ALGORITHMS FOR ROBUST POLE ASSIGNMENT IN SINGULAR SYSTEMS

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ABSTRACT

The solution of the pole assignment problem by feedback in singular systems is parameterized and conditions are given which guarantee the regularity and maximal degree of the closed loop pencil. A robustness measure is defined, and numerical procedures are described for selecting the free parameters in the feedback to give optimal robustness.

1. INTRODUCTION

For a robust solution to the problem of pole assignment by feedback in a multi-variable, linear, time-invariant control system it is necessary for the prescribed pōles to be insensitive to perturbations in the closed loop system matrices. In a non-degenerate system, robustness can be achieved by selecting the eigenvectors associated with the assigned eigenvalues of the closed loop system such that the 'condition number' of the modal matrix of eigenvectors is small [5]. The inverse of the condition number thus gives a measure of the robustness of the system, and optimizing this measure in the state space corresponds to maximizing a lower bound on the stability margin of the system [7]. Algorithms for selecting the eigenvectors to give a robust feedback are described in [5].

In a singular, or degenerate, system the eigenstructure is more complicated. For robustness it is necessary not only that the poles be insensitive to perturbations, but also that the system pencil remains regular, and that the degree, that is, the number of finite poles of the system remains unaltered under perturbations. In this paper we define an over-all measure of the conditioning of the generalized eigenproblem for the singular system and demonstrate that robustness can be achieved, with guaranteed regularity of the closed loop pencil, by selecting the eigenvectors associated with the given <u>finite</u> poles, <u>together</u> with a certain set of additional parameters, such as to optimize this measure. In the next section the problem is stated formally and background theory is given. In section 3 the robustness measure is defined and in section 4 algorithms for determining the feedback solution are described. These algorithms are based on the procedures developed in [5] for non-singular systems. A numerical example is given in section 5.

2. THE POLE ASSIGNMENT PROBLEM.

In singular systems the problem of pole assignment by state feedback gives a generalized inverse eigenvalue problem for a matrix pencil. The problem is:

Given system matrices $E, A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times m}$, find feedback matrix $F \in \mathbf{R}^{m\times n}$ such that the closed loop matrix pencil $\lambda E - M$, where $M = A + BF$, has as many prescribed finite eigenvalues as possible and remains <u>regular</u>, that is, such that

$$(A + BF)X_q = EX_q\Lambda_q , \tag{1}$$

where $\Lambda_q = \mathrm{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_q\}$ is given, and

$$\det(A + BF - \lambda E) \not\equiv 0 \tag{2}$$

with q maximal.

It is easily seen that $q \leq \mathrm{rank}\, E$, but equality cannot generally be achieved. Necessary and sufficient conditions for the existence of a solution to the pole assignment problem for any arbitrary self-conjugate set of (distinct) poles $L = \{\lambda_1, \lambda_2, \ldots, \lambda_q\}$ with $q = \mathrm{rank}\,(E)$ are shown in [3] and [4] to be given by:

<u>C1</u> $\mathrm{rank}[B, A - \lambda E] = n \quad \forall \lambda \in \mathbf{C}$;

<u>C2</u> $\mathrm{rank}[B, E + AS_\infty S_\infty^T] = n$,

where S_∞ gives an orthonormal basis for $\ker E$, that is: $ES_\infty = 0$, $S_\infty S_\infty^T = I$ and $\mathrm{rank}\, S_\infty = n - \mathrm{rank}\, E$. We note that C1 is just the finite pole controllability condition and C2 is the infinite pole controllability, or infinite pole shifting, condition [1] [2] [6] [10].

