Characterizing GHZ Correlations in Nondegenerate Parametric Oscillation via Phase Measurements

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We present a potential realization of the Greenberger, Horne and Zeilinger "*all or nothing*" contradiction of quantum mechanics with local realism using phase measurement techniques in a simple photon number triplet. Such a triplet could be generated using nondegenerate parametric oscillation.

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Many of the traditional tests of quantum mechanics (using the Bell Inequalities [1,2]) have used parametric amplifiers or oscillators to generate correlated photon number states [3–5]. When these correlated photon pairs (the signal and idler) are passed through polarisers or beamsplitter/phase-shifters and, measured by single photon detectors, a test of the Bell inequality can be achieved. Such tests however require auxiliary conditions [2] that lessen (or call into doubt) the contradiction. Such contradictions are very microscopic in nature as they involve single photon detection. Multi-particle tests of the Bell inequality have also been proposed using parametric amplification [6,7]. No multi-particle test of the Bell inequality has ever been experimentally considered.

The quantum states described by Greenberger, Horne and Zeilinger [8–12] (GHZ states) give predictions contrary to those of all classical theories based on the Einstein-Podolsky-Rosen [13] (EPR) premises of local realism. The spin GHZ state is an entangled state of three spins specified by stating that all spins are in the same direction. As this correlation can be realized in two ways, the state is the sum of the two amplitudes representing each way separately. The resulting interference between these two amplitudes ensures that there is a particular result for triple product spin measurements that can never occur. In contrast, a classical local hidden variable state exhibiting the same correlation (that is all spins the same) will produce this forbidden result with a non-zero probability. If this forbidden result were ever observed in a perfect experiment, the quantum prediction would be incorrect. On the contrary, never observing the forbidden result would verify quantum mechanics. Unfortunately not observing an event is a difficult way to test a theory experimentally. Detector inefficiencies may also lead to the non-observation of the forbidden result for reasons that have nothing to do with quantum entanglement.

The GHZ paradox can be formulated as follows: Consider three spin $\frac{1}{2}$ particles in a state $|\uparrow\rangle|\uparrow\rangle|\uparrow\rangle + |\downarrow\rangle|\downarrow\rangle|\downarrow\rangle$ where the $\uparrow$ or $\downarrow$ specifies spin up or down along the appropriate z axis. These particles originate in a spin conserving gendanken decay and fly apart along three different straight lines in the $x-y$ plane. Now because the spin vectors of distinct particles commute component by component, we can simultaneously measure the x component of one particle and the y components of the remaining two. In fact for the given initial state the product of the results of the three spin measurements $S_{x1}S_{y2}S_{y3}$, $S_{y1}S_{y2}S_{x3}$, $S_{y1}S_{x2}S_{y3}$, where S_{xi} and S_{yi} represent the spin along the horizontal and vertical directions, has to be $+1$ according to both quantum mechanics and local realism. According to local realism the spin product $S_{x1}S_{x2}S_{x3}$ must also be unity. Such a product can also be calculated quantum mechanically and in fact is found to be minus the product of all the three of them. To account for experimental situations where the spin product predictions are not unity in size, Mermin [14] derived the following inequality based on local realism arguments

$$F = |S_{x_1}S_{x_2}\,S_{x_3} - S_{y_1}S_{y_2}S_{x_3} - S_{y_1}S_{x_2}S_{y_3} - S_{x_1}S_{y_2}S_{y_3}| \leq 2. \tag{1}$$

To date there have been no tests of the GHZ inequality given by (1), due mostly to the difficult nature of generating a triple spin state of the form $|\uparrow\rangle|\uparrow\rangle|\uparrow\rangle + |\downarrow\rangle|\downarrow\rangle|\downarrow\rangle$. Recent developments by Laflamme *et. al* [15] have seen the generation of a GHZ state using the proton and carbon spins of trichloroethylene in NMR spectroscopy. They have shown using state tomography techniques that a 95% construction of the triple spin state can be achieved. Because such an experiment was done using a molecule no significant separation of the photon/carbon spins could be achieved and hence a test of the locality condition implicate in the GHZ paradox could not be made.

