

# Time Delays and Stimulus-Dependent Pattern Formation in Periodic Environments in Isolated Neurons

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**Abstract**—The dynamical characteristics of a single isolated Hopfield-type neuron with dissipation and time-delayed self-interaction under periodic stimuli are studied. Sufficient conditions for the heteroassociative stable encoding of periodic external stimuli are obtained. Both discrete and continuously distributed delays are included.

**Index Terms**—Global attractivity, Hopfield-type neural networks, periodic environments, time delays.

## I. INTRODUCTION

THERE IS evidence from the experimental and theoretical studies [21], [8] that a mammal's brain may be exploiting dynamic attractors for its encoding and subsequent associative recall rather than temporally static (equilibrium-type) attractors as it has been proposed in most studies of artificial neural networks. Limit cycles, strange attractors and other dynamical phenomena have been used by many authors to represent encoded temporal patterns as associative memories [7], [22], [4], [17], [14]. Most of the existing literature on theoretical studies of artificial neural networks is predominantly concerned with autonomous systems containing temporally uniform network parameters and external input stimuli. Literature dealing with time-varying stimuli or network parameters appears to be scarce; such studies are, however, important to understand the dynamical characteristics of neuron behavior in time-varying environments.

In this article, we study how a temporally varying, in particular a periodic environment, can influence the dynamics of a single effective neuron of the Hopfield-type; in addition to the temporal variation of the input, we incorporate time delays in the processing part of the neuron's architecture. We consider discrete delays and delays distributed over a finite and infinite interval. It will be found from the results of this article that while the neural dissipation dominates the gain, time-delays do not restrict the associative recall of the encoded patterns. It has been reported [5], [12], [6] that assemblies of cells in the visual cortex oscillate synchronously in response to external stimuli. Such a synchrony is a manifestation of the encoding process of temporally varying external stimuli.

The purpose of this paper is to obtain sufficient conditions for the existence (or encoding) of a globally attractive (heteroassociative recall) periodic solution (or a pattern) associated with a given periodic external stimulus. The neuronal parameters, dissipation and gain can either be temporally uniform or be periodic with the same period as that of the stimulus. In particular we study the dynamics of a single artificial effective neuron model in continuous time with time delays. The incorporation of delays in the formulation is motivated by the following; delays are naturally present biological networks through synaptic transmission, finite conduction velocity and neural processing of input stimuli. We refer to the articles of Gopalsamy and He [10], [11] and the references therein for literature related to the stability of neural networks with time delays in temporally uniform environments modeled by autonomous delay and integro-differential equations.

## II. MODEL SPECIFICATION

We formulate a model of a single artificial effective neuron with dissipation and a processing delay subjected to an external temporally periodic stimulus. We want to obtain sufficient conditions for the neuron to encode the stimulus in a temporally periodic pattern as the unique solution of a neuron equation of the Hopfield-type given by

$$\frac{dx(t)}{dt} = -a(t)x(t) + b(t) \tanh[x(t - \tau)] + f(t) \quad -\infty < t_0 < t \quad (1)$$

in which  $t_0$  is a fixed real number;  $x(t)$  denotes the membrane potential of the neuron modeled;  $a(\cdot)$ ,  $b(\cdot)$ ,  $f(\cdot)$  denote continuous real valued functions defined on  $(-\infty, \infty)$  and are periodic with period  $\omega > 0$  so that

$$\begin{aligned} a(t + \omega) &= a(t) \\ b(t + \omega) &= b(t) \\ f(t + \omega) &= f(t), \quad t \in \mathbb{R}. \end{aligned} \quad (2)$$

The authors recognize the fact that it is unlikely for all of  $a(\cdot)$ ,  $b(\cdot)$ ,  $f(\cdot)$  to have the same period  $\omega$ . One of the possible cases that such an assumption includes is the following; the parameters  $a(\cdot)$  and  $b(\cdot)$  are temporally uniform while the stimulus input  $f(\cdot)$  is periodic. Our analysis includes this case. Alternatively, the stimulus  $f(\cdot)$  can be temporally uniform while  $a(\cdot)$  or  $b(\cdot)$  or both  $a(\cdot)$  and  $b(\cdot)$  can be periodic; our analysis includes this case also. A more general problem is concerned with the case where  $a(\cdot)$ ,  $b(\cdot)$  and  $f(\cdot)$  have integrally independent

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periods. The authors intend to pursue this problem in a subsequent investigation; this general case needs different mathematical tools and such a system will lead to quasiperiodic or almost periodic neural responses. In this article we consider the above simpler case and establish that as a special case, if  $a(\cdot)$  and  $b(\cdot)$  are time invariant (constants) satisfying the required sufficient conditions while the stimulus is periodic, the neuron response will inherit the period of the stimulus; the primary motivation for this article has been the derivation of this result. The assumption of periodicity of  $a(\cdot)$  and  $b(\cdot)$  has been incorporated for generality of analysis only. It will follow from our analysis that the neuron response will inherit the period of any one of  $a(\cdot)$ ,  $b(\cdot)$  or  $f(\cdot)$  when at least one of them is periodic or when two or all of them are periodic with a common period.

It is believed that a self-connection in a single neuron is unlikely to occur. While this is plausible, there are some circumstances where self-interaction is possible as it is in the case of single effective neuron (see, for instance, [20], [23]). We provide some details in brief for the convenience of the reader. Consider for instance the deterministic Hopfield model [15]

$$C_i \frac{du_i}{dt} = -\frac{u_i}{R_i} + \sum_{j \neq i}^3 J_{ij} \tanh(u_j) + \alpha_i, \quad i = 1, 2, 3. \quad (3)$$

Suppose that the neurons 2 and 3 relax to a steady state at a much faster rate than neuron 1 in the sense that  $R_2 \ll R_1$ ,  $R_3 \ll R_1$  and furthermore suppose that  $R_2 \ll 1$ ,  $R_3 \ll 1$ ; we can then suppose that

$$\begin{aligned} C_2 \frac{du_2}{dt} \approx 0 &\implies u_2 \\ &\approx R_2 [J_{21} \tanh(u_1) + J_{23} \tanh(u_3) + \alpha_2] \\ C_3 \frac{du_3}{dt} \approx 0 &\implies u_3 \\ &\approx R_3 [J_{31} \tanh(u_1) + J_{32} \tanh(u_2) + \alpha_3]. \end{aligned} \quad (4)$$

By using the assumption that  $R_2$  and  $R_3$  are small, we have

$$\begin{aligned} \tanh(u_2) &\approx R_2 [J_{21} \tanh(u_1) + J_{23} \tanh(u_3) + \alpha_2] \\ \tanh(u_3) &\approx R_3 [J_{31} \tanh(u_1) + J_{32} \tanh(u_2) + \alpha_3]. \end{aligned} \quad (5)$$

The equations in (5) are linear in  $\tanh(u_2)$  and  $\tanh(u_3)$  and can be solved in terms of  $\tanh(u_1)$ ; these values can be substituted in the first equation of (3) so as to eliminate the presence of  $\tanh(u_2)$  and  $\tanh(u_3)$  and this procedure will lead to an equation for the dynamics of a single neuron with a self-connection term.

We briefly address the question, why should one consider time dependence in the coefficients  $a(\cdot)$  and  $b(\cdot)$  in (1)? We remark that there is no *a priori* reason for time dependence in  $a(\cdot)$  and  $b(\cdot)$ ; however effects of dynamical environments have been investigated in areas other than neural networks, especially in population dynamics where temporal variations of the environment have been incorporated in the parameters of systems like the Lotka–Volterra equation by assuming that the system parameters are time dependent, periodic or almost periodic. If a neuron is operating under a periodic environment such as being excited or inhibited by periodic inputs, it is not unreasonable to assume that the dissipation and gain are also periodic; an assumption

of this type does not preclude them from being temporally uniform. We remark that when the external input is time-varying, the neuron is operating in a time-dependent environment. For example, when one is listening to a piece of music, the temporal tones of the music can give rise to varying conditions of the brain. In an effort to investigate the effects of such varying conditions, we can let the dissipation and gain parameters time-dependent. There is another plausible reason for assuming that  $a(\cdot)$  and  $b(\cdot)$  in (1) are time-varying. Ott *et al.* [18] have shown theoretically that one can convert a chaotic attractor to one of possible attracting time periodic motions by making time dependent perturbations of system parameters. Our primary motivation for making  $a(\cdot)$  and  $b(\cdot)$  in (1) to be time dependent is one of generalization rather than specialization; all our results and analyzes are valid if  $a(\cdot)$  and  $b(\cdot)$  in (1) are temporally uniform.

