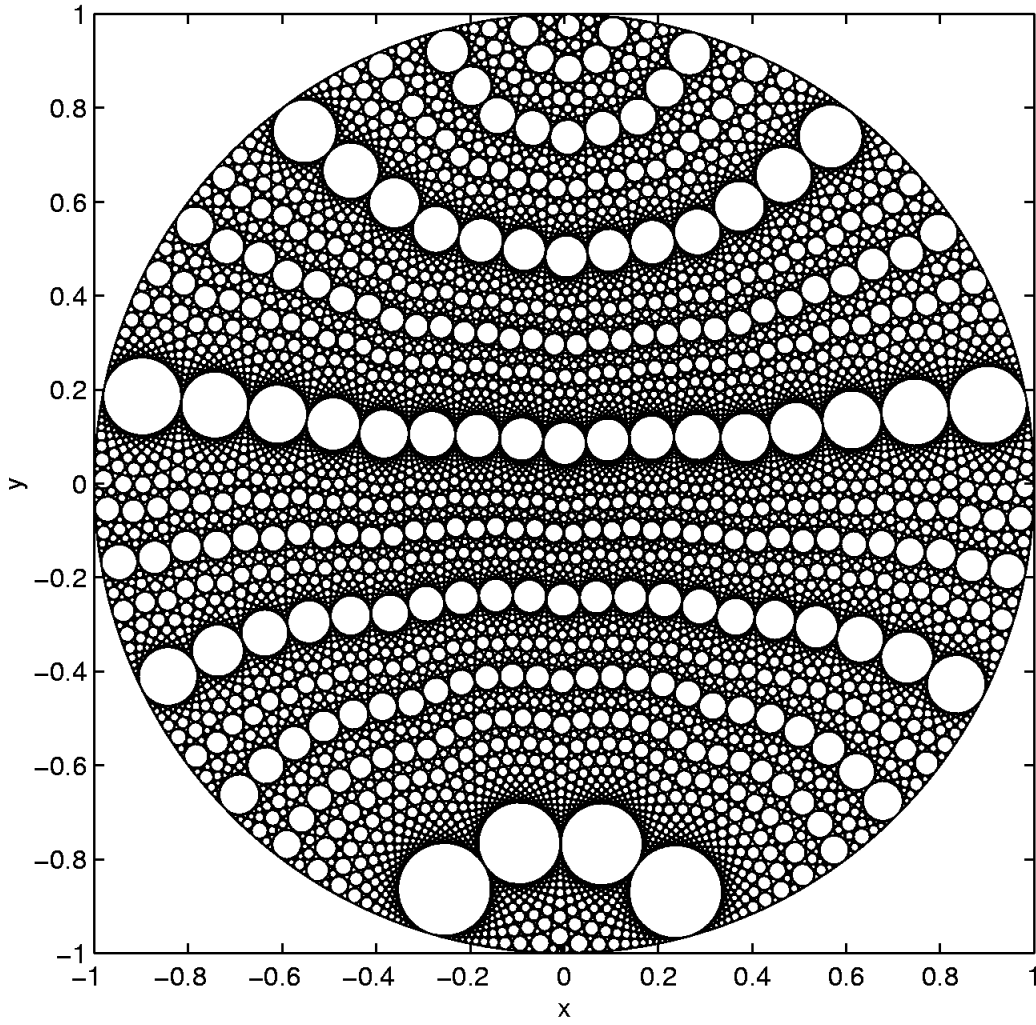FIG. 5: The phase space portrait for the planar map when $\omega/2\pi = 3/5$ and $\mu = 1$.