In this letter we propose a novel use of phase measurements to test the GHZ correlations. We will show how a simple correlated photon number triplet could be used to provide a definitive test. Such a state could be produced via nondegenerate parametric oscillation where we have signal, idler and pump modes. Discrete phase measurements are however difficult to realize experimentally and hence we consider how a homodyne quadrature phase amplitude measurement can provide a more realisable test. In a homodyne measurement the signal field is coupled to a strong local oscillator, hence providing very efficient detection. Current homodyne detection efficiency [16] can exceed 99%, thus providing a more stringent test by removing potential detection efficiency loopholes [17–19]. Also the use of the strong local oscillator field means that large intensities are incident on the highly efficient detectors.

A quantum entangled state shows correlations that cannot be explained in terms of the correlations between local classical properties of the subsystems. In this letter we will describe a pure entangled state of three modes in which the correlations are in photon number. More specifically, the nature of the correlation can be succinctly stated by saying that there are equal number of photons in each mode. As there are many different ways to realize this fact, the total state is the sum over amplitudes for all possible ways in which this correlation can be realized. This kind of sum over amplitudes for correlations is characteristic of an entangled state.

The question now arises as how best to see the quantum nature of the correlation. Obviously it is not enough to measure photon number as this would not distinguish a mixed state with equal photon numbers in each mode, from the equivalent entangled pure state. In some sense we need to measure an observable which carries as little information as possible about photon number in order to see the interference between all the possible ways in which the correlation in photon number can be realized. We conjecture that the best choice is the observable canonically conjugate to photon number: the canonical phase.

Pegg and Barnett [20,21] have shown that a set of $s+1$ orthonormal phase states, with values of θ differing by $2\pi/(s+1)$, can be generated from

$$|\theta_\mu\rangle = \exp\left[i\hat{N}\mu 2\pi/(s+1)\right]|\theta_0\rangle, \;\; \mu = 0,\ldots,s \qquad (2)$$

where $|\theta_0\rangle$ is the reference (or zero) phase state, $\hat{N}$ is the number operator and μ the particular discrete phase we are interested in. The values for θ_μ are given by

$$\theta_\mu = \theta_0 + \frac{2\mu\pi}{(s+1)} \qquad (3)$$

which are spread evenly over the range $\theta_0 \le \theta_\mu \le \theta_0 + 2\pi$, where θ_0 is the initial (or reference) phase.

The probability of finding a generalized system $|\Psi\rangle$ in a particular phase state $|\theta_\mu\rangle$ is

$$P_\mu(\theta_0) = |\langle\Psi|\theta_\mu\rangle|^2 \qquad (4)$$

where μ labels the particular phase state, and θ_0 is the choice of initial phase.

We require large s to describe an arbitrary phase for a general system. However, in the case of the measurement schemes required for various quantum violations of classical inequalities such as the Bell [1] and GHZ [8–11] (or Mermin higher spin [14]) inequalities, all that is required and necessary is a binary result. Thus a discrete phase measurement with $s=1$ suffices, that is two phase states are sufficient. If more phase states are chosen, for example $s=3$, a binary result is still required for these particular quantum inequalities, which could be achieved by dividing or binning the phase states into two discrete distinct sets. However this will not be ideal as to get this binary result we must discard information. Such a process must lessen (or destroy) our potential GHZ violation.

Production of a state of the form

$$|\Psi\rangle = \frac{1}{\sqrt{2}}|\uparrow\rangle|\uparrow\rangle|\uparrow\rangle + \frac{1}{\sqrt{2}}|\downarrow\rangle|\downarrow\rangle|\downarrow\rangle \qquad (5)$$

where $\uparrow, \downarrow$ represent the spin of the particle, has been difficult to achieve experimentally. Reid and Munro [23] have considered previously a photon triplet state

$$|\Psi\rangle = \frac{1}{\sqrt{2}}|0\rangle|0\rangle|0\rangle + \frac{1}{\sqrt{2}}|1\rangle|1\rangle|1\rangle \qquad (6)$$

which can also be used to test the GHZ inequality. Production of this triplet has yet to be realized. Potential for similar photon triplet state production exists in parametric oscillation. The ideal nondegenerate parametric oscillator may be specified by an interaction Hamiltonian of the form

$$H_{\rm int} = i\hbar\chi\left[c^\dagger ab - ca^\dagger b^\dagger\right] \qquad (7)$$

where $\hat{a}$, $\hat{b}$ and $\hat{c}$ are the boson annihilation operators for the signal, idler, and pump modes, respectively and χ is the parametric coupling constant. Initially preparing the pump mode in a single Fock state $|1\rangle$, with the signal and idler modes initially in vacuum states, it can be easily shown that

$$|\Psi\rangle = c_0|0\rangle|0\rangle|1\rangle + c_1|1\rangle|1\rangle|0\rangle \qquad (8)$$

can be generated where normalization requires $|c_0|^2 + |c_1|^2 = 1$. The state (8) is also a stable soliton solution [22] when the system is driven by a classical pump field coupled to mode $\hat{c}$.