The nonnegative number  $\tau$  in (1) denotes a neural processing or synaptic transmission delay; several possible types of delays will be considered in this article. In (1),  $a(\cdot)$  denotes a quantitative measure of the neuronal dissipation or a negative feedback term;  $b(\cdot)$  denotes the neuron gain and  $f(\cdot)$  denotes the external stimulus or input. We have assumed that the activation or response of the neuron is given by the monotonic function  $\tanh(\cdot)$ . Using (2) and the continuity of  $a$ ,  $b$ ,  $f$ , we define  $a_*$ ,  $a^*$ ,  $b_*$ ,  $b^*$ ,  $f_*$ ,  $f^*$  by the following:

$$\begin{Bmatrix} a_* \\ b_* \\ f_* \end{Bmatrix} = \min_{0 \leq t \leq \omega} \begin{Bmatrix} a(t) \\ |b(t)| \\ |f(t)| \end{Bmatrix}$$

and

$$\begin{Bmatrix} a^* \\ b^* \\ f^* \end{Bmatrix} = \max_{0 \leq t \leq \omega} \begin{Bmatrix} a(t) \\ |b(t)| \\ |f(t)| \end{Bmatrix}. \quad (6)$$

We shall assume that  $a_* > 0$ ; it is elementary to see from (1) that

$$\frac{d^+|x(t)|}{dt} \leq -a_*|x(t)| + b^* + f^*, \quad t > t_0$$

and hence

$$\frac{d^+|x(t)|}{dt} \leq a_* \left\{ \frac{b^* + f^*}{a_*} - |x(t)| \right\}, \quad t > t_0 \quad (7)$$

where  $d^+(\cdot)/dt$  denotes the upper right derivative. It follows from (7) that

$$x(t_0) < \frac{b^* + f^*}{a_*} \implies x(t) \leq \frac{b^* + f^*}{a_*} \quad \text{for } t > t_0. \quad (8)$$

Similarly we have from (1)

$$\begin{aligned} \frac{d^+|x(t)|}{dt} &\geq -a^*|x(t)| - b^* - f^* \\ &= -a^* \left\{ \frac{b^* + f^*}{a^*} + |x(t)| \right\}, \quad t > t_0 \end{aligned} \quad (9)$$

and hence

$$x(t_0) > -\frac{b^* + f^*}{a^*} \implies x(t) \geq -\frac{b^* + f^*}{a^*} \quad \text{for } t > t_0. \quad (10)$$

The interval

$$\left[ -\frac{b^* + f^*}{a^*}, \frac{b^* + f^*}{a_*} \right]$$

is invariant with respect to (1) in the sense that if the initial value  $x(s) = \varphi(s)$ ,  $s \in [t_0 - \tau, t_0]$  belongs to the invariant interval, then the corresponding solution  $x(t)$  belongs to the same interval for  $t \geq t_0$ . In the next section we obtain sufficient conditions for the uniform stability of solutions of (1) in the following sense (see [13]).

**Definition 2.1:** A solution  $x_0(t)$  of (1) is said to be uniformly stable if for every  $\epsilon > 0$  there exists a  $\delta = \delta(\epsilon) > 0$  such that

$$\begin{aligned} |\varphi(s) - x_0(s)| < \delta \quad \text{for } s \in [t_0 - \tau, t_0] \\ \implies |x(t, t_0, \varphi) - x_0(t)| < \epsilon \quad \text{for } t \geq t_0 \end{aligned}$$

where  $x(t, t_0, \varphi)$  denotes a solution of (1) satisfying  $x(s, t_0, \varphi) = \varphi(s)$ ,  $s \in [t_0 - \tau, t_0]$ .

### III. DISCRETE DELAYS AND ENCODING OF PERIODIC STIMULI

In this section, we establish sufficient conditions for the associative encoding of a given external periodic stimulus in the form of a periodic solution of the neuron equation. We first consider the uniform stability of solutions of equations of the form

$$\frac{dx(t)}{dt} = -a(t)x(t) + b(t) \tanh[x(t - \tau)] + f(t), \quad t > 0 \quad (11)$$

where  $a(\cdot)$ ,  $b(\cdot)$ ,  $f(\cdot)$  are continuous real-valued and bounded functions defined on  $\mathbb{R} = (-\infty, \infty)$ ;  $\tau$  denotes a nonnegative constant. We assume that (11) is supplemented with an initial condition of the form

$$x(s) = \varphi(s) \in \mathbb{R}, \quad s \in [-\tau, 0], \quad \varphi \in C[-\tau, 0] \quad (12)$$

where  $C[-\tau, 0]$  denotes the space of all continuous real valued functions defined on  $[-\tau, 0]$  endowed with the supremum norm defined by

$$\text{for } x \in C[-\tau, 0], \quad \|x(t)\| = \sup_{s \in [-\tau, 0]} |x(t + s)|.$$

Note that if  $x(\cdot)$  is defined on  $[-\tau, \infty)$ , then we can associate with such an  $x$  an element of  $C[-\tau, 0]$  denoted by  $x_t$  where  $x_t = x(t + s)$ ,  $s \in [-\tau, 0]$  for  $t \geq 0$ . The next result provides sufficient conditions for the uniform stability of all solutions of (11).

**Theorem 3.1:** Assume that the coefficients  $a(\cdot)$ ,  $b(\cdot)$  and the delay  $\tau$  satisfy

$$a_* > b^* \exp[a_* \tau] \geq 0 \quad (13)$$

where

$$a_* = \inf_{t \in \mathbb{R}} a(t), \quad b^* = \sup_{t \in \mathbb{R}} |b(t)|.$$

Then every solution of (11) and (12) is uniformly stable.

Before we proceed to the proof of the above result, some remarks regarding the implication of (13) are provided. We note that the requirement (13) is only a sufficient condition and it appears that (13) is not a necessary condition. A very much improved sufficient condition is the following:

$$\int_0^\omega (a(s) - |b(s)|) ds > 0$$

which itself is satisfied if

$$\inf_{t \in \mathbb{R}} a(t) > \sup_{t \in \mathbb{R}} |b(t)| > 0.$$

Note that these conditions are delay independent and show that if the dissipation  $a(\cdot)$  dominates the gain  $b(\cdot)$  then the coding of the input stimulus is delay independent. The authors' preliminary results toward these are promising and will be elaborated in a subsequent article.

**Proof:** Let  $x(t)$  and  $y(t)$  denote any two solutions of (11) corresponding to the respective initial values  $\varphi$  and  $\psi$ . We obtain from

$$\begin{aligned} \frac{dx(t)}{dt} &= -a(t)x(t) + b(t) \tanh[x(t - \tau)] + f(t) \\ \frac{dy(t)}{dt} &= -a(t)y(t) + b(t) \tanh[y(t - \tau)] + f(t) \end{aligned}$$

with an application of the mean value theorem of differential calculus that

$$\begin{aligned} \frac{d}{dt}[x(t) - y(t)] &= -a(t)[x(t) - y(t)] + b(t) \operatorname{sech}^2[\theta(t - \tau)] \\ &\quad \cdot [x(t - \tau) - y(t - \tau)], \quad t > 0 \quad (14) \end{aligned}$$

where  $\theta(t - \tau)$  lies between  $x(t - \tau)$  and  $y(t - \tau)$ . By using the positivity of  $a(\cdot)$ , one can simplify (14) to the form

$$\begin{aligned} \frac{d^+}{dt} |x(t) - y(t)| \\ \leq -a_* |x(t) - y(t)| + b^* |x(t - \tau) - y(t - \tau)|, \quad t > 0. \quad (15) \end{aligned}$$

We can rewrite (15) in the form

$$\frac{d^+}{dt} [|x(t) - y(t)| e^{a_* t}] \leq b^* e^{a_* t} |x(t - \tau) - y(t - \tau)|, \quad t > 0$$

with the implication

$$\begin{aligned} |x(t) - y(t)| e^{a_* t} \\ \leq |x(0) - y(0)| + b^* e^{a_* \tau} \int_0^t |x(s - \tau) - y(s - \tau)| \\ \cdot e^{a_*(s - \tau)} ds \\ \leq |x(0) - y(0)| + b^* e^{a_* \tau} \int_{-\tau}^{t - \tau} |x(u) - y(u)| e^{a_* u} du \\ \leq |x(0) - y(0)| + b^* e^{a_* \tau} \int_{-\tau}^0 |x(u) - y(u)| e^{a_* u} du \\ + b^* e^{a_* \tau} \int_0^t |x(u) - y(u)| e^{a_* u} du, \quad t > 0. \quad (16) \end{aligned}$$

We let

$$\|x(t) - y(t)\| = \sup_{s \in [t - \tau, t]} |x(s) - y(s)| \quad (17)$$

and derive from (16) and (17) that

$$\begin{aligned} |x(t) - y(t)| e^{a_* t} &\leq (1 + b^* \tau e^{a_* \tau}) \|x(0) - y(0)\| \\ &\quad + b^* e^{a_* \tau} \int_0^t |x(u) - y(u)| e^{a_* u} du, \quad t > 0. \quad (18) \end{aligned}$$

An application of Gronwall's inequality in (18) leads to

$$\begin{aligned} |x(t) - y(t)| &\leq (1 + b^* \tau e^{a_* \tau}) \|x(0) - y(0)\| \\ &\quad \cdot \exp\{-(a_* - b^* e^{a_* \tau}) t\}, \quad t > 0. \quad (19) \end{aligned}$$

Now by using (13) in (19), we obtain

$$\begin{aligned} & \sup_{s \in [t-\tau, t]} |x(s) - y(s)| \\ & \leq (1 + b^* \tau e^{a_* \tau}) \|x(0) - y(0)\| e^{-(a_* - b^* e^{a_* \tau})(t-\tau)} \\ & \quad t > 0 \\ & \leq \left[ (1 + b^* \tau e^{a_* \tau}) e^{(a_* - b^* e^{a_* \tau})\tau} \right] \|x(0) - y(0)\| \\ & \quad \cdot e^{-(a_* - b^* e^{a_* \tau})t}, \quad t > 0. \end{aligned} \quad (20)$$

Let  $\epsilon > 0$  be arbitrary. Choose a  $\delta = \delta(\epsilon)$  as follows:

$$\delta(\epsilon) = \epsilon \{ (1 + b^* \tau e^{a_* \tau}) \exp[(a_* - b^* e^{a_* \tau})\tau] \}^{-1}. \quad (21)$$

The uniform stability of an arbitrary solution, say  $y(t)$ , follows from (20) and (21) on using the hypothesis (13). This completes the proof.  $\square$

*Corollary 3.2:* If instead of (13) one assumes that

$$a_* > b^* \geq 0 \quad (22)$$

then

$$|x(t) - y(t)| \rightarrow 0 \quad \text{as } t \rightarrow \infty \quad (23)$$

where  $x(t)$  and  $y(t)$  denote arbitrary solutions of (21) for a given  $f$ .

