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Asymptotic analysis of stochastic manufacturing system with slow and fast machines

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Abstract

A dynamic flow-shop is considered consisting of one slow and one fast machine. Production capacaties of both machines vary randomly according to a Markov chain whose transition rates are consistent with the time scale of the fast machine. Problem of minimization of a discounted cost of manufacturing is formulated. A conjecture is presented regarding the asymptotic behaviour of the value function for the above problem when the separation of slow and fast time scales becomes singular. Natural conditions are formulated under which the conjecture is might be satisfied, hence this note is a report on work in progress.

1 Formulation of the problem

Notation and terminology used throughout this note are similar to those used in the book by Sethi and Zhang [5].
We consider a two-machine dynamic flow-shop characterized by the presence of slow and fast operating machines. The vector process of capacities on both machines is denoted by $k(t) = (k_1(t), k_2(t))$, and is assumed to be a Markov process such that $k_i(t) \in \{0, m_i\}, m_i > 0, i \in 1, 2$. The infinitesimal generator matrix for the process $k(\cdot)$ is denoted by $Q = [q_{ij}]$. By $v(t) = (v_1(t), v_2(t))$ we denote a vector process of production intensities for the two machines, with $v_i(t) \in [0,1], i = 1, 2$. Now, let $\varepsilon > 0$ be a small constant representing the ratio between slow and fast time regimes in our system. We additionally assume that the capacities of both machines change with rate that is consistent with the fast time regime. Considering the first machine as the fast one, and second machine as the slow one we obtain the following dynamical model of the flow-shop:

$$\begin{cases} \varepsilon \frac{dx_1(t)}{dt} & = & u_1(t) - \gamma u_2(t), x_1(0) = x_1 \\ \frac{dx_2(t)}{dt} & = & u_2(t) - z, x_2(0) = x_2, \end{cases} \tag{1.1}$$

where $x_i(t), i = 1, 2$ represent state of the corresponding buffers at time $t \geq 0, u_i(t) \triangleq k_i(\frac{t}{\varepsilon})v_i(t), i = 1, 2$, are the production rates, γ is a compatibility constant, and z is a constant demand rate.

We impose the following constraint on the state of the first buffer:

(A1) $\quad 0 \leq x_1(t) \leq \overline{x}_1, \forall t \geq 0.$

For $k = (k_1, k_2) \in K \triangleq \{0, m_1\} \times \{0, m_2\}$, representing the state of $k(t)$, and for $x = (x_1, x_2) \in X \triangleq [0, \overline{x}_1] \times R$, representing the states of the two buffers we define

$$V(x,k) \triangleq \{v \in V \triangleq [0,1]^2 : k_1v_1 - k_2v_2 \geq 0 \text{ if } x_1 = 0, \\ k_1v_1 - k_2v_2 \leq 0 \text{ if } x_1 = \overline{x}_1\}.$$

Let also $\mathcal{F}_t^\varepsilon = \sigma\{k(s), s \leq \frac{t}{\varepsilon}\}$, for $t \geq 0$.

Definition 1.1 (Admissible controls)
A control process $v(\cdot)$ with values in V is admissible with respect to the initial conditions $x \in X$ and $k \in K$ if

(i) $v(\cdot)$ is $(\mathcal{F}_t^\varepsilon)_{t \geq 0}$ adapted

(ii) the corresponding state process $x(\cdot)$, solution to (1.1) satisfies (A1).

The class of such admissible control processes is denoted by $\mathcal{A}^\varepsilon(x, k)$. □

Remark 1.1
Similarly to [5] we can define admissible feedback controls, except that now, if $v(\cdot)$ is a feedback control, then $v(t) = \nu(x(t), k(\frac{t}{\varepsilon}))$, for some functions $\nu(\cdot,\cdot)$ and $t \geq 0$. Note that under the control $\nu(\cdot,\cdot)$ the corresponding state process satisfies (A1) if and only if $\nu(x(t), k(\frac{t}{\varepsilon})) \in V(x(t), k(\frac{t}{\varepsilon})), t \geq 0$. □

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Let $(\Omega, \mathcal{F}, \mathcal{P})$ be the underlying probability space and define a discounted cost functional

$$\mathcal{J}_\alpha^\varepsilon(x,k,v(\cdot)) \triangleq E[\int_0^\infty e^{-\alpha s}(h(x(s)) + c(u(s)))ds \mid x(0) = x, k(0) = k], \quad (1.2)$$

where $\alpha > 0$, and consider the following optimization problem,

$$\inf_{v(\cdot)\in\mathcal{A}^\varepsilon(x,k)} \mathcal{J}_\alpha^\varepsilon(x,k,v(\cdot)), \forall (x,k) \in X \times K, \quad (P_\varepsilon^\alpha)$$

subject to (1.1).