Given the state (8) one can calculate the probability of obtaining the phase states $\theta_{\mu_1}, \theta_{\mu_2}, \theta_{\mu_3}$ (where the labels μ_1, μ_2, μ_3 corresponds to the $\hat{a}$, $\hat{b}$ and $\hat{c}$ modes respectively). For $s=1$, a choice of only two phase states, we have

$$\begin{aligned} P_{\mu_1\mu_2\mu_3} \quad &(\theta_{0,1},\theta_{0,2},\theta_{0,3}) = |\langle\Psi|\theta_{\mu_1}\rangle|\theta_{\mu_2}\rangle|\theta_{\mu_3}\rangle|^2 \\ &= \frac{1}{8} + \frac{1}{4}c_0c_1\cos\left[(\mu_1+\mu_2-\mu_3)\,\pi + \psi_0\right] \end{aligned} \qquad (9)$$

where μ_i is zero or one, and $\psi_0 = \theta_{0,1}+\theta_{0,2}-\theta_{0,3}$. We explicitly note that our initial phases for the three particles

$\theta_{0,i}$ can be expressed as one ψ_0. To classify our binary result we say that $\mu_i = 1$ corresponds to a "1" measurement, while $\mu_i = 0$ corresponds to a "0" measurement (for each of the particles). If we consider a single particle, then there is a probability of detecting it in the "1" (labeling the probability P_1) or the "0" state (labeling this P_0). Hence the probability of all three particles being in a "1" state is

$$P_{111}(\psi_0) = \frac{1}{8} - \frac{1}{4} c_0 c_1 \cos[\psi_0] \tag{10}$$

Similarly we can calculate the probability P_{000} of all particles being in a "0" state

$$P_{000}(\psi_0) = \frac{1}{8} + \frac{1}{4} c_0 c_1 \cos[\psi_0] \tag{11}$$

Other probabilities such as P_{001} can be calculated in an identical manner. It is necessary to point out that probabilities such as P_{001} and P_{010} are not identical due to the asymmetric initial state (8).

We define the spin of a single particle i as

$$S_i(\theta_{0,i}) = P_1(\theta_{0,i}) - P_0(\theta_{0,i}) \tag{12}$$

where we are explicitly indicating that the spin depends on the initial reference angle choice $\theta_{0,i}$. The spin product of the three particles is then the product of each of the spins. Hence the triple spin product is

$$S_1 S_2 S_3(\psi_0) = -2 c_0 c_1 \cos[\psi_0] \tag{13}$$

where we use the label S_i to represent the spin of the i^{th} particle and the angle ψ_0 to represent the total simplified initial phase choice. Given a triple spin product it is now possible to examine the GHZ paradox.

Generally, previous authors [14,23] have considered a GHZ inequality of the form (1). However because of our asymmetric initial state, we will consider the following inequality

$$\begin{aligned} F = |S_{y_1} S_{y_2}\ S_{x_3} - & S_{x_1} S_{x_2} S_{x_3} \\ & - \ S_{y_1} S_{x_2} S_{y_3} - S_{x_1} S_{y_2} S_{y_3}| \leq 2 \end{aligned} \tag{14}$$

which can be derived in an identical way to (1). We note that according to local realism

$$S_{y_1} S_{y_2} S_{x_3} = S_{x_1} S_{x_2} S_{x_3} \times S_{y_1} S_{x_2} S_{y_3} \times S_{x_1} S_{y_2} S_{y_3} \tag{15}$$

provided the magnitude of each spin product is one. Now according to local realism, the triple spin product $S_{y_1} S_{y_2} S_{x_3}$ has the same sign as the product of the other three triple products. It can be shown that the three triple spin products $S_{x_1} S_{x_2} S_{x_3}$, $S_{y_1} S_{x_2} S_{y_3}$, $S_{x_1} S_{y_2} S_{y_3}$ all have the same negative sign and hence $S_{y_1} S_{y_2} S_{x_3}$ should be negative. Hence adding all four spin products together according to (14) will give $F \leq 2$.