*Proof:* Consider a Lyapunov functional  $V(x, y)(t)$  of the form

$$V(x, y)(t) = |x(t) - y(t)| + b^* \int_{t-\tau}^t |x(s) - y(s)| ds, \quad t > 0. \quad (24)$$

A direct calculation of the upper right derivative  $d^+V/dt$  in (24) along the solutions of (11) leads to

$$\frac{d^+V}{dt} \leq -(a_* - b^*)|x(t) - y(t)|, \quad t > 0 \quad (25)$$

which implies that

$$\begin{aligned} V(x, y)(t) + (a_* - b^*) \int_0^t |x(s) - y(s)| ds \\ \leq V(x, y)(0), \quad t > 0. \end{aligned} \quad (26)$$

From the boundedness of solutions (Section II) of (11), the boundedness of the derivatives of solutions of (11) follows implying the uniform continuity of solutions of (11) on  $[0, \infty)$ . One can conclude from (26) that  $V$  is bounded and nonnegative for  $t \geq 0$ ; also (26) implies that

$$|x(\cdot) - y(\cdot)| \in L_1(0, \infty). \quad (27)$$

By a lemma due to Barbalat (see [9]), we can conclude that (23) holds and the proof is complete.  $\square$

We now proceed to prove the existence of a periodic solution of (11). In our proof of the existence of a periodic solution of (11) we use a fixed point theorem due to [3] and our formulation of this result is extracted from [1, p. 248].

*Caristi's Fixed Point Theorem:* Let  $X$  denote a complete metric space with a metric  $\mu$ . Let  $F: X \rightarrow X$  be a single valued map and let  $U: X \rightarrow \mathbb{R}$  be a bounded lower semicontinuous function such that

$$\mu(x, F(x)) \leq U(x) - U(F(x)) \quad \text{for all } x \in X. \quad (28)$$

Then  $F$  has a fixed point  $\tilde{x} \in X$  such that  $F(\tilde{x}) = \tilde{x}$ .

*Theorem 3.3:* Assume that the neuron parameters  $a(t)$ ,  $b(t)$  and the external stimulus  $f(t)$  are periodic in  $t$  with period  $\omega$  so that

$$\begin{aligned} a(t + \omega) &= a(t) \\ b(t + \omega) &= b(t) \\ f(t + \omega) &= f(t), \quad t \in \mathbb{R}. \end{aligned}$$

Let  $\tau$  be a nonnegative real number such that  $\omega > \tau$ . Suppose further that

$$a_* > b^* \exp[a_* \tau] \geq 0 \quad (29)$$

$$(1 + b^* \tau e^{a_* \tau}) \exp[-(a_* - b^* e^{a_* \tau})(\omega - \tau)] = \rho < 1. \quad (30)$$

Then (11) has a periodic solution of period  $\omega$  which is a global attractor.

*Proof:* The global attractivity of a periodic solution of (11), if it exists, follows from Theorem 3.1. Hence we will prove only the existence of a periodic solution. Let  $x(t, \varphi)$  and  $y(t, \psi)$  denote the solutions of (11) corresponding to the initial values  $\varphi$  and  $\psi$ , respectively, satisfying

$$x(0, \varphi)(s) = \varphi(s), \quad y(0, \psi)(s) = \psi(s), \quad \varphi, \psi \in C[-\tau, 0].$$

Proceeding as in the proof of Theorem 3.1, we obtain

$$\begin{aligned} & \sup_{s \in [t-\tau, t]} |x(s, \varphi) - y(s, \psi)| \\ & \leq (1 + b^* \tau e^{a_* \tau}) e^{-(a_* - b^* e^{a_* \tau})(t-\tau)} \|\varphi - \psi\|, \quad t > 0 \end{aligned} \quad (31)$$

which implies that

$$\begin{aligned} & \|x(t, \varphi) - y(t, \psi)\| \\ & \leq (1 + b^* \tau e^{a_* \tau}) e^{-(a_* - b^* e^{a_* \tau})(t-\tau)} \|\varphi - \psi\|, \quad t > 0. \end{aligned} \quad (32)$$

We note by hypothesis that  $\omega > \tau$  and we let  $t = \omega$  in (32) to obtain

$$\|x(\omega, \varphi) - y(\omega, \psi)\| \leq \rho \|\varphi - \psi\| \quad (33)$$

$$\begin{aligned} \|x(\omega, \varphi) - y(\omega, x(\omega, \varphi))\| &= \|x(\omega, \varphi) - x(2\omega, \varphi)\| \\ &\leq \rho \|\varphi - x(\omega, \varphi)\|. \end{aligned} \quad (34)$$

From (34), we derive

$$\begin{aligned} \|\varphi - x(\omega, \varphi)\| &\leq \frac{1}{1 - \rho} \|\varphi - x(\omega, \varphi)\| \\ &\quad - \frac{1}{1 - \rho} \|x(\omega, \varphi) - x(2\omega, \varphi)\|. \end{aligned} \quad (35)$$

We define  $F: C[-\tau, 0] \rightarrow \mathbb{R}$  and  $P: C[-\tau, 0] \rightarrow C[-\tau, 0]$  as follows:

$$F(\varphi) = \frac{1}{1 - \rho} \|\varphi - x(\omega, \varphi)\| \quad (36)$$

$$P(\varphi) = x(\omega, \varphi). \quad (37)$$

As a consequence of (35)–(37)

$$\|\varphi - P(\varphi)\| \leq F(\varphi) - F(P(\varphi)), \quad \varphi \in C[-\tau, 0]. \quad (38)$$

It follows from (38) and Caristi's fixed point theorem that  $P: C[-\tau, 0] \rightarrow C[-\tau, 0]$  has a fixed point, say  $\varphi^*$  such that

$P(\varphi^*) = \varphi^*$ . Consider now the solutions  $x(t, \varphi^*)$  and  $x(t + \omega, \varphi^*)$ ; these two solutions satisfy

$$x(0, \varphi^*) = \varphi^*, \quad x(\omega, \varphi^*) = P(\varphi^*) = \varphi^*$$

and hence by the uniqueness of solutions of (11) we have

$$x(t, \varphi^*) = x(t + \omega, \varphi^*), \quad \text{for all } t > 0.$$

Hence the solution  $x(t, \varphi^*)$  is periodic with period  $\omega$ . The uniqueness and global attractivity of this periodic solution follow from Theorem 3.1.  $\square$

#### IV. DISTRIBUTED DELAYS

Time delays in the dynamics of neural networks need not necessarily be discrete and temporally uniform as it has been proposed in (11). It is quite plausible for the time delay in synaptic transmission or processing to be continuously distributed over a finite or infinite duration; the intensity of influence of the delay-effects can vary over the length of the time delay and this influence is usually modeled by a delay kernel. Accordingly we consider a modification of (11) given by

$$\frac{dx(t)}{dt} = -a(t)x(t) + b(t) \cdot \tanh \left\{ \int_0^T K(s)x(t-s) ds \right\} + f(t), \quad t > 0 \quad (39)$$

together with an initial value for the membrane potential in the form

$$x(s) = \varphi(s), \quad s \in [-T, 0], \quad \varphi \in C[-T, 0] \quad (40)$$

where  $T$  is a finite positive number;  $a(\cdot)$ ,  $b(\cdot)$ ,  $f(\cdot)$  are continuous and bounded functions defined on  $\mathbb{R} = (-\infty, \infty)$ . The delay kernel  $K: [0, T] \mapsto [0, \infty)$  is assumed to be continuous. We assume that the state space of (39) is the space of real-valued functions defined on  $[-T, 0]$  which are continuous and endowed with the supremum norm; in conventional notation this space is  $C[-T, 0]$  where

$$C[-T, 0] = \{\varphi: [-T, 0] \mapsto \mathbb{R} \mid \varphi \text{ is continuous}\}$$

and

$$\|\varphi\| = \sup_{s \in [-T, 0]} |\varphi(s)|.$$

The existence of a globally attracting periodic solution of (39) is established in the following. We use the notation of the previous section in the following.

