An Extended Kalman Filter for Demodulation of Polynomial Phase Signals*

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Abstract

This letter presents a new formulation of the extended Kalman filter (EKF) for use in frequency tracking. A brief summary of previous EKF approaches is given and the new approach detailed. Simulation studies of the standard and new algorithms show that a significant improvement in tracking and threshold performance is achieved.

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1 Introduction

The EKF for frequency tracking proposed by La Scala and Bitmead [1] represents an EKF design which has evolved from work in the early to mid-70's on phase-locked loops [2]. Moore and Tam [3] updated these in an EKF framework which was also reported in Anderson and Moore [4]. Further work by Parker and Anderson [5] encompassed periodic, rather than sinusoidal, frequency tracking.

These formulations allow tracking of the signal

$$y_t = A_t \exp(j \sum_{n=0}^{t} \omega_t) + \epsilon_t$$

where ω_t is the phase increment or instantaneous frequency, and ϵ_t is complex, white gaussian noise with independent real and imaginary parts.

The problem with this approach is that the phenomenon of thresholding [6] occurs at noise levels where the signal-to-noise ratio (SNR; $20 \log_{10} A^2/2\mathrm{var}(\epsilon_t)$) is greater than about 6dB. Thresholding occurs when a small increase in the noise level leads to a dramatic decrease in performance.

Other recent work [7] has examined the case of estimating the instantaneous phase or frequency of polynomial phase signals:

$$y_t = A_t \exp(j \sum_{m=0}^{M} \alpha_m t^m) + \epsilon_t$$

The aim of this letter is to formulate a new, polynomial phase, EKF approach which works in moderate noise levels.

The next section states the EKF formulation of the frequency tracking problem used by La Scala and Bitmead [1]. A new signal model which extends the usable region of the EKF by avoiding early thresholding is detailed in Section 3. The performance of the standard and new models are compared in Section 4 and the letter is concluded in Section 5.

2 The Standard EKF Formulation

The EKF for sinusoidal frequency tracking of La Scala and Bitmead [1] uses the following signal model:

$$x_{t+1} = \begin{bmatrix} \cos x_t^{(3)} & -\sin x_t^{(3)} & 0 \\ \sin x_t^{(3)} & \cos x_t^{(3)} & 0 \\ 0 & 0 & \gamma \end{bmatrix} x_t + \begin{bmatrix} 0 \\ 0 \\ w_t \end{bmatrix} \quad ; \quad y_t = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x_t + v_t \tag{1}$$

where the three components of the state are the in-phase signal, the quadrature signal and the frequency to be tracked.

The EKF equations may then be found directly from the equations given by, for example, Anderson and Moore [4].

3 A New EKF Formulation

Peleg and Porat [7] and Peleg, Porat and Friedlander [8] derived Cramer-Rao lower bounds on the variance of unbiased estimators of the phase of polynomial phase signals. Rather than estimating the α_m coefficients in $\phi(t) = \sum_{m=0}^{M} \alpha_m t^m$ which is generally sensitive to noise errors [9], they assumed that the β_m coefficients in $\phi(t) = \sum_{m=0}^{M} \beta_m P_m(t)$ were estimated. The $P_m(t)$ are specially chosen polynomials; Peleg and Porat [7] selected the Legendre polynomials due to their orthogonality on the interval $[-1, 1]$. Peleg and Porat [7] used a simple transformation which allows other intervals $[T_1, T_2]$ to be used $P_m(t; T_1, T_2) = \sqrt{\frac{2}{T}} P_m(\frac{2t - T_1 - T_2}{T_2 - T_1})$.

Assume that the phase is given by $\phi(t) = \sum_{m=0}^{M} \beta_m P_m(t; T_1, T_2)$. For formulation as an extended Kalman filter problem with the phase assumed to be of this form, we can use the following state-space signal model where the coefficients of the Legendre polynomials, β_m $(m = 0, 1, \ldots, M)$ and the amplitude A form the state of the system:

$$x_{t+1} = F_t x_t + G w_t \quad ; \quad y_t = \begin{bmatrix} x_t^{(M+2)} \cos(x_t^T \boldsymbol{P}) \\ x_t^{(M+2)} \sin(x_t^T \boldsymbol{P}) \end{bmatrix} + v_t$$

where $F_t = I_{M+2}$ and is the $M{+}2 \times M{+}2$ identity matrix, $E[w_t w_t^T] = Q_t$, $E[v_t v_t^T] = R_t$ and T is the signal length. The vector of Legendre polynomials, $\boldsymbol{P}$, is given by $\boldsymbol{P}^T := [P_0(t;T_1,T_2)\ P_1(t;T_1,T_2)\ \dots\ P_M(t;T_1,T_2)\ 0]$. The initial state is $x_0^T = [\beta_0\ \beta_1\ \dots\ \beta_M\ A]$.

For the true polynomial phase case G should be zero so that the phase parameters do not change. The EKF formulation allows this parameter variation to be catered for, which is a useful ability if the signal being tracked changes. The EKF equations [4] are again applied to yield the state estimator.

4 Comparisons

In this section we highlight two cases. First, the relative threshold performances of the old and new formulations are compared by estimating the frequency of a constant frequency signal. Second, a chirp signal with step-changes in the initial phase, frequency and frequency rate is used to illustrate the tracking performance of the new formulation.

4.1 Constant Frequency

One major problem with previous EKF formulations has been that the threshold point occurs at quite low noise levels. To illustrate a strength of the new formulation, the following simulation was undertaken. One thousand realisations of a 256 sample, noisy, constant frequency (β_1), random initial phase (β_0) signal were generated for each SNR between -20dB and 20dB. The frequency estimated by the new and standard EKF formulations was compared with the true frequency and a root mean square error calculated for each SNR. Each tracker was initialized using the maximizer of the 256-point FFT of the signal under consideration. The root mean square error for this estimator was also calculated.