Next we need to relate these S_x and S_y to our $S(\theta_{0,i})$. We specify that $S_x = S(0)$ and $S_y = S(\pi/2)$. It can be easily shown using (13), our quantum mechanical triple spin product result, that

$$F = 8 c_0 c_1 \tag{16}$$

Therefore $F > 2$ if $c_0 c_1 > 1/4$ and a violation is possible. For the equal superposition in (8) we have $c_0 = c_1 = 1/\sqrt{2}$. Therefore $F = 4$ giving a maximal violation. If we have instead used the inequality given by (1), then $F = 4 c_0 c_1 \leq 2$ for all c_0, c_1.

The scheme presented here requires a discrete phase measurement, which has yet to be experimentally realized in the ultra high detector efficiency limit. However, recent work by Gilchrist *et. al* [24], and Yurke and Stoler [25] has suggested how quadrature phase amplitude measurements may be used to test the Bell inequality in the high detector efficiency limit. A homodyne based scheme is considered next to provide a feasible phase measurement.

A quadrature phase-amplitude homodyne measurement $X(\theta)$ can achieved by combining a signal field (say $\hat{a}$) with a local oscillator field (say $\hat{b}$) to form two new fields given by $\hat{c}_\pm = \left[\hat{a} \pm \hat{b} \exp(i\theta)\right] / \sqrt{2}$. Here θ is a phase shift which allows the choice of particular observable to be measured, for instance choosing θ as 0 or $\pi/2$ allows the measurement of the conjugate phase variables $X(0)$ and $X(\pi/2)$ respectively. The homodyne measurement is achieved by measuring using photodectors the intensities of both the beams c_+ and c_-, and then subtracting them to give a photocurrent difference as $I_d = c_+^\dagger c_+ - c_-^\dagger c_-$. Using the definition for $c_\pm$ the photocurrent difference can be rewritten in terms of the original signal and oscillator modes as

$$I_d = \hat{b}^\dagger \hat{a} e^{-i\theta} + \hat{b} \hat{a}^\dagger e^{i\theta}. \tag{17}$$

In the limit of a large oscillator field we can make a replacement of the b mode by a real classical field ϵ. Hence

$$I_d = |\epsilon| \left(\hat{a} e^{-i\theta} + \hat{a}^\dagger e^{-i\theta}\right) = |\epsilon| X(\theta) \tag{18}$$

Thus performing a measurement on the quadrature phase amplitude $X(\theta)$ yields a result $x(\theta)$ which ranges in size and sign. For our state (8), the probability of obtaining $x_1(\theta_1), x_2(\theta_2), x_3(\theta_3)$ (abbreviated as x_1, x_2, x_3) for the three particles measured by individual homodyne measurements is

$$P_{x_1 x_2 x_3}(\psi_0) = |\langle x_1 | \langle x_2 | \langle x_3 | \Psi \rangle|^2 \tag{19}$$

For a given quadrature measurement x_i, we classify the result as "1" if $x_i > 0$ and "0" if $x_i < 0$. The probability of obtaining the result "1" for all three particles is then

$$P_{111}(\psi_0) = \int_0^\infty \int_0^\infty \int_0^\infty dx_1 dx_2 dx_3 P_{x_1x_2x_3}(\psi_0)$$
$$= \frac{1}{4} - \frac{c_0c_1}{4}\sqrt{\left(\frac{2}{\pi}\right)^3}\cos[\psi_0] \quad (20)$$

Other probabilities such as P_{001} can be calculated in a similar fashion.

Defining the spin S_i in terms of P_1 and P_0 as before, we can show that the triple spin product is given by

$$S_1S_2S_3(\psi_0) = -2c_0c_1\sqrt{\left(\frac{2}{\pi}\right)^3}\cos[\psi_0] \quad (21)$$

and hence F given by (14) reduces to

$$F = 8c_0c_1\sqrt{\left(\frac{2}{\pi}\right)^3} \quad (22)$$

We maximize the discrepancy between quantum mechanics and local realism by choosing an equal superposition ($c_0 = c_1 = 1/\sqrt{2}$, and hence $F = 8\sqrt{2}/\sqrt{\pi^3} \sim 2.0318 > 2$. Though this is a small violation, it is still a violation of the GHZ inequality in the high detection efficiency limit.

