Xu, S. and Ostrikov, K.N., 2000. Series in vector spherical harmonics: an efficient tool for solution of nonlinear problems in spherical plasmas. Physics of Plasmas, 7, 3101.

and may be found at:
http://pop.aip.org/resource/1/phpaen/v7/i7/p3101_s1

DOI: http://dx.doi.org/10.1063/1.874166

AIP | Physics of Plasmas

Series in vector spherical harmonics: An efficient tool for solution of nonlinear problems in spherical plasmas

S. Xu and K. N. Ostrikov

Citation: Phys. Plasmas **7**, 3101 (2000); doi: 10.1063/1.874166
View online: http://dx.doi.org/10.1063/1.874166
View Table of Contents: http://pop.aip.org/resource/1/PHPAEN/v7/i7
Published by the American Institute of Physics.

Series in vector spherical harmonics: An efficient tool for solution of nonlinear problems in spherical plasmas

S. Xu[a)] and K. N. Ostrikov[b)]
Plasma Processing Laboratory, School of Science, Nanyang Technological University, 469 Bukit Timah Road, 259756 Singapore and Department of Physics, Flinders University of South Australia, GPO Box 2100, Adelaide SA 5001, Australia

(Received 4 January 2000; accepted 27 March 2000)

The series expansion of the plasma fields and currents in vector spherical harmonics has been demonstrated to be an efficient technique for solution of nonlinear problems in spherically bounded plasmas. Using this technique, it is possible to describe the nonlinear plasma response to the rotating high-frequency magnetic field applied to the magnetically confined plasma sphere. The effect of the external magnetic field on the current drive and field configuration is studied. The results obtained are important for continuous current drive experiments in compact toruses. [S1070-664X(00)02307-7]

The nonlinear plasma response is widely recognized as an intrinsic feature of spherically bounded plasmas. It controls the inertial plasma confinement and heating by intense lasers,[1] and the current drive and plasma stability in compact toroidal devices.[2,3] The self-consistent description of the nonlinear plasma response is thus an important issue of modern plasma theory. Recently, several analytical and numerical techniques have been applied for description of nonlinear collective processes in spherically bounded plasmas.[4–6] It has been reported that under certain conditions a generation of strongly nonlinear electromagnetic field structures appears possible.[7,8] In many cases, even for a small perturbation, the nonlinear effects can be important.[9] Due to the geometrical complexity of the problem, it is often difficult to obtain the closed-form solutions for the plasma currents and fields. Usually, such a problem is solved either by seeking for a specific subclass of exact nonlinear plasma responses, satisfying the initial nonlinear equations with the appropriate boundary conditions,[7–9] or by an expansion of all plasma perturbations in series of the field/current powers assuming the nonlinearity weak.[10] In the latter case, using the conventional Legendre polynomial representation for the plasma parameters, it turns out possible to obtain feasible solutions for the first-order nonlinear plasma responses analytically. Further derivation of higher-order nonlinear solutions in most cases becomes inappropriate due to overwhelming complexity.

In this Brief Communication, we demonstrate the effectiveness of the series expansion of plasma fields and currents in vector spherical harmonics for an analytical derivation of closed-form first- and second-order nonlinear responses in spherically bounded magnetized plasmas. As an example, we apply the above technique for solution of the nonlinear problem of steady-state plasma current drive in a spherical Rotamak with an externally applied rotating magnetic field (RMF) and a steady vertical magnetic field (VMF).[11,12]

In a Rotamak, a dielectric spherical vessel with the internal radius r_w is immersed in rotating $\tilde{\mathbf{B}}_R$ and vertical steady $\mathbf{B}_v$ magnetic fields

$$\tilde{\mathbf{B}}_R = B_\omega\{[\sin(\theta)\mathbf{r} + \cos(\theta)\boldsymbol{\theta}]\cos\psi - \boldsymbol{\phi}\sin\psi\}, \tag{1}$$

$$\mathbf{B}_v = B_v[\cos(\theta)\mathbf{r} - \sin(\theta)\boldsymbol{\theta}], \tag{2}$$

where $\psi = \omega t - \phi$, (r,θ,ϕ) are standard right-handed spherical coordinates, ω and B_ω are the frequency and amplitude of the RMF, and B_v is the amplitude of the VMF. The plasma is treated in a fluid approximation, with immobile ions. The fields and currents in the plasma are described by Maxwell's equations

$$\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t, \tag{3}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}, \tag{4}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{5}$$

and the generalized Ohm's law

$$\mathbf{E} = \eta \mathbf{J} + (1/n_e e)\mathbf{J} \times \mathbf{B}, \tag{6}$$

which is identical to the equation of motion of the electron fluid.[11,12] Here n_e is the electron density, η is the plasma resistivity, and $\mathbf{E}$, $\mathbf{B}$, and $\mathbf{J}$ are the electric, magnetic fields, and screening currents in the plasma, respectively. The electron density and plasma resistivity have been assumed spatially and temporally constant.