**Theorem 4.1:** Suppose  $a(\cdot)$ ,  $b(\cdot)$ ,  $f(\cdot)$  are continuous and periodic with period  $\omega$ ; we assume that  $\omega > T$  in (39) and

$$a_* > b^* \int_0^T K(s)e^{a_* s} ds \geq 0 \quad (41)$$

$$\left( 1 + b^* T \int_0^T K(s)e^{a_* s} ds \right) \cdot \exp \left[ - \left( a_* - b^* T \int_0^T K(s)e^{a_* s} ds \right) (\omega - T) \right]$$

$$= \rho < 1 \quad (42)$$

then (39) has a globally attracting periodic solution of period  $\omega$ .

*Proof:* The details of proof are analogous to those of Theorems 3.1 and 3.3; we shall be brief. Let  $x(t) = x(t, \varphi)$  and  $y(t) = y(t, \psi)$  be any two solutions of (39) with initial values  $\varphi, \psi$ , respectively. We can then obtain from (39)

$$\begin{aligned} & |x(t) - y(t)|e^{a_* t} \\ & \leq |x(0) - y(0)| + b^* \int_0^t \\ & \quad \cdot e^{a_* u} \left\{ \int_0^T K(s)|x(u-s) - y(u-s)| ds \right\} du, \\ & \quad t > 0 \\ & \leq |x(0) - y(0)| + b^* \int_0^T K(s) \\ & \quad \cdot \left\{ \int_{-s}^{t-s} e^{a_*(s+v)} |x(v) - y(v)| dv \right\} ds \\ & \leq |x(0) - y(0)| + b^* \int_0^T K(s) \\ & \quad \cdot \left\{ \int_{-s}^0 e^{a_*(s+v)} |x(v) - y(v)| dv \right\} ds \\ & \quad + b^* \int_0^T K(s) \left\{ \int_0^t e^{a_*(s+v)} |x(v) - y(v)| dv \right\} ds \\ & \leq |x(0) - y(0)| + b^* T \left( \int_0^T K(s)e^{a_* s} ds \right) \|\varphi - \psi\| \\ & \quad + \left( b^* \int_0^T K(s)e^{a_* s} ds \right) \int_0^t e^{a_* v} |x(v) - y(v)| dv \\ & \leq \left( 1 + b^* T \int_0^T K(s)e^{a_* s} ds \right) \|\varphi - \psi\| \\ & \quad + \left( b^* \int_0^T K(s)e^{a_* s} ds \right) \int_0^t e^{a_* v} |x(v) - y(v)| dv \\ & \quad t > 0. \end{aligned} \quad (43)$$

By using Gronwall's inequality in (43)

$$|x(t) - y(t)|e^{a_* t} \leq \left( 1 + b^* T \int_0^T K(s)e^{a_* s} ds \right) \|\varphi - \psi\| \cdot e^{(b^* \int_0^T K(s)e^{a_* s} ds)t}, \quad t > 0 \quad (44)$$

and hence

$$|x(t) - y(t)| \leq \left( 1 + b^* T \int_0^T K(s)e^{a_* s} ds \right) \|\varphi - \psi\| \cdot e^{-[a_* - b^* \int_0^T K(s)e^{a_* s} ds]t}, \quad t > 0. \quad (45)$$

Now by using (41) in (45)

$$\|x(t) - y(t)\| \leq \|\varphi - \psi\| \left( 1 + b^* T \int_0^T K(s)e^{a_* s} ds \right) \cdot e^{[-(a_* - b^* \int_0^T K(s)e^{a_* s} ds)(t-T)]}, \quad t > 0. \quad (46)$$

By assumption  $\omega > T$  and hence we have from (46)

$$\begin{aligned} & \|x(\omega) - y(\omega)\| \\ & \leq \|\varphi - \psi\| \left( 1 + b^* T \int_0^T K(s) e^{a_* s} ds \right) \\ & \quad \cdot e^{[-(a_* - b^* \int_0^T K(s) e^{a_* s} ds)(\omega - T)]} \\ & \leq \rho \|\varphi - \psi\|. \end{aligned} \quad (47)$$

The remaining details of proof are similar to those of Theorem 3.3 and we omit these to avoid repetition. The global attractivity of the periodic solution is a consequence of (45).  $\square$

In the next result we consider the case of delays distributed over an infinite (unbounded) interval  $(-\infty, 0)$  characterized by a delay kernel  $K_1: [0, \infty) \mapsto [0, \infty)$  satisfying certain conditions to be specified. In particular we consider an equation of the form

$$\frac{dx(t)}{dt} = -a(t)x(t) + b(t) \tanh\left(\int_0^\infty K_1(s)x(t-s)ds\right) + f(t), \quad t > 0 \quad (48)$$

together with an initial condition of the form

$$\begin{aligned} x(s) &= \varphi(s) \\ \varphi: (-\infty, 0] &\mapsto \mathbb{R}, \varphi \text{ is bounded and continuous.} \end{aligned} \quad (49)$$

We assume as before that  $a(\cdot), b(\cdot), f(\cdot)$  are periodic with period  $\omega > 0$ . The delay kernel  $K_1$  is assumed to satisfy the integrability condition

$$\int_0^\infty K_1(s) e^{a_* s} ds < \infty. \quad (50)$$

We need two preliminary results in order to prove the existence of periodic solutions of (48).

**Lemma 4.2:** Assume that  $K_1: [0, \infty) \mapsto [0, \infty)$  is such that the infinite series

$$\sum_{j=0}^\infty K_1(u + j\omega)$$

converges uniformly in  $u \in [0, \infty)$  and we let

$$H(u) = \sum_{j=0}^\infty K_1(u + j\omega), \quad u \in [0, \infty). \quad (51)$$

Then  $x(t)$  is a periodic solution of (48) if and only if  $x(t)$  is a periodic solution of period  $\omega$  of

$$\begin{aligned} \frac{dx(t)}{dt} &= -a(t)x(t) + b(t) \tanh\left(\int_0^\omega H(u)x(t-u)du\right) + f(t) \\ & \quad t > 0. \end{aligned} \quad (52)$$

*Proof:* Suppose  $x(t)$  is a periodic solution of period  $\omega$  of (48). Then

$$\begin{aligned} & \int_0^\infty K_1(s)x(t-s)ds \\ &= \sum_{j=0}^\infty \int_{j\omega}^{(j+1)\omega} K_1(s)x(t-s)ds \\ &= \sum_{j=0}^\infty \int_0^\omega K_1(u + j\omega)x(t-u-j\omega)du, \quad t > 0. \end{aligned}$$

By the periodicity of  $x(\cdot)$  and the uniform convergence in (51), we have

$$\int_0^\infty K_1(s)x(t-s)ds = \int_0^\omega H(u)x(t-u)du, \quad t > 0 \quad (53)$$

implying that  $x(t)$  is a solution of (52). Now if  $x(t)$  is a periodic solution of period  $\omega$  of (52), then one can reverse the above sequence of steps and show that  $x(t)$  is also a periodic solution of (48). We omit these details.  $\square$

**Lemma 4.3:** If the delay kernel  $K_1: [0, \infty) \mapsto [0, \infty)$  satisfies (50) and (51) then

$$b^* \omega \int_0^\omega H(s) e^{a_* s} ds < b^* \omega \int_0^\infty K_1(v) e^{a_* v} dv. \quad (54)$$

*Proof:* The result follows from the uniform convergence in (51) and

$$\begin{aligned} \int_0^\omega H(s) e^{a_* s} ds &= \int_0^\omega \sum_{j=0}^\infty K_1(s + j\omega) e^{a_* s} ds \\ &= \sum_{j=0}^\infty \int_0^\omega K_1(s + j\omega) e^{a_* s} ds \\ &= \sum_{j=0}^\infty \int_{j\omega}^{(j+1)\omega} K_1(v) e^{a_* (v-j\omega)} dv \\ &< \int_0^\infty K_1(v) e^{a_* v} dv. \end{aligned}$$

$\square$

**Theorem 4.4:** Assume that  $a(\cdot), b(\cdot)$  and  $K_1(\cdot)$  satisfy

$$a_* > b^* \int_0^\infty K_1(v) e^{a_* v} dv \geq 0 \quad (55)$$

$$\begin{aligned} & \left( 1 + b^* \omega \int_0^\omega H(s) e^{a_* s} ds \right) \\ & \quad \cdot \exp\left[-\left(a_* - b^* \int_0^\omega H(s) e^{a_* s} ds\right)\omega\right] \\ &= \rho_1 < 1. \end{aligned} \quad (56)$$

Then (52) has a periodic solution of period  $\omega$ .

