could be increased to about 55 μHz (Δf_{new}) using only two GAL's (GAL 16V8) without any reduction of the spectral purity of the output signal.

V. Conclusion

By using the Fine Tuning Circuit presented in this paper it is possible to improve the performance of Direct Digital Synthesizers containing Numerically Controlled Oscillators (NCO's). The solution is easy to implement and offers the advantage that a higher frequency resolution can be achieved by simply cascading single tuning stages. The special advantage of the presented serial structure is the fact that it is easy to implement even at very high clock frequencies. Due to the structure of the circuit it is possible to connect as many single stages as necessary to obtain the desired frequency resolution. The output sequence of the Fine Tuning Circuit is identical to the carry output of a phase accumulator. That means that there are no differences in the spectral properties of the generated sinewave.

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FIR Approximation of Fractional Sample Delay Systems

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Abstract—**This brief examines the approximation of fractional delays by FIR systems using various techniques which have previously been reported in the literature. In particular, the equivalence of the time-domain Lagrangian interpolation, the frequency-domain maximally flat error criterion and the window method is shown, provided maximal flatness is specified at $\omega_0 = 0$ and the window used is a scaled binomial function. The first of these has been known before, the second equivalence is new.**

I. Introduction

In sensor array beamsteering, it is sometimes necessary to apply fractional sample delays to discrete-time signals. This note examines the design of discrete-time finite impulse response filters to approximate fractional sample delay systems.

Polyphase filterbank implementations of fractional delay systems are not considered. This is because of the need for increased sampling rates in such schemes. We wish only to consider single sampling rate implementations.

A. Ideal Discrete-Time Fractional Sample Delay

The approximation of delays in continuous-time systems [1] is a well-studied problem. There are several major differences between the continuous-time approximation problem and the discrete-time analog we address. For a start, causal continuous-time delays are always causal and BIBO stable whereas in discrete-time problem neither of these properties hold (for fractional sample delays).

The ideal discrete-time fractional sample delay problem has the following characteristics.

1) The frequency response of an ideal delay system $H_{\text{ideal}}(e^{j\omega})$ with a delay of τ_d samples is given by $H_{\text{ideal}}(e^{j\omega}) = e^{-j\omega\tau_d}$.
2) The impulse response of the system is

$$h_n^{\tau_d} = \frac{\sin[\pi(n - \tau_d)]}{\pi(n - \tau_d)}, \; n \in \mathbb{Z}. \tag{1}$$

If $\tau_d \in \mathbb{Z}$ then (1) simplifies to $h_n^{\text{int}} = \delta[n - \tau_d]$, where $\delta[\cdot]$ is the Kronecker delta function.

3) The system is all-pass $|H_{\text{ideal}}(e^{j\omega})| = 1, \quad \forall\omega$.
4) The group delay is constant $\tau_g(\omega) := -(d/d\omega)\arg[H_{\text{ideal}}(e^{j\omega})] = \tau_d$.
5) For $\tau_d \notin \mathbb{Z}$, the system is noncausal and is not BIBO stable ($h_n^{\tau_d}$ is not absolutely summable).

B. Problem Definition

This brief examines the problem where we wish to implement an approximation to H_{ideal} as a finite impulse response (FIR) filter. The problem is specified as follows:

Problem 1. FIR Approximation of Fractional Delay Systems: Given a discrete-time fractional delay system with frequency response

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$e^{-j\omega\tau_d}$, we wish to choose the coefficients $\hat{h}_n$ of a finite impulse response filter so that the error

$$E(e^{j\omega}) := e^{-j\omega\tau_d} - \sum_{n=0}^{N-1} \hat{h}_n e^{-j\omega n}$$

is acceptable. □

There are several possible interpretations of "acceptable." It may mean the p-norm of E is minimized with respect to the coefficients $\hat{h}_n$ or, in a more classical treatment, it may mean that the coefficients are chosen so that the error function is maximally flat [2].

Following Cain *et al.* [3] we have the following result for the infinity-norm error of approximants with real coefficients.

Proposition 1. Infinity-Norm Error Bound—Real Coefficients: Given $\hat{h}_k \in \mathbb{R}$ then

$$\max_{\omega} |E(e^{j\omega})| \geq |\sin(\pi\tau_d)|.$$

□

Proof: The proof (from [3]) is obtained by noting that

$$\hat{H}(\omega)|_{\omega=\pi} \in \mathbb{R}$$

and that, at this frequency, $\mathrm{Im}(e^{-j\omega\tau_d}) = -\sin(\pi\tau_d)$. ■

Remark 1: This result says that, no matter how long an FIR filter is used to approximate the ideal delay, there will always be an infinity-norm error of $|\sin(\pi\tau_d)|$. Thus, the infinity-norm is not a sensible criterion by which to compare various methods for approximating fractional sample delay systems.