The results are plotted in Figure 1.

The standard and new trackers each assumed a process noise covariance of $10^{-5}I$ and a measurement noise covariance of I.

The new formulation gives a threshold extension of approximately 12dB over the old formulation. The price paid is that the performance of the new estimator is not as good as the original below the threshold point.

4.2 Step Change in Frequency and Frequency Rate

A 500 sample chirp signal with step changes in initial phase, frequency and chirp rate with 6dB noise was generated. The true and estimated values of β_0, β_1 and β_2 are shown in Figure 2.

Note that the new formulation allows tracking of changes in all the chirp signal parameters.

5 Conclusions

A new EKF formulation for tracking parameters in polynomial phase signals was given. The new formulation allows tracking of these parameters and overcomes threshold problems noted with previous formulations.

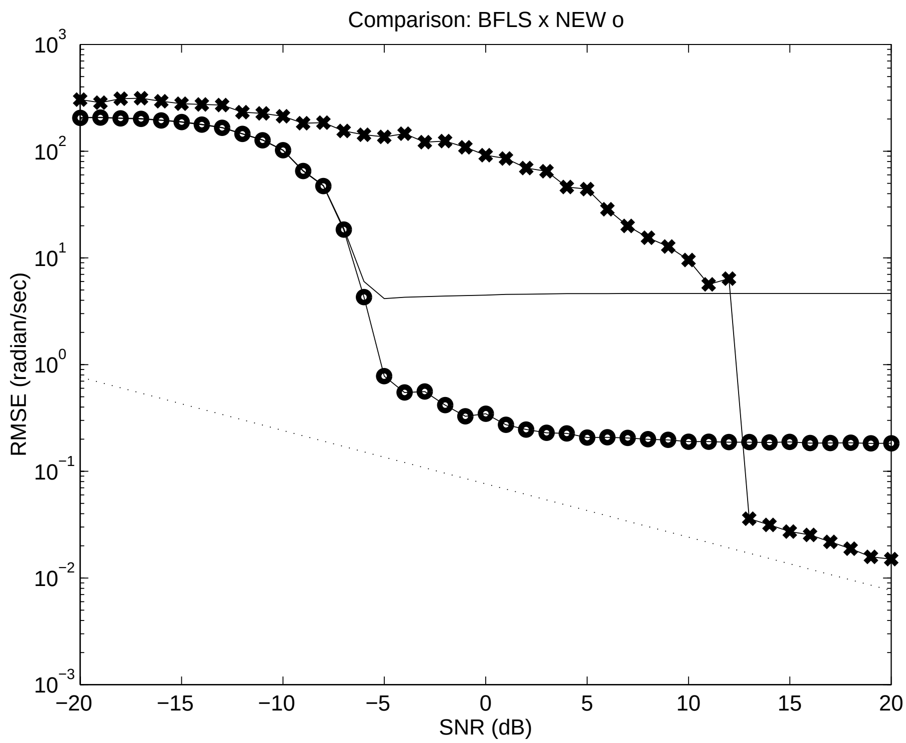

Figure 1: Thresholding performance of the new EKF (solid and ∘ line) versus that of La Scala and Bitmead [1] (solid and × line). Each tracker was initialized using the maximiser of the FFT (solid-only line). The Cramer-Rao lower bound for the frequency is plotted as the dotted line.

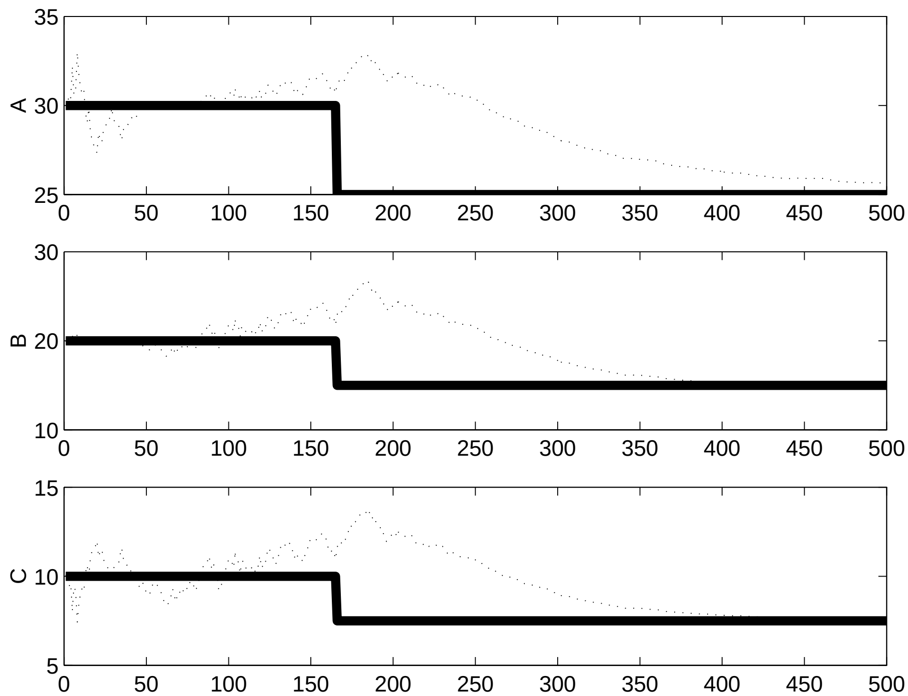

Figure 2: Tracking performance of new EKF. True (solid) and estimated (dotted) **A.** β_2, **B.** β_1 and **C.** β_0.

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