In [3] and [4] it is shown, furthermore, that condtion C2 may be used to establish the regularity of the closed loop pencil. We have the following structure theorem:

<u>Theorem 1: Given the set</u> $L = \{\lambda_1, \lambda_2, \ldots, \lambda_q\}$ <u>of (distinct) self-conjugate complex numbers, where</u> $q = \mathrm{rank}\, E$, <u>there exist vectors</u>

$$\underline{x}_i \in S_i \equiv \{\underline{x} \mid (A-\lambda_i E)\underline{x} \in \mathcal{R}(B)\} , \quad i = 1, 2, \ldots, q , \tag{3}$$

such that

$$\text{rank}[X_q, S_\infty] = n , \qquad (4)$$

where $X_q = [\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_q]$, and a matrix W satisfying

$$\text{rank}[E + AS_\infty S_\infty^T + BWS_\infty^T] = n \qquad (5)$$

if and only if conditions C1 and C2 hold. If (3)-(5) hold, then the matrix F given by

$$F = [B^+(EX_q\Lambda_q - AX_q), W][X_q, S_\infty]^{-1} \qquad (6)$$

solves the pole assignment problem, and (1) and (2) are satisfied. (Here B^+ denotes the Moore-Penrose pseudo-inverse of matrix B).

□

We remark that if multiple poles are to be assigned, then C1 and C2 are necessary, but may not be sufficient, for (3)-(5) to hold.

From Theorem 1 we conclude that the required feedback matrix F can be parameterized in terms of the eigenvectors $\underline{x}_i$, $i = 1, 2, \ldots, q$, associated with the prescribed finite poles and the components of matrix W . For a robust solution to the pole placement problem we must, therefore, select the freedom in the vectors $\underline{x}_i$ and the matrix W such that the poles of the closed loop pencil are insensitive to perturbations and such that the rank of the matrices in (4) and (5) is also insensitive to perturbations. In the next section we give an over-all measure of robustness which quantifies these properties.

3. ROBUSTNESS MEASURE

In non-singular systems, (where $E \equiv I$ can be taken), the sensitivity of an eigenvalue is well-known [9] to depend on a condition number proportional to the 'angle' between its associated right and left eigenvectors. For non-defective matrices, the square of the Frobenius condition number of the modal matrix of eigenvectors is then equal to a weighted sum of the squares of the condition numbers, and techniques for selecting the eigenvectors to minimize this sensitivity measure have been developed [5].

In singular systems, the sensitivity of the finite poles can be measured similarly, but the influence of perturbations on the infinite poles must also be taken into consideration. Following [8] we may define a generalized eigenvalue of the matrix pencil $\lambda E - M$ to be a pair $(\lambda, \delta) \in \mathbf{C} \times \mathbf{R}$, where the pole takes the finite 'value' λ/δ for $\delta \neq 0$, and becomes infinite for $\delta = 0$. We denote the right and left eigenvectors associated with (λ, δ) by $\underline{x}, \underline{y}$, so that $\underline{x}, \underline{y}$ satisfy

$$\delta M\underline{x} = \lambda E\underline{x} , \qquad \delta \underline{y}^T M = \lambda \underline{y}^T E .$$

If the pencil is non-defective, that is, it has a full set of n linearly independent eigenvectors, then perturbations of order $O(\varepsilon)$ in the coefficients of M and E cause perturbations of order $O(\varepsilon c(\lambda, \delta))$ in a simple eigenvalue, where the conditon number $c(\lambda, \delta)$ is defined by

$$c(\lambda, \delta) = \|\underline{y}\|_2 \|\underline{x}\|_2 / (|\lambda|^2 + \delta^2)^{\frac{1}{2}} . \qquad (7)$$

Here $\|\cdot\|$ denotes the L_2-vector norm and the eigenvectors $\underline{x}, \underline{y}$ are assumed to be normalized such that

$$\underline{y}^T E\underline{x} = \delta , \qquad \underline{y}^T M\underline{x} = \lambda .$$