Denoting by $w_\varepsilon^\alpha(\cdot,\cdot)$ the value function for the problem $(\mathcal{P}_\varepsilon^\alpha)$, that is

$$w_\varepsilon^\alpha(x,k) \triangleq \inf_{v(\cdot)\in\mathcal{A}^\varepsilon(x,k)} \mathcal{J}_\alpha^\varepsilon(x,k,v(\cdot)),$$

we are interested in limiting behaviour of w_ε^α when $\varepsilon \to 0$.

Remark 1.2
Due to space limitation we are not specifying here any assumptions on $Q, h(\cdot)$ and $C(\cdot)$. Such assumptions are implicitly implied by conditions (C1)-(C3) listed in Section 3 below. □

2 Infinitesimal problem

Fix $x_2, l \in R$ and consider the following infinitesimal problem (in the terminology of Artstein and Gaitsgory [1]):

$$\limsup_{T\to\infty} \inf_{V_I(\cdot)\in\mathcal{A}_I(x_1,k)} \mathcal{J}_T(x_1,k,V_I(\cdot);x_2,l) \quad (P_I^{x_2,l})$$

subject to

$$\begin{cases} \frac{dx_1(t)}{dt} &= k_1(t)V_{I1}(t) - k_2(t)V_{I2}(t) \\ x_1(0) &= x_1, \end{cases} \quad (2.1)$$

where, for $(x_1,k) \in [0,\overline{x}_1] \times K$,

$\mathcal{A}_I(x_1,k) \triangleq \{V_I(\cdot) : V_I(\cdot)$ is a $V-$valued process, adapted to $(\mathcal{F}_t)_{t\geq 0} \triangleq (\sigma\{k(s), s \leq t\})_{t\geq 0}$, and such that the corresponding solution of (2.1) satisfies (A1), $x_1(0) = x_1, k(0) = k\}$ (2.2)

and

$$\mathcal{J}_T(x_1,k,V_I(\cdot);x_2,l) \triangleq \frac{1}{T}E[\int_0^T [h.(x_1(s),x_2) + l(k_2(s)V_{I2}(s) - z) + c(k_1(s)V_{I1}(s), k_2(s)V_{I2}(s))]ds \mid x_1(0) = x, k(0) = k]. \quad (2.3)$$

Let us denote the value of the limsup in the formulation of $(P_I^{x_2,l})$ by $h(x_2,l)$.

3 A limit conjecture

We impose the following conditions,

(C1) $\lim_{t\to\infty} \inf_{V_I(\cdot)\in\mathcal{A}_I(x_1,k)} \mathcal{J}_T(x_1,k,V_I(\cdot);x_2,l) = h(x_2,l)$ uniformly with respect to $x_1 \in [0,\overline{x}_1]$ and x_2, l in any compact subset of R^2.

(C2) There exist a constant $C_1 > 0$ and a function $C_2(\varepsilon), \lim_{\varepsilon\downarrow 0} C_2(\varepsilon) = 0$, such that
$\sup_{x_1,x_1'\in[0,\overline{x}_1]} \sup_{k,k'\in K} \mid w_\varepsilon^\alpha(x_1,x_2,k) - w_\varepsilon^\alpha(x_1',x_2',k') \mid \leq C_1 \mid x_2 - x_2' \mid + C_2(\varepsilon)$
for any x_2, x_2' in a compact subset of R.

(C3) There exists a unique viscosity solution $w^\alpha(x_2)$ to

$$h(x_2, \frac{dw^\alpha(x_2)}{dx_2}) = \alpha w^\alpha(x_2).$$

It appears that using the techniques similar to [1, 3, 4] it might be possible to establish the validity of the following conjecture.

Conjecture 3.1
Assume (A1) and (C1) - (C3). Then

$$\lim_{\varepsilon\downarrow 0} w_\varepsilon^\alpha(x_1,x_2) \to w^\alpha(x_2)$$

uniformly in $x_1 \in [0,\overline{x}_1]$ and x_2 in any compact subset of R. □

Remark 3.1
Under some extra conditions the function $w^\alpha(x_2)$ can be interpreted as the value function for an appropriately defined so called limit control problem (similarly as in Bensoussan and Blakenship [2], and Artstein and Gaitsgory [1, 3, 4].)

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