*Proof:* Let  $x(t) = x(t, \varphi)$  and  $y(t) = y(t, \psi)$  be two solutions of (52) corresponding to the initial values

$$x(s) = \varphi(s), \quad y(s) = \psi(s), \quad s \in [-\omega, 0].$$

Proceeding as in the proof of Theorem 4.1, one can obtain

$$|x(t) - y(t)| \leq \|\varphi - \psi\| \left( 1 + b^* \omega \int_0^\omega H(s) e^{a_* s} ds \right) \cdot e^{[-(a_* - b^*) \int_0^\omega H(s) e^{a_* s} ds] t}, \quad t > 0. \quad (57)$$

We let

$$\|x(t) - y(t)\| = \sup_{s \in [t-\omega, t]} |x(s) - y(s)| \quad (58)$$

and obtain from (55)–(58) that

$$\|x(\omega) - y(\omega)\| \leq \rho_1 \|\varphi - \psi\|. \quad (59)$$

One can now proceed as in the proof of Theorem 4.1 and complete the remaining details to establish the existence of a periodic solution of (52). We omit further details.  $\square$

**Theorem 4.5:** Assume that the hypotheses of Theorem 4.4 hold. Then (48) has a periodic solution which is globally attractive.

*Proof:* Let  $x(t)$  and  $y(t)$  denote any two solutions of (48). Then we have from (48) after an application of the mean value theorem of differential calculus and simplification

$$\frac{d^+}{dt} |x(t) - y(t)| \leq -a_* |x(t) - y(t)| + b^* \int_0^\infty K_1(s) |x(t-s) - y(t-s)| ds, \quad t > 0. \quad (60)$$

Let us consider a Lyapunov functional  $V(t) = V(x, y)(t)$  defined by

$$V(t) = |x(t) - y(t)| + b^* \int_0^\infty K_1(s) \cdot \left( \int_{t-s}^t |x(u) - y(u)| du \right) ds, \quad t > 0. \quad (61)$$

From (55) we have

$$b^* \int_0^\infty K_1(v) v dv < b^* \int_0^\infty K_1(v) e^{a_* v} dv < a_* \quad (62)$$

and hence  $\int_0^\infty K(v) v dv < \infty$  from which it will follow that

$$V(0) \leq \left( \sup_{s > -\infty} |x(s) - y(s)| \right) \left( 1 + b^* \int_0^\infty s K(s) ds \right) < \infty \quad (63)$$

due to the boundedness of initial values and the solutions of (48) on  $(-\infty, \infty)$ . Calculating the upper right derivative  $(d^+ V(t))/dt$  of  $V(t)$  along the solutions of (48) and by using (60)

$$\frac{d^+ V(t)}{dt} \leq - \left( a_* - b^* \int_0^\infty K_1(s) ds \right) |x(t) - y(t)| \quad t > 0 \quad (64)$$

leading to

$$V(t) + \left( a_* - b^* \int_0^\infty K_1(s) ds \right) \int_0^t |x(s) - y(s)| ds \leq V(0) < \infty. \quad (65)$$

It follows from (65) that

$$|x(\cdot) - y(\cdot)| \in L_1(0, \infty). \quad (66)$$

The boundedness of solutions of (48) implies that of their derivatives on  $(0, \infty)$  and hence the solutions of (48) are uniformly continuous on  $(0, \infty)$ . By Barbalat's lemma we can conclude from (66) that

$$|x(t) - y(t)| \rightarrow 0 \quad \text{as } t \rightarrow \infty. \quad (67)$$

Now if  $y(t)$  denotes a periodic solution and  $x(t)$  denotes any arbitrary solution, then (67) implies the global attractivity of the periodic solution  $y(t)$ . The global attractivity of the periodic solution also implies its uniqueness. The proof is complete.  $\square$

## V. PERIODICALLY VARYING DELAYS

In this section we consider the dynamics of

$$\frac{dx(t)}{dt} = -a(t)x(t) + b(t) \tanh[x(t - \tau(t))] + f(t), \quad t > 0 \quad (68)$$

where  $a(\cdot)$ ,  $b(\cdot)$ ,  $f(\cdot)$  are as before continuous real-valued functions defined on  $(-\infty, \infty)$  and are periodic with a common period  $\omega > 0$ . The time delay  $\tau(\cdot)$  is defined on  $(-\infty, \infty)$  and is assumed to be continuous and periodic with period  $\omega > 0$  satisfying

$$\begin{aligned} \tau(t) &\geq 0, \quad t - \tau(t) > 0, \quad t - \tau(t) \rightarrow \infty \quad \text{as } t \rightarrow \infty \\ 0 &\leq \tau_* = \inf_{t \in \mathbb{R}} \tau(t) \leq \tau^* = \sup_{t \in \mathbb{R}} \tau(t) < \infty. \end{aligned} \quad (69)$$

The initial values associated with (68) are of the form

$$x(s) = \varphi(s), \quad s \in [-\tau^*, 0], \quad \varphi \in C[-\tau^*, 0]$$

where  $C[-\tau^*, 0]$  denotes the space of continuous real-valued functions defined on  $[-\tau^*, 0]$  endowed with the supremum norm. In this section we establish the existence of a globally attractive periodic solution of (68). First we establish an improved version of inequality due to [13].

**Lemma 5.1:** Let  $\alpha$ ,  $\beta$  and  $\tau^*$  be positive numbers and let  $u(\cdot)$  be a nonnegative solution of

$$\frac{du(t)}{dt} \leq -\alpha u(t) + \beta \left( \sup_{s \in [t-\tau^*, t]} u(s) \right), \quad t > t_0. \quad (70)$$

If  $\alpha > \beta$ , then there exists a positive real number  $\eta$  such that

$$u(t) \leq \left( \sup_{s \in [t_0-\tau^*, t_0]} u(s) \right) e^{-\eta(t-t_0)}, \quad t > t_0. \quad (71)$$

**Note:** The inequality established by Halanay [13] for (70) when  $\alpha > \beta$  is of the form

$$u(t) \leq C e^{-\eta(t-t_0)}, \quad t > t_0$$

for some positive numbers  $C$  and  $\eta$ . We need detailed information about the constant  $C$  in our application.

*Proof:* By assumption  $\alpha > \beta$  and we define  $g$  as follows:

$$g(\lambda) = -\alpha + \lambda + \beta e^{\lambda \tau^*}, \quad \lambda \in \mathbb{R}. \quad (72)$$

Note that  $g(0) = -\alpha + \beta < 0$ ; due to the continuity of  $g$ , there exists a positive number say  $\mu$  satisfying

$$g(\mu) = -\alpha + \mu + \beta e^{\mu \tau^*} < 0. \quad (73)$$

Define  $\tilde{u}$  by the following:

$$\tilde{u}(t) = u(t) e^{\mu(t-t_0)}, \quad t > t_0. \quad (74)$$

From (70) and (74)

$$\begin{aligned} \frac{d}{dt} \tilde{u}(t) &= \frac{du}{dt} e^{\mu(t-t_0)} + \mu u(t) e^{\mu(t-t_0)} \\ &\leq (-\alpha + \mu) u(t) e^{\mu(t-t_0)} + \beta e^{\mu(t-t_0)} \left( \sup_{s \in [t-\tau^*, t]} u(s) \right) \\ &\leq (-\alpha + \mu) \tilde{u}(t) + \beta e^{\mu\tau^*} \left( \sup_{s \in [t-\tau^*, t]} \tilde{u}(s) \right) \\ t &> 0, \end{aligned} \quad (75)$$

Let  $M$  be defined by

$$M = \sup_{s \in [t_0 - \tau^*, t_0]} u(s) \quad (76)$$

and let  $\delta$  denote an arbitrary number such that  $\delta > 1$ . First we show that

$$\tilde{u}(t) < M\delta \quad \text{for } t > t_0 - \tau^*. \quad (77)$$

It is true from the definition of  $M$  that

$$\tilde{u}(t) < M\delta \quad \text{for } t \in [t_0 - \tau^*, t_0].$$

If (77) is not valid, then there exists a  $t_1 > t_0$  such that

$$\tilde{u}(t) < M\delta \quad \text{for } t \in [t_0 - \tau^*, t_1], \quad \tilde{u}(t_1) = M\delta, \quad \frac{d\tilde{u}}{dt} \Big|_{t=t_1} \geq 0. \quad (78)$$

But we have from (75) and (78)

$$\begin{aligned} 0 &\leq \frac{d\tilde{u}}{dt} \Big|_{t=t_1} \\ &\leq (-\alpha + \mu) \tilde{u}(t_1) + \beta e^{\mu\tau^*} \left( \sup_{s \in [t_1 - \tau^*, t_1]} \tilde{u}(s) \right) \\ &\leq M\delta [-\alpha + \mu + \beta e^{\mu\tau^*}] \\ &< 0 \end{aligned}$$

which is not possible. Hence (77) holds. Since  $\delta > 1$  is arbitrary, we have from (76)

$$\tilde{u}(t) < M\delta \leq M \quad \text{as } \delta \rightarrow 1 + \quad \text{for } t > t_0. \quad (79)$$

From the definition of  $M$  and  $\tilde{u}$ , we derive that

$$\tilde{u}(t) \leq M \implies u(t) \leq \left( \sup_{s \in [t_0 - \tau^*, t_0]} u(s) \right) e^{-\mu(t-t_0)}, \quad t > t_0 \quad (80)$$

and the proof is complete.  $\square$

We note that if  $x(t)$  denotes a solution of (68) defined for  $t > 0$ , then  $x(t + \omega)$  is also a solution of (68) defined for  $t > 0$  due to the periodicity of  $a(\cdot)$ ,  $b(\cdot)$ ,  $f(\cdot)$  and  $\tau(\cdot)$ . Therefore in order to prove the existence of a periodic solution of (68) it is sufficient to prove the existence of a fixed point of a Poincare map. Let  $x(t) = x(t, \varphi)$  denote a solution of (68) satisfying

$$x(s) = \varphi(s), \quad s \in [-\tau^*, 0], \quad \varphi \in C[-\tau^*, 0].$$

We define the Poincare map  $P: C[-\tau^*, 0] \mapsto C[-\tau^*, 0]$  by the following:

$$P(\varphi)(s) = x(\omega, \varphi)(s), \quad s \in [-\tau^*, 0] \quad (81)$$

where  $x(\omega, \varphi)(s)$  denotes the value of  $x(\omega) = x(\omega, \varphi)$  of (68) with an initial value  $\varphi$  from  $C[-\tau^*, 0]$ .