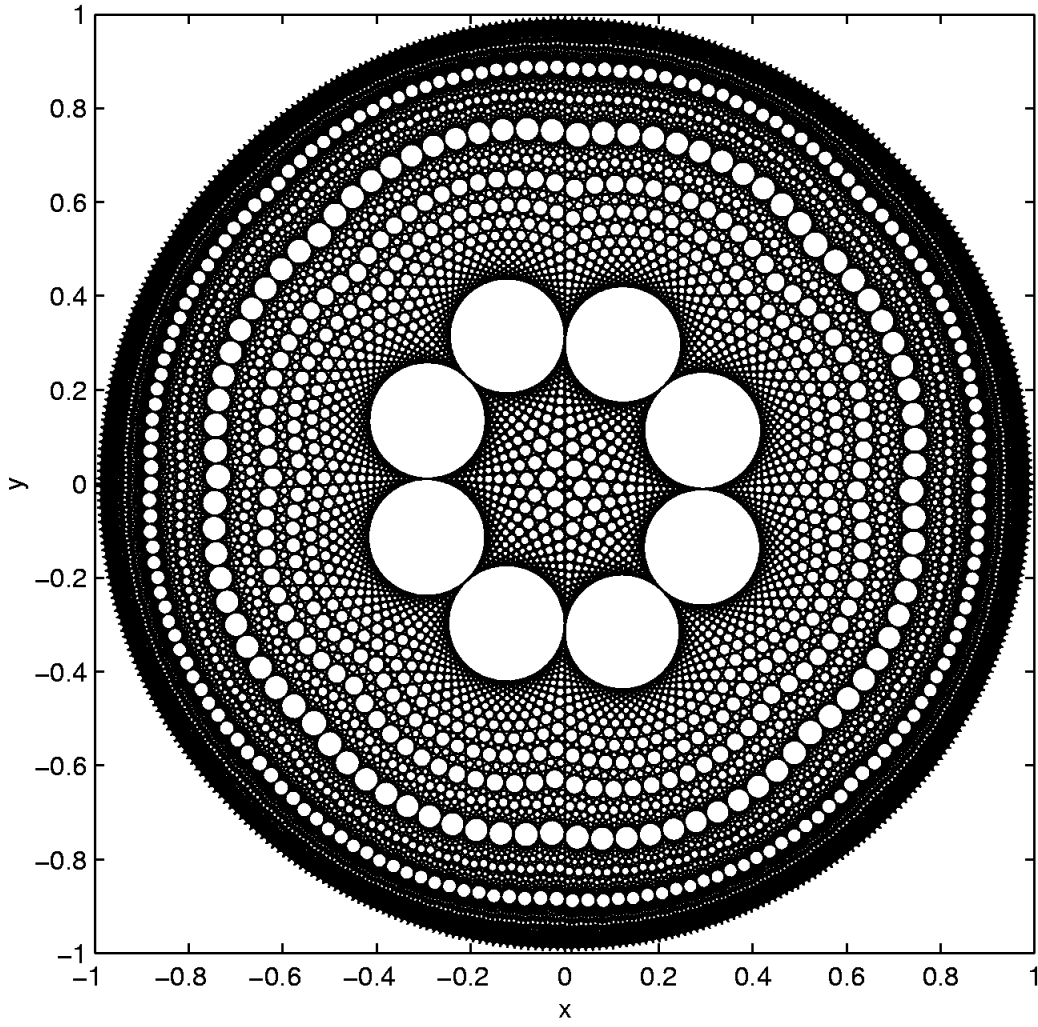
FIG. 6: The phase space portrait for the hyperbolic map when $\omega = \pi(\sqrt{5} - 1)$ and $\mu = 0.04$.

FIG. 7: The phase space portrait for the hyperbolic map when $\omega = \pi(\sqrt{5}-1)$ and $\mu = 0.2$.