Renormalizing the variables $r = r/r_w$, $\mathbf{B} = \mathbf{B}/B_\omega$, $\mathbf{J} = \mathbf{J}(r_w\mu_0/B_\omega)$, and $\mathbf{E} = \mathbf{E}(\omega r_w B_\omega)^{-1}$, and assuming the dependence of all perturbations on time and azimuthal angle as $\sim \exp(i\psi)$, (1)–(6) can be combined to

$$2\lambda^2 \frac{\partial \mathbf{B}}{\partial \phi} = \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \mathbf{B} + \gamma \boldsymbol{\nabla} \times ((\boldsymbol{\nabla} \times \mathbf{B}) \times \mathbf{B}), \tag{7}$$

where we have defined $\gamma = \omega_{ce}/\nu_{\text{eff}} = B_\omega / n_e e \eta$ and $\lambda = r_w/\delta = (\omega \mu_0 r_w^2 / 2\eta)^{1/2}$. Here $\omega_{ce} = \omega_{ce}(B_\omega)$ is the electron cyclotron frequency in the RMF, ν_{eff} is the effective frequency of electron collisions, and δ is the classical collisional skin depth.

Assuming the nonlinearity is weak ($\gamma \ll 1$), we expand any vector field $\mathbf{F}$ in a series

$$\mathbf{F} = \mathbf{F}^{(0)} + \mathbf{F}^{(1)}\gamma + \mathbf{F}^{(2)}\gamma^2 + \cdots, \tag{8}$$

where **F** can either be **B** or **J**.

Substituting (8) into (7) and combining the terms with the same powers in γ, we obtain

$$2\lambda^2 \frac{\partial \mathbf{B}^{(0)}}{\partial \phi} - \nabla \times \nabla \times \mathbf{B}^{(0)} = 0, \tag{9}$$

which is a linear equation describing the magnetic field penetration in the plasma sphere. To the first-order approximation, the magnetic field $\mathbf{B}^{(1)}$ is governed by

$$2\lambda^2 \frac{\partial \mathbf{B}^{(1)}}{\partial \phi} - \nabla \times \nabla \times \mathbf{B}^{(1)} = \mathcal{G}_{00}, \tag{10}$$

where quadratic in the field amplitude's nonlinear plasma responses have been accounted for. To the second order in γ, we obtain

$$2\lambda^2 \frac{\partial \mathbf{B}^{(2)}}{\partial \phi} - \nabla \times \nabla \times \mathbf{B}^{(2)} = \mathcal{G}_{10} + \mathcal{G}_{01}, \tag{11}$$

which is the nonlinear self-action equation.[13] In (9)–(11), differential operator $\mathcal{G}_{lk} = \nabla \times ((\nabla \times \mathbf{B}^{(l)}) \times \mathbf{B}^{(k)})$ represents the source of the nonlinearity.

To solve (9)–(11), the vector fields $\mathbf{F}^{(j)}$ have been expanded in series

$$\mathbf{F}^{(j)} = \sum_{\alpha=1}^{3} \sum_{l=0}^{N} \sum_{m=-l}^{l} f_{\alpha l m} \boldsymbol{\mathcal{Y}}_{\alpha l m}(\theta, \phi), \tag{12}$$

where

$$\boldsymbol{\mathcal{Y}}_{1lm} = \hat{\mathbf{r}} Y_{lm},$$

$$\boldsymbol{\mathcal{Y}}_{2lm} = (r/\sqrt{l(l+1)}) \nabla Y_{lm} = \hat{\mathbf{r}} \times \boldsymbol{\mathcal{Y}}_{3lm},$$