**Theorem 5.2:** Let  $a(\cdot)$ ,  $b(\cdot)$ ,  $f(\cdot)$  and  $\tau(\cdot)$  be continuous and be periodic in  $t \in \mathbb{R}$  with period  $\omega > 0$ ; suppose that  $a(t) \geq a_* > 0$ ,  $t \in \mathbb{R}$ ; let  $\tau(\cdot)$  be differentiable satisfying

$$0 \leq \tau(t) \leq \tau^* < \infty, \quad t \in \mathbb{R} \text{ and } \omega > \tau^*$$

$$\frac{d\tau(t)}{dt} \leq \alpha < 1, \quad t \in \mathbb{R}$$

$$t - \tau(t) > 0, \quad t - \tau(t) \rightarrow \infty \quad \text{as } t \rightarrow \infty.$$

Let  $|b(t)| \leq b^*$  for  $t \in \mathbb{R}$ . If

$$a_* > \frac{b^* e^{a_* \tau^*}}{1 - \alpha} \geq 0 \quad (82)$$

$$\begin{aligned} &\left( 1 + \frac{b^* e^{a_* \tau^*}}{1 - \alpha} \tau^* \right) \exp \left[ - \left( a_* - \frac{b^* e^{a_* \tau^*}}{1 - \alpha} \right) (\omega - \tau^*) \right] \\ &= \rho < 1 \end{aligned} \quad (83)$$

then (68) has a globally attractive periodic solution of period  $\omega$ .

*Proof:* Let  $x(t) = x(t, \varphi)$ ,  $y(t) = x(t, \psi)$  denote two solutions of (68) corresponding to the initial values  $x(s) = x(0, \varphi)(s)$ ,  $y(s) = y(0, \psi)(s)$ ,  $s \in [-\tau^*, 0]$ . We derive from (68)

$$\begin{aligned} \frac{d}{dt} [x(t) - y(t)] &= -a(t)[x(t) - y(t)] \\ &\quad + b(t)[\tanh(x(t - \tau(t))) - \tanh(y(t - \tau(t)))] \end{aligned}$$

which implies

$$\begin{aligned} \frac{d^+}{dt} |x(t) - y(t)| &\leq -a_* |x(t) - y(t)| \\ &\quad + b^* |x(t - \tau(t)) - y(t - \tau(t))| \end{aligned}$$

and leads to

$$\begin{aligned} \frac{d^+}{dt} [|x(t) - y(t)| e^{a_* t}] &\leq b^* e^{a_* t} |x(t - \tau(t)) - y(t - \tau(t))|, \quad t > 0. \end{aligned} \quad (84)$$

We derive from (84) that

$$\begin{aligned} |x(t) - y(t)| e^{a_* t} &\leq |x(0) - y(0)| + b^* \int_0^t \\ &\quad \cdot e^{a_* s} |x(s - \tau(s)) - y(s - \tau(s))| ds, \quad t > 0. \end{aligned} \quad (85)$$

By a change of variable in the integral in (85) in the form  $s - \tau(s) = u$ , we obtain

$$\begin{aligned} |x(t) - y(t)| e^{a_* t} &\leq |x(0) - y(0)| + b^* \int_{-\tau(0)}^{t - \tau(t)} \\ &\quad \cdot e^{a_* (u + \tau^*)} |x(u) - y(u)| \frac{du}{1 - \alpha} \\ &\leq |x(0) - y(0)| + \frac{b^* e^{a_* \tau^*}}{1 - \alpha} \int_{-\tau}^0 e^{a_* u} |x(u) - y(u)| du \\ &\quad + \frac{b^* e^{a_* \tau^*}}{1 - \alpha} \int_0^t e^{a_* u} |x(u) - y(u)| du \\ &\leq \left( 1 + \frac{b^* e^{a_* \tau^*}}{1 - \alpha} \tau^* \right) |x(0) - y(0)| \\ &\quad + \frac{b^* e^{a_* \tau^*}}{1 - \alpha} \int_0^t e^{a_* u} |x(u) - y(u)| du, \quad t > 0. \end{aligned} \quad (86)$$



By applying Gronwall's inequality in (86)

$$|x(t) - y(t)| \leq \left(1 + \frac{b^* e^{a_* \tau^*}}{1 - \alpha} \tau^*\right) \|x(0) - y(0)\| \cdot \exp\left[-\left(a_* - \frac{b^* e^{a_* \tau^*}}{1 - \alpha}\right)t\right], \quad t > 0. \quad (87)$$

Taking the supremum on both sides of (87) over the interval  $[t - \tau^*, t]$ , we obtain

$$\begin{aligned} & \|x(t) - y(t)\| \\ & \leq \left(1 + \frac{b^* e^{a_* \tau^*}}{1 - \alpha}\right) \exp\left[-\left(a_* - \frac{b^* e^{a_* \tau^*}}{1 - \alpha}\right)(t - \tau^*)\right] \\ & \cdot \|x(0) - y(0)\|, \quad t > 0. \end{aligned} \quad (88)$$

By hypothesis  $\omega > \tau^*$  and hence we have from (88)

$$\|x(\omega) - y(\omega)\| \leq \rho \|x(0) - y(0)\| \quad (89)$$

where  $\rho$  is defined in (83). One can now proceed as in Theorem 4.1 to show the existence of a  $\varphi^* \in C[-\tau^*, 0]$  satisfying  $x(\omega) = x(\omega, \varphi^*)(s) = \varphi^*(s)$  from which it will follow that (68) has a periodic solution of period  $\omega$ . The global attractivity of the periodic solution follows from Lemma 5.1. For instance if  $\tilde{y}(t)$  is a periodic solution of (68) and  $x(t)$  is any arbitrary solution of (68), then we have from

$$\begin{aligned} \frac{d^+}{dt} |x(t) - \tilde{y}(t)| & \leq -a_* |x(t) - \tilde{y}(t)| \\ & + b^* |x(t - \tau(t)) - \tilde{y}(t - \tau(t))|, \quad t > 0 \end{aligned} \quad (90)$$

that

$$\begin{aligned} \frac{d^+}{dt} |x(t) - \tilde{y}(t)| & \leq -a_* |x(t) - \tilde{y}(t)| \\ & + b^* \left( \sup_{s \in [t - \tau^*, t]} |x(s) - \tilde{y}(s)| \right), \quad t > 0. \end{aligned} \quad (91)$$

It follows from (82) that  $a_* > b^*$  and hence we obtain by virtue of Lemma 5.1

$$|x(t) - \tilde{y}(t)| \leq \left( \sup_{s \in [-\tau^*, 0]} |x(s) - \tilde{y}(s)| \right) e^{-\mu t}, \quad t > 0 \quad (92)$$

for some positive number  $\mu$ . The global attractivity of  $\tilde{y}$  together with its uniform stability will follow from (92). This completes the proof.  $\square$

## VI. NUMERICAL SIMULATIONS

We consider (11) with the delay  $\tau = 5$

$$\frac{dx(t)}{dt} = -a(t)x(t) + b(t) \tanh[x(t - 5)] + f(t), \quad t > 0. \quad (93)$$

The neuron parameters  $a(t)$  and  $b(t)$  and the external stimulus  $f(t)$  are, respectively, given by

$$a(t) = 2 + \sin\left(\frac{2\pi t}{7}\right) \quad (94)$$

$$b(t) = e^{-6} \cos\left(\frac{2\pi t}{7}\right) \quad (95)$$

$$f(t) = 2 \sin\left(\frac{2\pi t}{7}\right) - 3 \cos\left(\frac{2\pi t}{7}\right). \quad (96)$$

These parameters are periodic functions of period  $\omega = 7$ . The sufficient conditions for the existence of periodic solution of period 7 are satisfied, in fact we have with  $a_* = 1$ ,  $b^* = e^{-6}$ ,  $\tau = 5$ ,  $\omega = 7$

$$a_* > b^* e^{a_* \tau} > 0$$

and

$$(1 + b^* \tau e^{a_* \tau}) \exp[-(a_* - b^* e^{a_* \tau})(\omega - \tau)] = 0.8019978599 < 1.$$

Numerical simulations of (93) are generated from the discrete-time analogs of (93) in the form

$$\begin{aligned} x((n+1)h) & = \frac{1}{1 + a(nh)h} x(nh) + \frac{b(nh)h}{1 + a(nh)h} \\ & \cdot \tanh\left\{x\left(nh - \left[\frac{\tau}{h}\right]h\right)\right\} + \frac{f(nh)h}{1 + a(nh)h}, \quad n \in \mathbb{Z} \end{aligned} \quad (97)$$

where  $[t]$  denotes the greatest integer contained in  $t$  and  $h$  denotes a positive real number such that  $h \in (0, 1]$ . In the discrete form, (94)–(96) become

$$a(nh) = 2 + \sin\left(\frac{2\pi nh}{7}\right) \quad (98)$$

$$b(nh) = e^{-6} \cos\left(\frac{2\pi nh}{7}\right) \quad (99)$$

$$f(nh) = 2 \sin\left(\frac{2\pi nh}{7}\right) - 3 \cos\left(\frac{2\pi nh}{7}\right). \quad (100)$$

In Fig. 1 we demonstrate the periodic solution of (97) with  $\tau = 5$ ,  $h = 0.2$  and initial value  $\varphi(s) = 2$ ,  $s \in [-5, 0]$ .