FIG. 8: The phase space portrait for the hyperbolic map when $\omega = 4.652..$ and $\mu = 0.5$.

FIG. 9: The phase space portrait for the spherical map when $\omega = \pi(\sqrt{5} - 1)$ and $\mu = 0.02$.
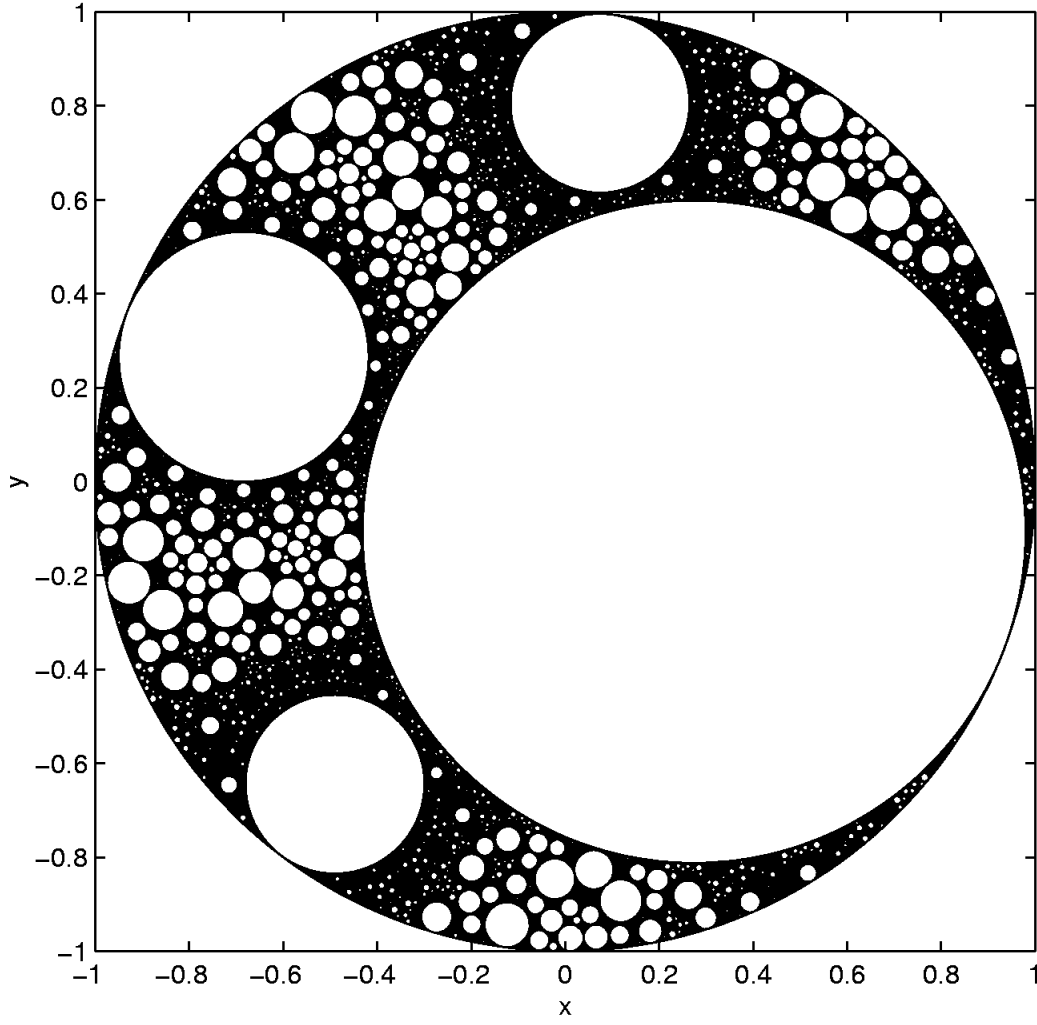
FIG. 10: The phase space portrait for the spherical map