$$\boldsymbol{\mathcal{Y}}_{3lm} = (1/\sqrt{l(l+1)}) \nabla \times (\mathbf{r} Y_{lm}) = -\hat{\mathbf{r}} \times \boldsymbol{\mathcal{Y}}_{2lm},$$

are the vector spherical harmonics,[14] $Y_{lm}(\theta,\phi) = (-1)^m \sigma_{lm} P_l^m(\cos\theta) e^{im\phi}$ are the scalar spherical harmonics, $\sigma_{lm} = [(2l+1)(l-m)!/4\pi(l+m)!]^{1/2}$ and $P_l^m(\cos\theta)$ are the Legendre polynomials. In calculations, the orthogonality and symmetry properties of functions $\boldsymbol{\mathcal{Y}}_{\alpha lm}(\theta,\phi)$ have been used. Additionally, the physical constraint for all the fields $\mathbf{F}(r,\theta,\phi)$ to be real dictates that $f^*_{\alpha lm} = (-1)^m f_{\alpha l -m}$.

The external fields (1) and (2) can also be presented in terms of vector spherical harmonics in a nondimensional form:

$$\tilde{\mathbf{B}}_R = -[\sqrt{2\pi/3} \boldsymbol{\mathcal{Y}}_{111} + \sqrt{(4\pi/3}\boldsymbol{\mathcal{Y}}_{211}] + \text{c.c.}, \tag{13}$$

and

$$\mathbf{B}_v = \beta_z [\sqrt{4\pi/3} \boldsymbol{\mathcal{Y}}_{110} + \sqrt{8\pi/3} \boldsymbol{\mathcal{Y}}_{210}], \tag{14}$$

where $\beta_z = B_z / B_\omega$, and c.c. stands for complex conjugate. Boundary conditions of continuity of the radial field and its derivative at the sphere surface $r=1$ yield

$$\frac{\partial b_{1lm}}{\partial r} + (2+l) \frac{b_{1lm}}{r} = (2l+1) \alpha_{lm} r^{l-2} r_w^{l-1}, \tag{15}$$

$$b_{3lm} = 0, \tag{16}$$

where $\alpha_{00}=0$, $\alpha_{11} = -\sqrt{2\pi/3}$, $\alpha_{10} = \sqrt{4\pi/3}\beta_z$, and $\alpha_{lm}=0$ for $l>1$.

The zeroth- and first-order fields and screening currents can nontrivially be derived from (9), (10), and (12)–(16). The harmonics of the steady magnetic field ($\gamma=0$) are $b_{110}^{(0)} = \beta_z(4\pi/3)^{1/2}$, and $b_{210}^{(0)} = \beta_z(8\pi/3)^{1/2}$, respectively. The most important first-order magnetic field harmonics ($m>0$) are

$$b_{111}^{(1)} = -\sqrt{6\pi}\lambda^2 \rho_1 \vartheta^{-1} \sum_{k=1}^{\infty} (k\zeta_1/k_1)(\vartheta r)^{2k-2}, \tag{17}$$

$$b_{211}^{(1)} = 2\sqrt{3\pi}\lambda^2 \rho_1 \vartheta^{-1} \sum_{k=1}^{\infty} (k^2\zeta_1/k_1)(\vartheta r)^{2k-2}, \tag{18}$$

$$b_{321}^{(1)} = 6\sqrt{\pi/5}\vartheta \rho_1 \sum_{\alpha=1}^{\infty} (\alpha_1 h_1/\alpha_3)(\vartheta r)^{2\alpha}, \tag{19}$$

where $\rho_1 = \beta_z/\sinh\vartheta$, $\vartheta = (1-i)\lambda$, $\zeta_1 = \vartheta\coth\vartheta - 2k + 1$, $k_1 = (2k+1)!$, $\alpha_1 = \alpha(\alpha+1)$, $\alpha_2 = \alpha_1(\alpha+2)$, $\alpha_3 = (2\alpha+3)!$, $h_1 = \mathcal{H} + \alpha - 1$, and $\mathcal{H} = (1/2)\{5 + [\vartheta^2(\sinh\vartheta - \vartheta\cosh\vartheta)]/[(\vartheta^2+3)\sinh\vartheta - 3\vartheta\cosh\vartheta]\}$. The associated first-order current density harmonics are

$$j_{311}^{(1)} = 3\pi\lambda^2 \rho_1 \sum_{k=2}^{\infty} ([(k-1)/(2k-1)!]\zeta_1)(\vartheta r)^{2k-3}, \tag{20}$$