For a robust solution to the pole placement problem, we must minimize some over-all measure of the condition numbers (7) of the eigenvalues of the closed loop pencil. Without loss of generality we may assume that the eigenvalues (λ_j, δ_j) of the pencil are scaled and ordered such that $\delta_j = 1$ for $j = 1, 2, \ldots, q$, and $\lambda_j = 1, \delta_j = 0$ for $j = q+1, \ldots, n$. We also let $X = [\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_n]$ and $Y = [\underline{y}_1, \underline{y}_2, \ldots, \underline{y}_n]$ denote the modal matrices of the associated right and left eigenvectors $\underline{x}_j, \underline{y}_j$, where $\underline{x}_j$ is normalized to unit length ($\|\underline{x}_j\|_2 = 1$), and we assume that X and Y satisfy

$$Y^T EX = \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix} , \qquad Y^T MX = \begin{bmatrix} \Lambda_q & 0 \\ 0 & I_{n-q} \end{bmatrix} .$$

It follows [4] that

$$X = [X_q, S_\infty] , \qquad Y^T = [EX_q, MS_\infty]^{-1} . \qquad (8)$$

If we now define

$$\nu(\underline{\omega}) \equiv \| D_{\underline{\omega}} Y^T \|_F \equiv \Big(\sum_{j=1}^{n} d_j^2 \|\underline{y}_j\|_2^2 \Big)^{\frac{1}{2}} , \qquad (9)$$

where $\|\cdot\|_F$ denotes the Frobenius (Euclidean) matrix norm, and $D_{\underline{\omega}} = \text{diag}\{d_j\}$ with $d_j = \omega_j / (|\lambda_j|^2 + \delta_j^2)^{\frac{1}{2}}$, $\omega_j > 0$, and $\Sigma \omega_j^2 = 1$, then, by the assumption $\|\underline{x}_j\|_2 = 1$, we have

$$\begin{aligned} \nu(\underline{\omega})^2 &= \sum_{j=1}^{n} \omega_j^2 \|\underline{y}_j\|_2^2 \|\underline{x}_j\|_2^2 / (|\lambda_j|^2 + \delta^2) \\ &\equiv \sum_{j=1}^{n} \omega_j^2 c^2(\lambda_j, \delta_j) . \end{aligned} \qquad (10)$$

The measure $\nu(\underline{\omega})^2$ is, therefore, equal to a weighted sum of the squares of the condition numbers. In the case of multiple eigenvalues the condition numbers are defined similarly and the square of the measure $\nu(\underline{\omega})$, defined by (9), gives a bound on the weighted sum of squares of the condition numbers [4]. To minimize the sensitivity of the eigenvalues of the pencil, then we aim to minimize $\nu(\underline{\omega})$.

For robustness the rank conditions (4) and (5) must also be insensitive to perturbations. A general measure of the distance of a matrix H from singularity is given by the matrix condition number $\kappa = \|H\|\|H^{-1}\|$, and, therefore, to ensure that the matrices $[X_q, S_\infty]$ and $[E + MS_\infty S_\infty^T]$, where $MS_\infty = AS_\infty + BW$, remain non-singular, we require their respective condition numbers κ_1, κ_2 to be small. It can be shown [4] that the measure $\nu(\underline{\omega})$ also provides bounds on κ_1 and κ_2 ; specifically, we have from (8) that

$$\alpha_1(\kappa_1\kappa_2)^{\frac{1}{2}} \le \nu(\underline{\omega})\|E + MS_\infty S_\infty^T\| \le \alpha_2(\kappa_1\kappa_2) \ , \quad (11)$$

where α_1 and α_2 are fixed constants. Thus, if we minimize $\nu(\underline{\omega})$, subject to $\|E + MS_\infty S_\infty^T\|$ remaining bounded, then κ_1 , κ_2 are also minimized, and conversely. The robustness of the closed loop pencil is thus measured by the inverse of $\nu(\underline{\omega})$, or, equivalently, the product $\kappa_1\kappa_2$, where $\|E + MS_\infty S_\infty^T\|$ is bounded. In practice we choose to minimize κ_1 and κ_2 separately since the free parameters in each of these measures are independent.

4. NUMERICAL ALGORITHMS

The numerical procedures consist of four basic steps.

<u>Step A</u>: Compute the orthonormal basis S_∞ of ker E , and compute orthonormal bases S_j for the subspaces $\mathcal{S}_j$, defined in (3), j = 1, 2, ..., q .