Fig. 2 shows the solutions of (97) corresponding to two initial values  $\varphi_1(s) = -e^{-0.1s}$  and  $\varphi_2(s) = -0.1s$ ,  $s \in [-5, 0]$ . As in Fig. 1, we use  $\tau = 5$  and  $h = 0.2$ .

As another example, we choose the parameters  $a(nh)$  and  $b(nh)$  to be constant and  $f(nh)$  as in (100) with the delay  $\tau = 2$ . For instance, we have

$$a(nh) = 0.25 \quad (101)$$

$$b(nh) = 0.10 \quad (102)$$

$$f(nh) = 2 \sin\left(\frac{2\pi nh}{7}\right) - 3 \cos\left(\frac{2\pi nh}{7}\right). \quad (103)$$

In Fig. 3, we plot the solution of (97) subjected to these parameters and initial value  $\varphi(s) = s + 2$ ,  $s \in [-2, 0]$ . Here we take  $\tau = 2$  and  $h = 0.2$ .

We now consider (11) with a delay kernel as in (48)

$$\begin{aligned} \frac{dx(t)}{dt} & = -a(t)x(t) + b(t) \tanh\left(\int_0^\infty K_1(s)x(t-s)ds\right) \\ & + f(t), \quad t > 0 \end{aligned} \quad (104)$$

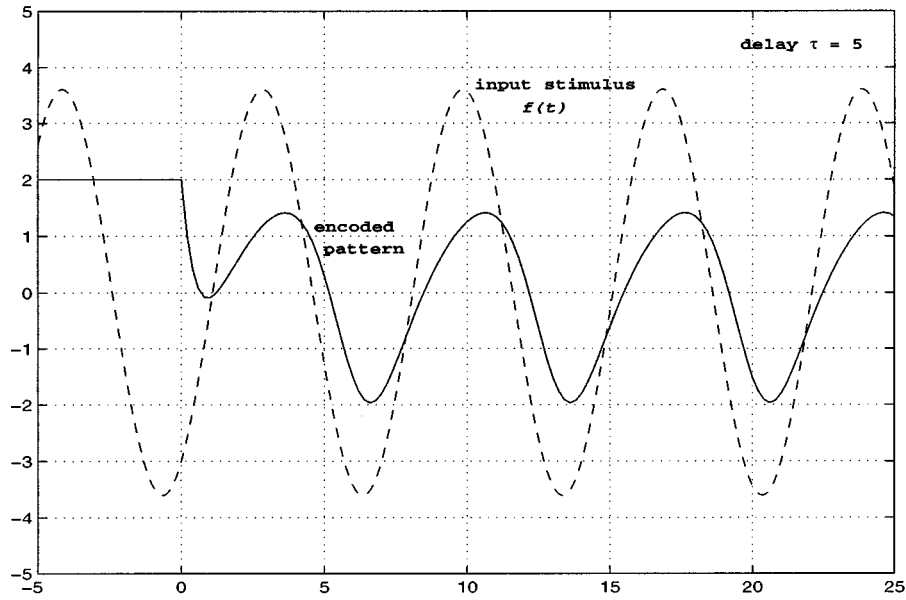


Fig. 1. Encoded pattern of period 7 (solid line) and the input stimulus  $f(t)$  of period 7 (dashed line).

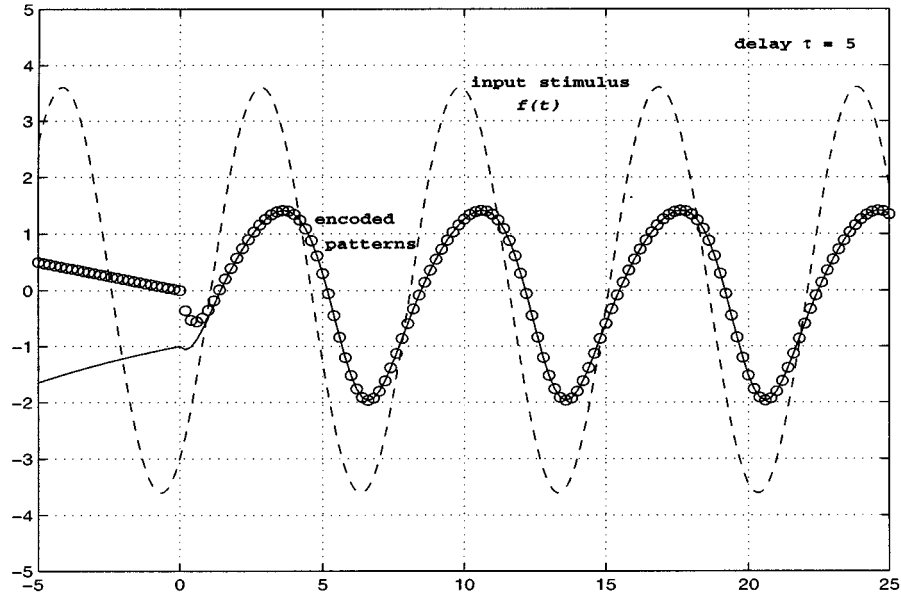


Fig. 2. Two encoded patterns of period 7 corresponding to two initial values  $\varphi_1(s)$  (solid line) and  $\varphi_2(s)$  (circle) merge. The input stimulus  $f(t)$  of period 7 is shown (dashed line).

in which  $a(\cdot)$ ,  $b(\cdot)$ ,  $f(\cdot)$  are periodic with period  $\omega > 0$ . The delay kernel  $K_1(\cdot)$  is assumed to satisfy the integrability condition

$$\int_0^\infty K_1(s)e^{a_*s} ds < \infty. \quad (105)$$

We assume the delay kernel  $K_1$  are of the form

$$K_1(s) = e^{-\alpha s}, \quad \alpha > 0. \quad (106)$$

Using this kernel, equation (104) can be written in the form

$$\frac{dx(t)}{dt} = -a(t)x(t) + b(t) \tanh\left(\int_0^\infty e^{-\alpha s} x(t-s) ds\right) + f(t), \quad t > 0. \quad (107)$$

The equation (107) can be converted into a system of ordinary differential equations by introducing an auxiliary variable  $u$  defined by

$$u(t) = \int_0^\infty e^{-\alpha s} x(t-s) ds = \int_{-\infty}^t e^{-\alpha(t-s)} x(s) ds. \quad (108)$$

It is found from (107) and (108) that  $x$  and  $u$  are governed by

$$\left. \begin{aligned} \frac{dx(t)}{dt} &= -a(t)x(t) + b(t) \tanh[u(t)] + f(t) \\ \frac{du(t)}{dt} &= x(t) - \alpha u(t) \end{aligned} \right\}. \quad (109)$$

This system is then solved numerically by means of Runge-Kutta scheme. Assuming the neuron parameters as in (94)–(96) and the delay kernel as in (106) with  $\alpha = 2$ , it is

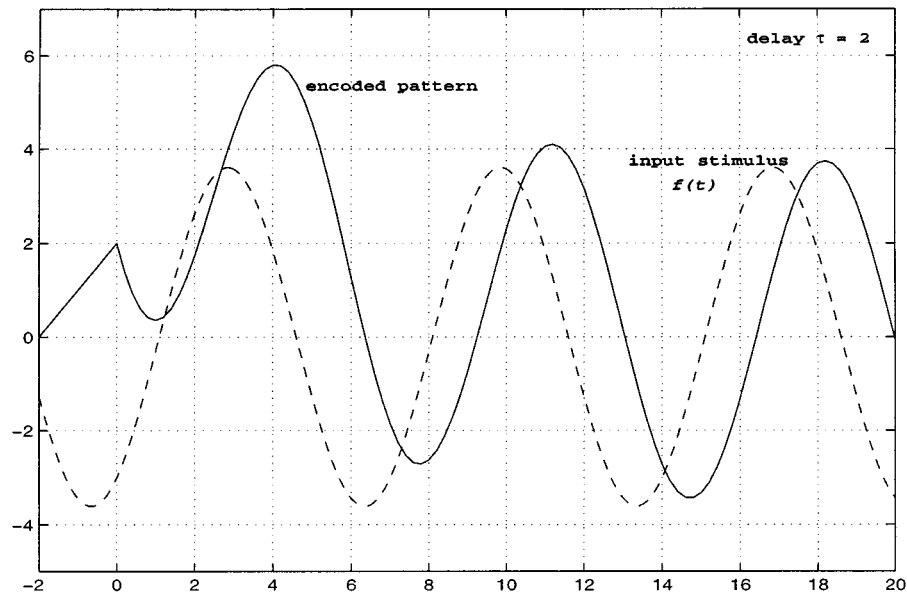


Fig. 3. Encoded pattern of period 7 (solid line) along with the input stimulus  $f(t)$  of period 7 (dashed line).