$$j_{121}^{(1)} = 6\sqrt{6\pi/5}\rho_1 \vartheta^2 \sum_{\alpha=1}^{\infty} (\alpha_1 h_1/\alpha_3)(\vartheta r)^{2\alpha-1}, \tag{21}$$

$$j_{221}^{(1)} = 6\sqrt{\pi/5}\rho_1 \vartheta^2 \sum_{\alpha=1}^{\infty} (\alpha_4 h_1/\alpha_3)(\vartheta r)^{2\alpha-1}, \tag{22}$$

where $\alpha_4 = \alpha_1(2\alpha+1)$.

In the second-order approximation, one can derive the relatively simple nonlinear equation for the magnetic field harmonic $b_{320}^{(2)}$

$$\frac{\partial^2}{\partial r^2}(r b_{320}^{(2)}) - \frac{6 b_{320}^{(2)}}{r} = \sqrt{3/20\pi}\{T_{310}^{(1)} b_{110}^{(0)} + \text{Re}[T_{311}^{(0)} b_{11-1}^{(1)} + T_{311}^{(1)} b_{11-1}^{(0)}]\}, \tag{23}$$

where $T_{\alpha lm} = (r\partial j_{\alpha lm}^{(j)}/\partial r - j_{\alpha lm}^{(j)})$. After substitution of the zeroth- and first-order solutions for magnetic fields and plasma currents into (23), we obtain

$$b_{320}^{(2)} = 12\sqrt{6\pi/5}\rho_2 \text{Re}\left\{ \sum_{k=2}^{\infty} [4k/(4k+2)!](2\lambda)^{4k-2}\rho_3 + \sum_{k,j=1}^{\infty} (kj/\zeta_3)(j+1)\zeta_4 \chi_1 \rho_4 \right\}, \tag{24}$$

where $\rho_2 = \beta_z \lambda^2/(\cosh(2\lambda) - \cos(2\lambda))$, $\zeta_3 = (2j+3)!(2k+1)!(k+j-1)(2k+2j+3)$, $\zeta_4 = i\,\text{Im}(\vartheta\coth\vartheta) + k - j - 2$, $\rho_3 = r^2 - r^{4k-2}$, $\rho_4 = r^{2(k+j)} - r^2$, and $\chi_1 = \xi^{2k-1}\vartheta^{2j+1}$.

Similarly, we derive

$$b_{110}^{(2)} = 4\sqrt{3\pi}\rho_2 \text{Im}\left\{ \sum_{k,j=1}^{\infty} kj(\zeta_1 - 1)(\zeta_5/\zeta_6)\chi_2 \right\}, \tag{25}$$

$$b_{210}^{(2)} = 4\sqrt{6\pi}\rho_2 \text{Im}\left\{ \sum_{k,j=1}^{\infty} kj(\zeta_1 - 1)(\zeta_7/\zeta_6)\chi_2 \right\}, \tag{26}$$

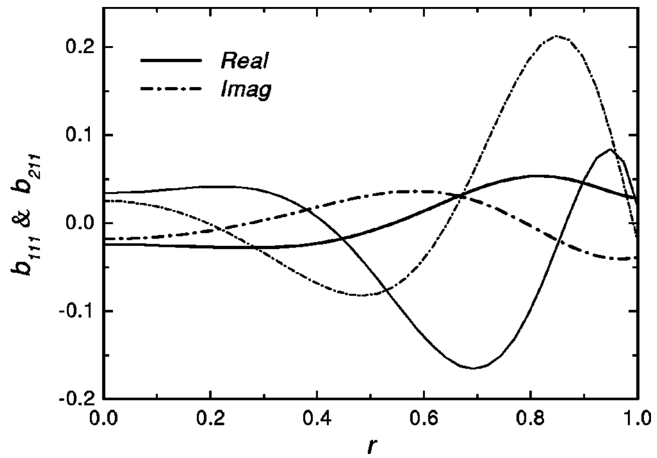

FIG. 1. Radial profiles of the time-varying magnetic field harmonics $b_{111}^{(1)}$ (thick lines) and $b_{211}^{(1)}$ (thin lines) calculated for $\lambda=5$ and $\beta_z=0.5$.