<u>Step W</u>: Select matrix W to minimize $\|[E + AS_\infty S_\infty^T + BWS_\infty^T]^{-1}\|$ subject to $\|E + AS_\infty S_\infty^T + BWS_\infty^T\| \le tol$, where tol is some given tolerance.

<u>Step X</u>: Select vectors $\underline{x}_j = S_j\underline{v}_j \in \mathcal{S}_j$ with $\|\underline{x}_j\|_2 = 1$, j = 1, 2, ..., q , to minimize $\kappa_1 = \|[X_q, S_\infty]\|\|[X_q, S_\infty]^{-1}\|$.

<u>Step F</u>: Determine F by solving the equation

$$F[X_q, S_\infty] = [B^+(EX_q\Lambda_q - AX_q), W] \ .$$

Standard library software with reliable procedures for computing QR, SVD and LU matrix decompositions is used to accomplish these steps. <u>Step X</u> uses one of the iterative methods described in [5] for selecting vectors $\underline{x}_j \in \mathcal{S}_j$ such that matrix $X = \{\underline{x}_j\}$ is <u>well-conditioned</u>. <u>Step W</u> essentially constructs matrix W such as to maximize the smallest singular value of $\|E + MS_\infty S_\infty^T\|$, subject to the largest singular value remaining bounded. In practice the result is only achieved approximately. Details of the computational procedures are given in [4].

5. EXAMPLE

To illustrate the form of the robust solutions determined by the algorithm, we reproduce here the results of an example given in [4]. For this example n = 5, m = 3, q = 3 and E, A, B are given by

$$E = \begin{bmatrix} 0 & 0 & 0 & 1.72 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0.82 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.0 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1.1 & 0 & 0 & 0 \\ 0 & 0 & 1.56 & 0 & 0 \\ 1.23 & 0 & 0 & 1.98 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.01 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 1.55 & 0 & 0 & 0 \\ 0 & 0 & 1.07 & 0 & -2.5 \\ 0 & 0 & 0 & -1.11 & 0 \end{bmatrix}$$

We assign the stable eigenvalue set $L = \{-0.5, -1.0, -2.0\}$. The condition numbers obtained for the computed solution are $\kappa_1 = 9.70$ and $\kappa_2 = 7.10$, where $\|E + MS_\infty S_\infty^T\| = 2.05$, and the results are reasonably robust. The computed feedback matrix F has magnitude $\|F\|_2 = 1.7806$ and is given to four figures by

$$F = \begin{bmatrix} 0.2700 & 0.0 & 0.0 & 0.7994 & 1.471 \\ -0.1843 & 0.0 & 0.0 & -0.7196 & -0.1306 \\ 0.07257 & 0.3099 & 0.0 & 0.5288 & -0.15686 \end{bmatrix}$$

To demonstrate the effects of perturbations, the feedback matrix F is rounded to three figures, introducing random errors of maximum order $\pm 10^{-3}$ into the system matrix. The errors in the assigned eigenvalues due to these perturbations are $\{1_{10^{-3}}, 4_{10^{-4}}, 3_{10^{-4}}\}$, giving a maximum relative error of 0.2% , and the magnitudes of the errors are well within the predicted range.

6. CONCLUSIONS

Conditions are given for the solution of the pole assignment problem by feedback in singular systems such that regularity and maximal degree are guaranteed. The feedback is parameterized in terms of the eigenvectors of the prescribed finite poles and the components of a matrix W which is selected to ensure regularity of the closed loop pencil. A measure of the sensitivity of the assigned eigenvalues of the system is given which also bounds the sensitivity of the regularity and degree of the pencil to perturbations. This measure is, thus, inversely proportional to the over-all robustness of the closed loop system. Reliable numerical techniques for selecting the free parameters in the feedback to optimize the robustness measure are described, and a numerical example is given. Experimental evidence shows that the procedure provides a useful design tool.

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