A fundamental question that needs to be considered is why the magnitude of the triple spin product in (21) is not one as it is in the discrete phase case (for the case $c_0 = c_1 = 1/\sqrt{2}$)? The answer is quite simple. Our homodyne measurement, while it may have perfect detection efficiency, is not an accurate (or efficient) measurement of the discrete phase. This leads to a significant lessening of the size of the violation of the potential GHZ violation. The homodyne measurement does however have its advantages. First and foremost, current homodyne measurement technology allows detection efficiencies in excess of 99%. Our model for homodyne assumes perfect efficiency detectors. However, because of our small potential violation, the homodyne detection efficiency would have to exceed 99.5% in a real experiment provided the initial state could be produced accurately. A second advantage is that as the homodyne measurement involves a strong local oscillator via $I_d = \epsilon X(\theta)$ (with ϵ being the strength of the local oscillator), the potential GHZ inequality violation could have a macroscopic nature.

To summarize, we have investigated a triple photon correlated state (that may be able to be produced by nondegenerate parametric oscillation) that can be used to test the GHZ inequality proposed by Mermin. We have proposed how discrete phase measurements could provide an effective test of the inequality. In fact, a binary phase measurement could provide a maximal violation of the GHZ inequality. As an approximation to the binary phase measurement, we consider homodyne quadrature phase amplitude measurements. Again a violation of the GHZ inequality is possible although it is significantly reduced because it is an insensitive binary phase measurement. An advantage of the homodyne method however is that because it involves a strong local oscillator the detection efficiencies are extremely high.

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[1] J. S. Bell, Physics (N.Y.) **1**, 195(1965).
[2] J. F. Clauser, M. A. Horne, A. Shimony and R. A. Holt, Phys. Rev. Lett. **23**, 880 (1969).
[3] D. C. Burnham and D. L. Weinberg, Phys. Rev. Lett. **25**, 84 (1970)
[4] S. Friberg, C. K. Hong, and L. Mandel, Phys. Rev. Lett. **54**, 2011 (1985)
[5] R. Ghosh and L. Mandel, Phys. Rev. Lett. **59**, 1903 (1987).
[6] P. D. Drummond, Phys. Rev. Lett. **50**, 1407 (1983).
[7] W. J. Munro and M. D. Reid, Phys Rev A**50**, 3661(1994).
[8] D. M. Greenberger, M. A. Horne and A. Zeilinger, in "Bell's Theorem, Quantum Theory and Conceptions of the Universe", edited by M. Kafatos (Kluwer Academic, Dordrecht, 1989) p.73.
[9] D. M. Greenberger, M. A. Horne, A. Shimony and A. Zeilinger, Am. J. Phys. **58**, 1131 (1990).
[10] N. D. Mermin, Physics Today, June, p. 11 (1990).
[11] N. D. Mermin, Am. J. Phys. **58**, 731 (1990).
[12] R. K. Clifton, M. L. G. Redhead and J. N. Butterfield, Found. Phys. **21**, 149 (1991).
[13] A. Einstein, B. Podolsky and N. Rosen, Phys. Rev. **47**, 777 (1935).
[14] N. D. Mermin, Phys. Rev. Lett. **65**, 1838 (1990).
[15] R. Laflamme, E. Knill, W. H. Zurek, P. Catasti, and S. V. S Mariappan, NMR GHZ, quant-ph/9709025
[16] E. S. Polzik, J. Carry, and H. J. Kimble, Phys. Rev. Lett **68**, 3020 (1992)
[17] P.G. Kwiat, P.H. Eberhard, A.M. Steinberg and R.Y. Chiao, Phys. Rev. A **49**, 3209 (1994).
[18] M. Freyberger, P.K. Aravind, M.A. Horne and A. Shimony, Phys. Rev. A **53**, 1232 (1995).
[19] E.S. Fry, T. Walther, and S. Li, Phys. Rev. A **52**, 4381 (1996).
[20] D. T. Pegg, and S. M. Barnett, Europhys. Lett.**6**, 483 (1988)
[21] D. T. Pegg, and S. M. Barnett, Phys. Rev. A**39**, 1665 (1989)
[22] K. V. Kheruntayan and P. D. Drummond, manuscript being prepared for publication
[23] M. D. Reid and W. J. Munro, Phys. Rev. Lett. **69**, 997 (1992).
[24] A. Gilchrist, P. Deuar and M. D. Reid , Phys. Rev. Lett. **80**, 3169 (1998).
[25] B. Yurke and D. Stoler, Phys. Rev. Lett. **79**, 4941 (1997).