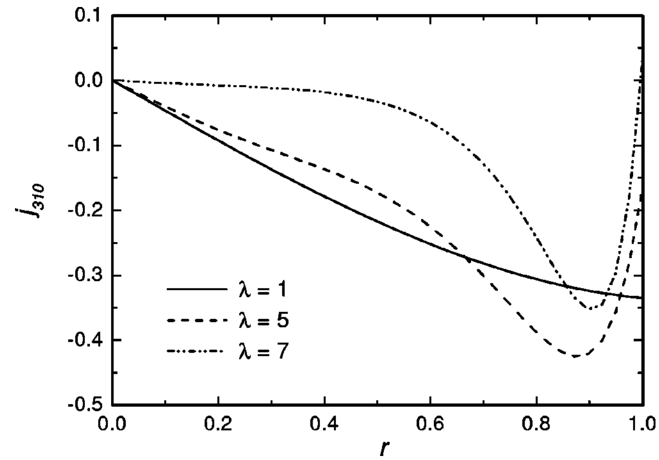

FIG. 2. Radial profiles of the steady toroidal current harmonics $j_{310}^{(2)}$ calculated for $\beta_z=0.5$ and $\lambda=1$, 5, and 7.

where $\zeta_5=2k+2j+1-3r^{2(k+j-1)}$, $\zeta_6=(2k+1)!(2j+1)!\times(k+j-1)(2k+2j+1)$, $\zeta_7=2k+2j+1-3(k+j)r^{2(k+j-1)}$, $\chi_2=\xi^{2j-1}\vartheta^{2k-1}$, and ξ is complex conjugate to ϑ.

The associated current density harmonics are

$$j_{120}^{(2)}=72\sqrt{\pi/5}\rho_2\mathrm{Re}\left\{\sum_{k=2}^{\infty}[4k/(4k+2)!](2\lambda)^{4k-2}\rho_8+\sum_{k,j=1}^{\infty}(kj/r\zeta_3)(j+1)\zeta_4\chi_1\rho_4\right\}, \tag{27}$$

$$j_{220}^{(2)}=12\sqrt{6\pi/5}\rho_2\mathrm{Re}\left\{\sum_{k=2}^{\infty}[4k/(4k+2)!](2\lambda)^{4k-2}\rho_9+\sum_{k,j=1}^{\infty}(kj/\zeta_3)(j+1)\zeta_4\chi_1\rho_{10}\right\}, \tag{28}$$

$$j_{310}^{(2)}=12\sqrt{6\pi}\rho_2\mathrm{Im}\left\{\sum_{k,j=1}^{\infty}kj(\zeta_1-1)(\rho_{11}/k_1j_1)\chi_2\right\}, \tag{29}$$

where $\rho_8=r-r^{4k-3}$, $\rho_9=3r-(4k-1)r^{4k-3}$, $\rho_{10}=(2k+2j+1)r^{2k+2j-1}-3r$, $\rho_{11}=r^{2k+2j-3}$, and $j_1=(2j+1)!$.

Below, using the analytical expressions for the nonlinear fields and currents, we discuss several important features of the RMF current drive in a spherical plasma with a finite vertical magnetic field ($\beta_z\neq 0$). Note that the VMF is required to achieve the magnetohydrodynamic (MHD) plasma equilibrium.[15,16] Our analytical and numerical results suggest that the steady VMF can strongly affect the current drive, the bi-directional toroidal field and the RMF. We emphasize that the application of the finite vertical magnetic field ($\beta_z\neq 0$) results in generation of certain current and field harmonics which do not exist in unmagnetized spherical plasma case.[11] The analyses show that the field harmonics (17)–(19) strongly affect the externally applied RMF. These field harmonics are driven by the nonlinear interaction of the steady VMF with the linear screening currents induced in the plasma, and vanish if $\beta_z=0$. The radial profiles of the field harmonics $b_{111}^{(1)}$ and $b_{211}^{(1)}$ are shown in Fig. 1. It is remarkable that the field harmonics (17) and (18) can be augmented by raising the VMF amplitude. Thus, the RMF that is essential for current drive in Rotamak devices,[11,12] can be amplified by applying the steady vertical magnetic field.