when $\omega = \pi(\sqrt{5}-1)$ and $\mu = \pi(\sqrt{5}-1)$.
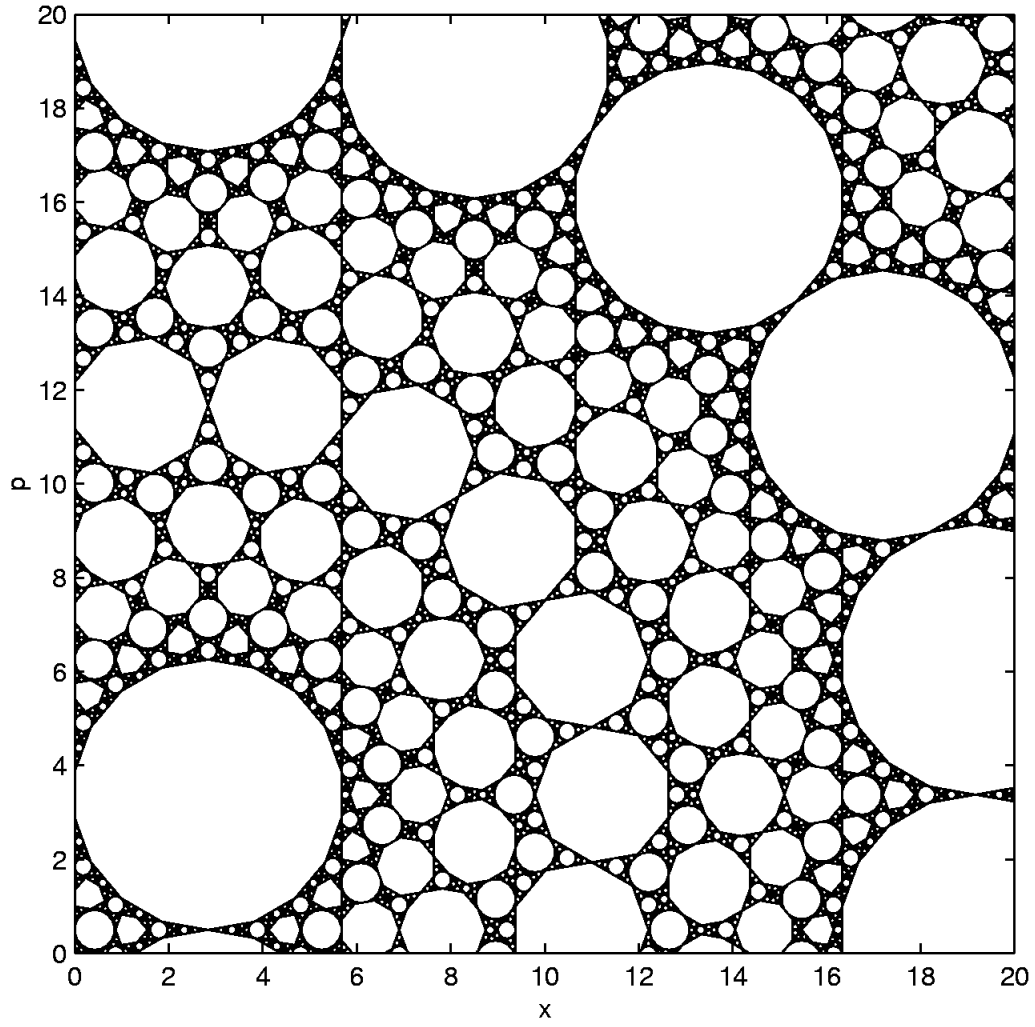
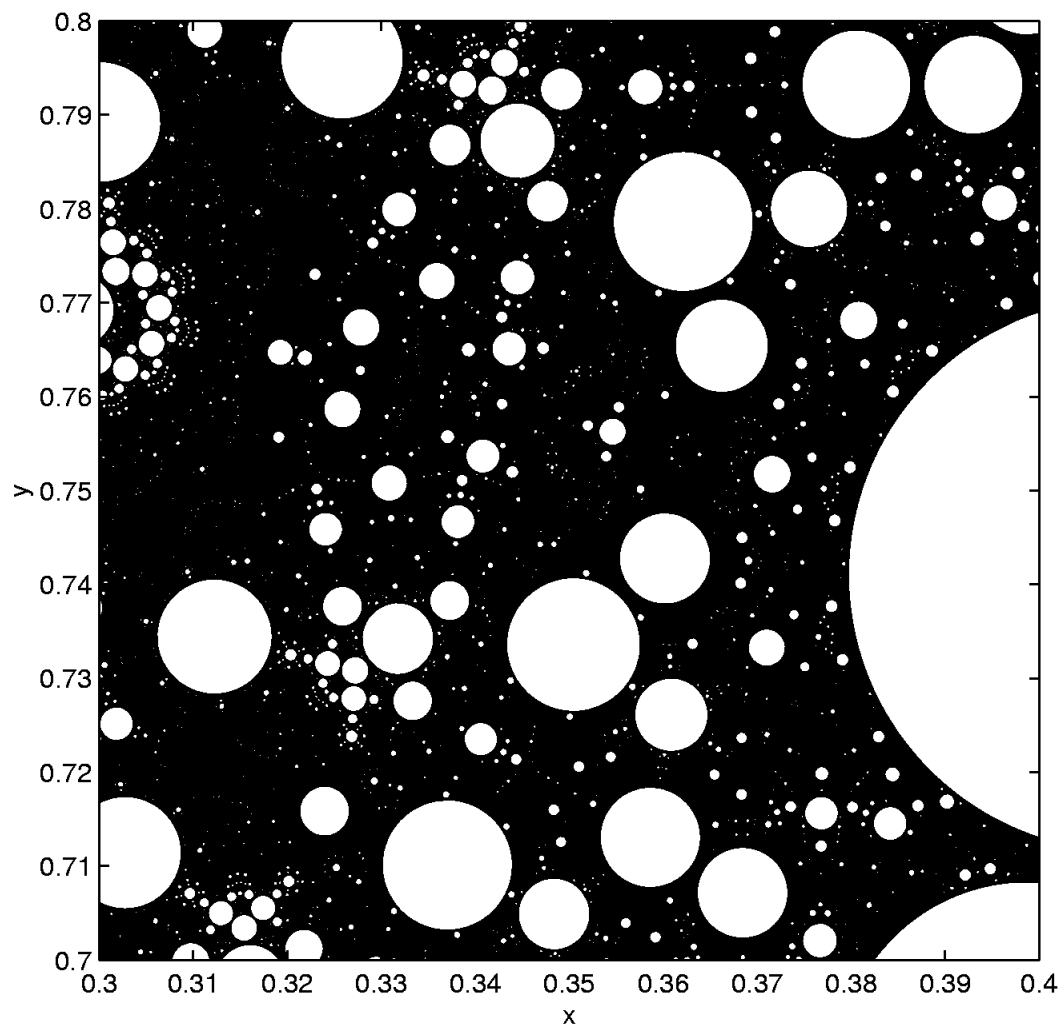FIG. 11: Detail of the phase plane when $\omega/2\pi = 2/9$ and $\mu = 1$ of the planar map.

FIG. 12: Detail of the phase space in Fig. 10.