It may be that a band-limited infinity-norm criterion, such as $\max_{\omega\in[0,\alpha)} |E(e^{j\omega})|$, where $\alpha < \pi$ is more suitable for use in designing real-valued FIR filters for the problem at hand.

II. Three Approximation Procedures

We examine the window, Lagrangian interpolation [4] and maximally flat error [4], [5] approaches to the problem. We show that, given certain design choices, these three approaches are equivalent.

A. The Window Method

Given a desired (possibly infinite duration and noncausal) impulse response $h_n^{\tau_d}$ from (1), the window method for FIR filter design generates the filter coefficients

$$A_n^{\tau_d} = \begin{cases} h_n^{\tau_d} & 0 \leq n \leq N-1 \\ 0 & \text{otherwise} \end{cases}$$

For this case, the "window" chosen is the length N rectangular window. The coefficients $A_n^{\tau_d}$ yield the ℓ_2-optimal N-coefficient FIR approximate to the desired sequence $h_n^{\tau_d}$.

Because of its simplicity and optimality, this technique is popular. However, to improve control of performance in specific frequency bands, the technique is usually modified so that

$$B_n^{\tau_d} = \begin{cases} W_n^{\tau_d} h_n^{\tau_d} & 0 \leq n \leq N-1 \\ 0 & \text{otherwise} \end{cases}$$

where $W_n^{\tau_d}$ is some more suitable window function. If the rectangular window is used (or $h_n^{\tau_d}$ represents an integer delay) then $A_n^{\tau_d} = B_n^{\tau_d}$.

B. A Time-Domain Lagrangian Interpolation Method

Minocha, Dutta Roy, and Kumar [4] find FIR filter coefficients $C_n^{\tau_d}$ such that

$$x_n^{\tau_d} = \sum_{k=0}^{N-1} C_k^{\tau_d} x_{n-k}$$

where the sequence $[x_n^{\tau_d}]$ approximates the sequence $[x_n]$ delayed by $\tau_d = \alpha + M$ samples with $|\alpha| < 1$. The sequence $[x_n^{\tau_d}]$ is obtained by interpolating $x(t)$ from $N = 2M + 1$ samples of $[x_n]$ using the Lagrange polynomials $L_k(t)$:

$$L_k(t) = \prod_{\substack{i=0 \\ i\neq k}}^{N-1} \frac{t-i}{k-i}$$

where

$$x(t) = \sum_{k=0}^{N-1} L_k(t) x_k.$$

The coefficients resulting from the derivations in [4] are

$$C_n^{\tau_d} := (-1)^{N-n-1} \binom{\tau_d}{n} \binom{\tau_d - n - 1}{N-n-1}, \quad n = 0, 1, \cdots, N-1. \tag{2}$$

This is slightly different from [4] due to our indexing starting from 0 rather than 1.

C. A Frequency Domain Maximally Flat Approach

Suppose the classical design criterion of maximal flatness (as is used to design Butterworth filters [2]) is applied to the approximation error:

$$E(e^{j\omega}) = e^{-j\omega\tau_d} - \sum_{n=0}^{N-1} D_n^{\tau_d} e^{-j\omega n}$$

This was done in [5] and expanded upon in [4]. The maximally flat approach chooses the FIR coefficients $D_n^{\tau_d}$ so that

$$E^{(k)}(e^{j\omega}) := \left.\frac{d^k E(e^{j\omega})}{d\omega^k}\right|_{\omega=\omega_0} = 0 \quad \text{for some specific } \omega_0,$$

where $k = 0, 1, \cdots, N-1$ and $E^{(0)}(e^{j\omega}) = E(e^{j\omega})$.

Minocha *et al.* [4] have shown that the coefficients for the maximally flat design satisfy the recursion:

$$D_n^{\tau_d} = e^{j\omega_0(n-\tau_d)} \frac{\tau_d^n}{n!} \prod_{p=1}^{n-1} \left(1 - \frac{p}{\tau_d}\right) - \sum_{r=1}^{N-n-1} \binom{n+r}{n} D_{n+r}^{\tau_d} e^{-j\omega_0(n+r)}, \tag{3}$$

where $n = N-1, N-2, \cdots, 0$ and summations $\sum_{k=M}^{N}(\cdot)$ for $N < M$ are zero. Explicitly, the coefficients are given by

$$D_n^{\tau_d} = e^{j\omega_0(n-\tau_d)} \left\{ \sum_{m=n+1}^{N} \left[\frac{(-1)^{m+n+1} \tau_d^{m-1}}{(m-n-1)!n!} \prod_{p=1}^{m-2} \left(1 - \frac{p}{\tau_d}\right) \right] \right\} \tag{4}$$

where $n = 0, 1, \cdots, N-1$.

III. Unification of the Methods

Considering the apparently disparate nature of the derivations of these approximation procedures, the following results are somewhat surprising. The first of these has been noted previously by Hermanowicz [6].