We now turn to the effect of the VMF on the toroidal current drive. The analytical analysis shows that the toroidal plasma current $j_{310}^{(2)}$ is excited as a result of the nonlinear interaction between the linear screening current $j_{311}^{(0)}$ and the RMF harmonics $b_{111}^{(1)}$ and $b_{211}^{(1)}$. To this order approximation, the toroidal current (29) vanishes if $\beta_z=0$. Figure 2 depicts the radial profiles of the steady current harmonic $j_{310}^{(2)}$ for different values of λ. Comparison with the case $\beta_z=0$ reveals that the VMF can noticeably strengthen the toroidally driven current in the plasma sphere.

From (24) one can notice that the steady bi-directional toroidal magnetic field is also controlled by the parameter β_z. Figure 3 displays the radial profiles of $b_{320}^{(2)}$ for different values of λ and β_z. In the small γ limit, the VMF of a direction favorable for MHD pressure balance ($\beta_z>0$)[15] augments the bi-directional toroidal field, while an oppositely directed field ($\beta_z<0$) diminishes it. One should note

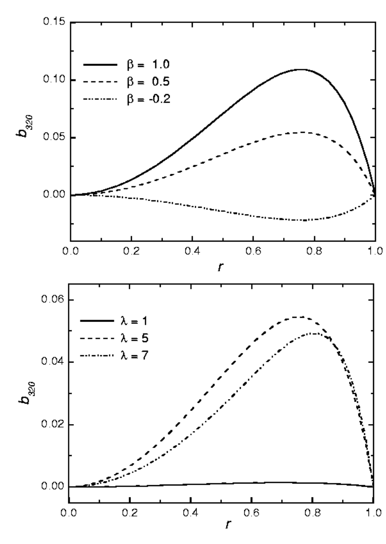

FIG. 3. Radial profiles of the bi-directional toroidal magnetic field harmonics $b_{320}^{(2)}$ calculated for $\lambda=5$ and $\beta_z=-0.2$, 0.5, and 1 (upper diagram), and $\beta_z=0.5$, $\lambda=1$, 5, and 7 (lower diagram).

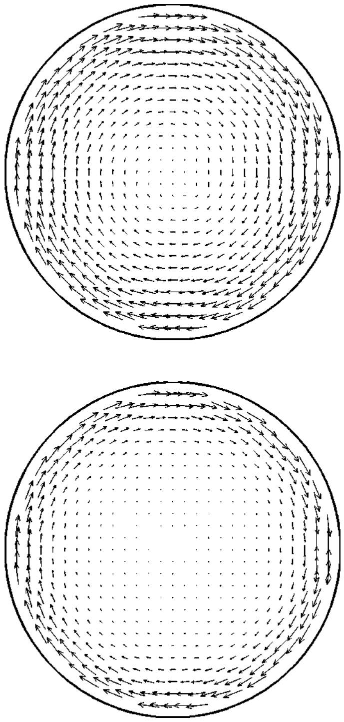

FIG. 4. Vector diagram of the steady toroidal current density in the equatorial plane for $\beta_z = 0.5$ and $\lambda = 5$ (a) and 7 (b), respectively.

that the self generated bi-directional toroidal magnetic field (24) is predominately produced by the interaction of azimuthal and altitudinal components of the screening current j_{311} with the radial component of the rotating magnetic field b_{11-1} and by the interaction of the steady toroidal plasma current j_{310} with its associated poloidal magnetic field b_{110}. Note that the above interactions are due to the nonlinear Hall force. Likewise, it is possible to deduce that the toroidal magnetic flux through the upper half of the poloidal cross section increases if $\beta_z > 0$, and decreases if the sign of the VMF reverses. It is worthwhile to pinpoint that relatively modest values of the applied vertical field are capable to substantially amplify the bi-directional toroidal magnetic field.

Combining the corresponding solutions, and using (8), one can numerically construct the topographies of the real fields and currents in a spherical plasma vessel. For instance, the steady toroidal current in the equatorial plane can be presented in a form of a vector plot depicted in Fig. 4. It is evident that the field penetration into the plasma sphere is weaker for a higher value of λ. Using the expressions for b_{320} we present in Fig. 5 the contour plots of the bi-directional toroidal magnetic field $b_{320}\mathcal{Y}_{320}$ in the meridian plane for $\lambda = 5$ and $\beta_z = 0.5$. Clearly, the toroidal field has opposite directions in the upper and lower hemispheres and thus possesses the net zero toroidal flux over the poloidal plane, as has been experimentally observed in Ref. 17.