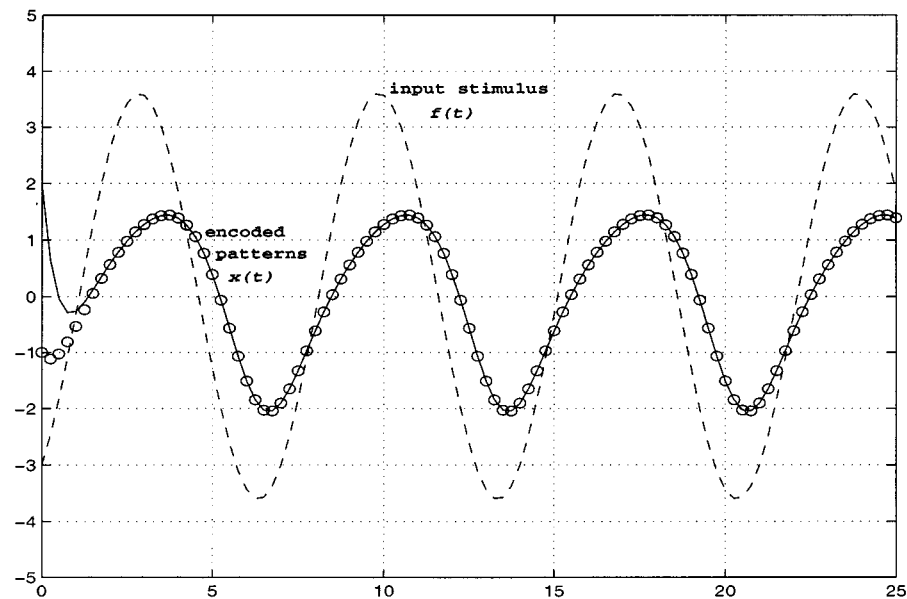


Fig. 4. Two encoded patterns  $x(t)$  of period 7 corresponding to two initial values  $x(0) = 2$  (solid line) and  $x(0) = -1$  (circle) along with the input stimulus  $f(t)$  of period 7 (dashed line).

found that the integrability condition in (105) and the assumptions of Theorem 4.4 are satisfied. Two periodic solutions of period  $\omega = 7$  of (107) corresponding to  $\alpha = 2$  and two initial values  $x(0) = 2$  and  $x(0) = -1$  is illustrated in Fig. 4.

In the following we consider another form of (104) given by

$$\frac{dx(t)}{dt} = -a(t)x(t) + b(t) \tanh\left(\int_0^\infty sK_1(s)x(t-s)ds\right) + f(t), \quad t > 0 \quad (110)$$

where  $a(\cdot)$ ,  $b(\cdot)$ ,  $f(\cdot)$  are periodic with period  $\omega > 0$  and the delay kernel  $K_1(\cdot)$  satisfies the integrability condition

$$\int_0^\infty sK_1(s)e^{a_*s}ds < \infty. \quad (111)$$

Upon substituting the delay kernel  $K_1$  given in (106), (110) becomes

$$\frac{dx(t)}{dt} = -a(t)x(t) + b(t) \tanh\left(\int_0^\infty se^{-\alpha s}x(t-s)ds\right) + f(t), \quad t > 0. \quad (112)$$

We let

$$\begin{aligned} u(t) &= \int_0^\infty se^{-\alpha s}x(t-s)ds, \\ &= \int_{-\infty}^t (t-s)e^{-\alpha(t-s)}x(s)ds \\ v(t) &= \int_0^\infty e^{-\alpha s}x(t-s)ds \end{aligned}$$

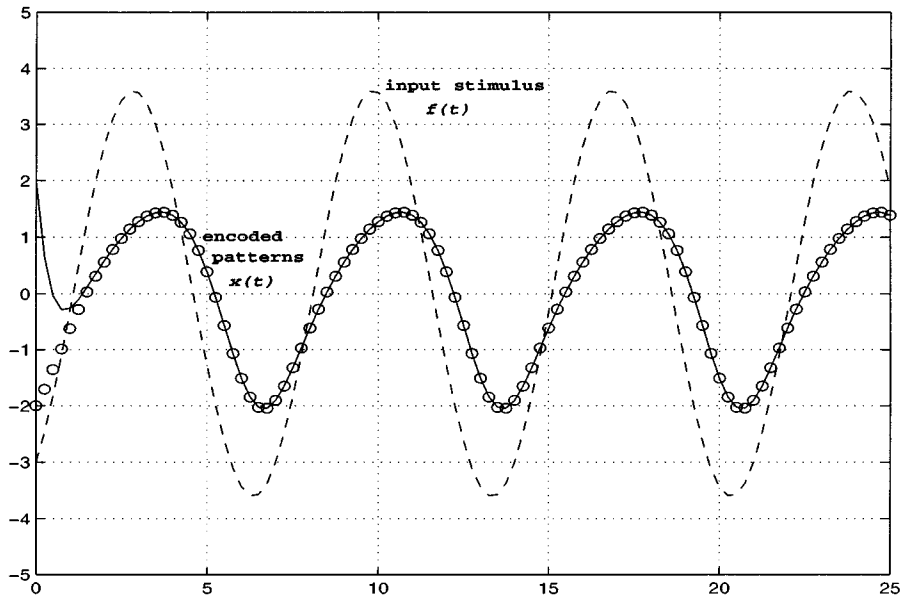


Fig. 5. Two encoded patterns  $x(t)$  of period 7 corresponding to two initial values  $x(0) = 2$  (solid line) and  $x(0) = -2$  (circle) along with the input stimulus  $f(t)$  of period 7 (dashed line).

$$= \int_{-\infty}^t e^{-\alpha(t-s)} x(s) ds$$

and obtain the following system of differential equations:

$$\left. \begin{aligned} \frac{dx(t)}{dt} &= -a(t)x(t) + b(t) \tanh[u(t)] + f(t) \\ \frac{du(t)}{dt} &= v(t) - \alpha u(t) \\ \frac{dv(t)}{dt} &= x(t) - \alpha v(t) \end{aligned} \right\}. \quad (113)$$

We again assume the neuron parameters as in (94)–(96) and the delay kernel as in (106) with  $\alpha = 2$ , it is found that the integrability condition in (111) and the assumptions of Theorem 4.4 are satisfied. Fig. 5 displays two periodic solutions of period  $\omega = 7$  of (112) corresponding to  $\alpha = 2$  and two initial values  $x(0) = 2$  and  $x(0) = -2$ .

## VII. CONCLUDING REMARKS

We note that if the recurrent stimulus in (1) is negligible or absent and if the dissipation rate  $a(\cdot)$  has exactly the same period as that of the input stimulus, our (1) reduces to the simpler equation

$$\frac{dv(t)}{dt} = -a(t)v(t) + f(t), \quad t > 0 \quad (114)$$

where

$$a(t + \omega) = a(t), \quad f(t + \omega) = f(t), \quad t > 0.$$

The linear equation (114) can be solved to obtain a periodic solution given by

$$v(t) = \left( \frac{\int_0^T f(s) e^{-\int_s^T a(u) du} ds}{1 - e^{-\int_0^T a(u) du}} \right) e^{-\int_0^t a(s) ds} + \int_0^t f(s) e^{-\int_s^t a(u) du} ds \quad (115)$$

whose existence is guaranteed by our hypothesis that

$$a(t) \geq a_* > 0. \quad (116)$$

Thus the dissipativity assumption on  $a(\cdot)$  guarantees the existence of a periodic solution given by (115) precluding any resonance type behavior in the absence of self-connection. Nonresonance is due to the dissipative nature of the system (114). If  $z(t)$  is any other solution of (114) then we have

$$\frac{d}{dt} [v(t) - z(t)] = -a(t)[v(t) - z(t)], \quad t > 0. \quad (117)$$

It is not difficult to see that (117) leads to

$$\begin{aligned} |v(t) - z(t)| &\leq |v(0) - z(0)| \exp \left[ -\int_0^t a(u) du \right] \\ &\leq |v(0) - z(0)| \exp[-a_* t] \\ &\rightarrow 0 \quad \text{as } t \rightarrow \infty \end{aligned}$$

implying that the periodic solution  $v$  is asymptotically stable.

The dynamics of neural networks subjected to time varying external stimuli have been considered by Rescigno *et al.* [19], König and Schillen [16], Bondarenko [2]. In this article we have obtained sufficient conditions for encoding of an external periodic stimulus by a single neuron-like processor having processing delays of various types. The dynamics of neural networks with two or more neurons with inhibitory and excitatory connections with transmission delays and external periodic and almost-periodic stimuli will be considered in the forthcoming articles.

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