Proposition 2. Equivalence of Interpolation and Maximally Flat Approximates: In the notation defined above

$$C_n^{\tau_d} = D_n^{\tau_d}, \quad \text{for } n = 0, 1, \cdots, N-1$$

when the maximally flat design for $D_n^{\tau_d}$ is carried out about $\omega_0 = 0$.

Proof: From (4),

$$\begin{aligned} D_n^{\tau_d} &= \sum_{m=n+1}^{N} \left\{ \frac{(-1)^{m+n+1}\tau_d^{m-1}}{(m-n-1)!n!} \prod_{p=1}^{m-2} [\tau_d^{-1}(\tau_d - p)] \right\} \\ &= \sum_{m=n+1}^{N} \frac{(-1)^{m+n+1}(m-1)!}{(m-n-1)!n!} \binom{\tau_d}{m-1} \\ &= \sum_{m=n+1}^{N} (-1)^{m+n+1} \binom{\tau_d}{m-1} \binom{m-1}{n} \\ &= \binom{\tau_d}{n} \sum_{m=n+1}^{N} (-1)^{m+n+1} \binom{\tau_d - n}{m-n-1} \end{aligned}$$

where the last manipulation makes use of the identity

$$\binom{r}{m}\binom{m}{n} = \binom{r}{n}\binom{r-n}{m-n} \quad \text{for } m, n \in \mathbb{Z}. \tag{5}$$

The binomial identities used in this and other proofs are all from Knuth [7].

Then, rewriting the summation limits yields

$$\begin{aligned} D_n^{\tau_d} &= (-1)^{2n} \binom{\tau_d}{n} \sum_{m=0}^{N-n-1} (-1)^m \binom{\tau_d - n}{m} \\ &= (-1)^{N-n-1} \binom{\tau_d}{n} \binom{\tau_d - n - 1}{N-n-1} \end{aligned}$$

which is identical to (2). ∎

Remark 2: An equivalent proof is available by showing that the Lagrangian interpolation coefficients $C_n^{\tau_d}$ satisfy the recursion relation of (3). □

Remark 3: Note that the condition that the expansion must take place about $\omega_0 = 0$ can be removed if we modify (2) to be

$$\tilde{C}_n^{\tau_d} := e^{-j\omega_0(n-\tau_d)}(-1)^{N-n-1}\binom{\tau_d}{n}\binom{\tau_d-n-1}{N-n-1}, \quad n = 0, 1, \cdots, N-1.$$

□

A more surprising result is the following.

Proposition 3. Equivalence of Interpolation and Window Method Approximates: In the notation defined above

$$C_n^{\tau_d} = B_n^{\tau_d}, \quad \text{for } n = 0, 1, \cdots, N-1,$$

where $B_n^{\tau_d} = W_n^{\tau_d} A_n^{\tau_d}$ and

$$W_n^{\tau_d} = \frac{\pi N}{\sin(\pi \tau_d)} \binom{\tau_d}{N} \binom{N-1}{n}. \tag{6}$$

□

Proof: First for $n = 0, 1, \cdots, N-1$, note that because $N = 2M + 1$ is odd

$$\begin{aligned} A_n^{\tau_d} &= \frac{\sin[\pi(n-\tau_d)]}{\pi(n-\tau_d)} \\ &= \frac{(-1)^{N-n}\sin(\pi\tau_d)}{\pi(n-\tau_d)}. \end{aligned}$$

Then, from (2)

$$\begin{aligned} C_n^{\tau_d} &= (-1)^{N-n-1}\binom{\tau_d}{n}\binom{\tau_d-n-1}{N-n-1} \\ &= (-1)^{N-n-1}\binom{\tau_d}{N}\binom{N-1}{n}\frac{N}{\tau_d-n} \\ &= \frac{(-1)^{N-n}\sin(\pi\tau_d)}{\pi(n-\tau_d)}\binom{\tau_d}{N}\binom{N-1}{n}\frac{\pi N}{\sin(\pi\tau_d)} \\ &= \frac{\pi N}{\sin(\pi\tau_d)}\binom{\tau_d}{N}\binom{N-1}{n}A_n^{\tau_d} \\ &= W_n^{\tau_d}A_n^{\tau_d} \end{aligned}$$

where the identities (5) and

$$\binom{r}{k} = \frac{r}{k}\binom{r-1}{k-1}$$

have been used to obtain the result. ∎

Remark 4: The interpretation of this result is that the Lagrangian interpolation and maximally flat approximants may be thought of as approximants formed using the window design technique with the window (6).

Remark 5: From an implementation point of view, the simple form of (6) means that only a small amount of storage would be required for a wide range of τ_d and N.

IV. Conclusion

We have shown that, under certain design choices, three apparently disparate approaches to the FIR approximation of fractional sample delays are equivalent. The equivalent approaches are: the window method, the Lagrangian interpolation method, and the maximally flat error method.

The equivalencies occur when the window method uses a scaled binomial window and when the maximally flat error method uses $\omega_0 = 0$ as the expansion point.

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