We should remark that the vector spherical harmonics have proved to be useful for data fitting in the spherical Rotamaks.[17] Some of them (b_{110}, b_{210}, b_{11-1}, b_{21-1}, b_{32-1}, b_{32-2}) and the associated current harmonics have been measured experimentally.[18]

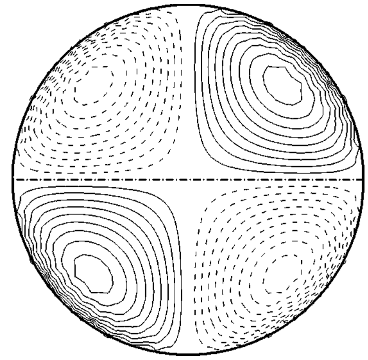

FIG. 5. Contour plot in the meridian plane for the bi-directional toroidal magnetic field B_ϕ ($b_{320}\mathcal{Y}_{320}$) for $\lambda = 5$ and $\beta_z = 0.5$. Continuous curves mean that B_ϕ is directed into the paper plane and broken curves correspond to the opposite direction of B_ϕ. The thick dashed line indicates the location of the equatorial plane.

To conclude, we emphasize that the expansion of the nonlinear fields and currents in the plasma appears to be an efficient tool for obtaining the feasible closed-form nonlinear solutions in spherically bounded plasmas. The effect of the external vertical magnetic field on the current drive and magnetic field configuration in spherical plasmas warrants further investigations. Finally, we emphasize that the results reported are important for future plasma confinement and current drive experiments in compact toroidal devices.

ACKNOWLEDGMENTS

Useful discussions with I. R. Jones, R. Storer, D. Brotherton-Ratcliffe, and S. Lee are kindly appreciated.

[a]Electronic mail: syxu@nie.edu.sg
[b]Author to whom correspondence should be addressed. Electronic mail: ekostrikov@ntu.edu.sg and kostrikov@nie.edu.sg. Also with School of Electrical and Electronic Engineering, Nanyang Technological University, Nanyang Avenue 639798, Singapore.
[1]T. K. Fowler, Rev. Mod. Phys. **71**, S456 (1999).
[2]A. Sykes, Phys. Plasmas **4**, 1665 (1997), Tech. Phys. **44**, 1047 (1999).
[3]J. Menard, R. Majeski, R. Kaita, M. Ono, T. Munsat, D. Stutman, and M. Finkenthal, Phys. Plasmas **6**, 2002 (1999).
[4]K. Katayama and M. Katsurai, Phys. Fluids **29**, 1939 (1986).
[5]T. Yano and Y. Inoue, Phys. Fluids **6**, 2831 (1994).
[6]A. Kageyama, M. M. Ochi, and T. Sato, Phys. Rev. Lett. **82**, 5409 (1999).
[7]M. Y. Yu and L. Stenflo, Nature (London) **384**, 224 (1996).
[8]L. Stenflo and M. Y. Yu, Phys. Plasmas **5**, 3122 (1998).
[9]L. Stenflo and M. Y. Yu, Phys. Rev. A **42**, 4894 (1990).
[10]V. I. Karpman, *Nonlinear Waves in Dispersive Media* (Pergamon, New York, 1973); P. K. Kaw, in *Advances in Plasma Physics*, edited by A. Simon and W. B. Thompson (Wiley, New York, 1976).
[11]I. R. Jones, Phys. Plasmas **6**, 1950 (1999).
[12]I. R. Jones, C. Deng, I. M. El-Fayoumi, and P. Euripides, Phys. Rev. Lett. **81**, 2072 (1998).
[13]I. Zhelyazkov, O. Stoyanov, and M. Y. Yu, Plasma Phys. Controlled Fusion **26**, 813 (1984).
[14]P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw Hill, New York, 1953); J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1975).
[15]S. Xu and D. Brotherton-Ratcliffe, Plasma Phys. Controlled Fusion **30**, 1905 (1988).
[16]D. Brotherton-Ratcliffe and R. G. Storer, Plasma Phys. Controlled Fusion **30**, 967 (1988).
[17]S. Xu and I. R. Jones, Plasma Phys. Controlled Fusion **35**, 531 (1993).
[18]S. Xu, AINSE Plasma Sci. Tech. Conf., Canberra, 12–15 July 1993, Proceedings (Australian Academy of Science, Canberra, 1993), P-II-10.