Delay-Independent Stability in Bidirectional Associative Memory Networks

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Abstract—**It is shown that if the neuronal gains are small compared with the synaptic connection weights, then a bidirectional associative memory network with axonal signal transmission delays converges to the equilibria associated with exogenous inputs to the network; both discrete and continuously distributed delays are considered; the asymptotic stability is global in the state space of neuronal activations and also is independent of the delays.**

I. Introduction

The stability characteristics of equilibria of continuous bidirectional associative memory networks of the type

$$\left.\begin{aligned}\frac{dx_i(t)}{dt} &= -x_i(t) + \sum_{j=1}^{n} m_{ij}S(y_j(t)) + I_i\\ \frac{dy_i(t)}{dt} &= -y_i(t) + \sum_{j=1}^{n} m_{ij}S(x_j(t)) + J_i\end{aligned}\right\} i = 1,2,\cdots,n \tag{1}$$

and some of their generalizations have been investigated by Kosko [9], [10]. Networks of the form (1) generalize the continuous Hopfield circuit model [8] and can be obtained as a special case from the model of Cohen and Grossberg [3]. If one assumes that the exogenous inputs I_i, J_i $(i = 1,2,\cdots,n)$ and the connection weights m_{ij} $(i,j = 1,2,\cdots,n)$ are constants while the neuronal output signal function S is a differentiable, monotonic nondecreasing real valued function on $(-\infty,\infty)$, then it is possible to introduce an energy function (or Lyapunov function) E such that

$$\begin{aligned}E(x,y)(t) = &\sum_{i=1}^{n}\int_0^{x_i} S'(x_i)x_i dx_i - \sum_{i=1}^{n}\sum_{j=1}^{n} S(x_i)S(y_j)m_{ij}\\ &- \sum_{i=1}^{n} S(x_i)I_i + \sum_{j=1}^{n} S'(y_j)y_j dy_j - \sum_{j=1}^{n} S(y_j)J_j\end{aligned} \tag{2}$$

where $S'(x) = \frac{dS(x)}{dx}$. It has been shown in [9] that

$$\frac{dE}{dt} = -\sum_{i=1}^{n} S'(x_i)\dot{x}_i^2 - \sum_{j=1}^{n} S'(y_j)\dot{y}_j^2 \le 0. \tag{3}$$

One can show from (3) that as $t \to \infty$, $\dot{x}_i(t) \to 0$, $\dot{y}_i(t) \to 0$, $i = 1,2,\cdots,n$ implying that the network (1) converges to an equilibrium corresponding to the constant external inputs I_i, J_i $(i = 1,2,\cdots,n)$. The equilibria are sometimes called patterns or memories associated with the external inputs I and J.

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It is possible to simplify bidirectional networks of the type in (1) to a single system of a network of the type

$$\frac{dx_i(t)}{dt} = -x_i(t) + f_i(x_1(t),\cdots,x_n(t)) \qquad i = 1,2,\cdots,n$$

for suitably defined nonlinear functions f_i, $i = 1,2,\cdots,n$. In fact, a referee suggested that we do such a simplification. The authors have to retain the model (1) as it stands since such a simplification will alter the bidirectional interplay of the input-output nature of the two layers of the system and will reduce the system to that of a single layer system. For a detailed investigation of single layer systems we refer to a recent article of Gopalsamy and He [7].

The purpose of this brief article is to investigate the existence and stability characteristics of the equilibria of networks of the form

$$\left.\begin{aligned}\frac{du_i(t)}{dt} &= -u_i(t) + \sum_{j=1}^{n} a_{ij}S(\lambda_j v_j(t-\sigma_{ij})) + I_i\\ \frac{dv_i(t)}{dt} &= -v_i(t) + \sum_{j=1}^{n} b_{ij}S(\mu_j u_j(t-\tau_{ij})) + J_i\end{aligned}\right\} \quad i = 1,2,\cdots,n \tag{4}$$

in which λ_j, μ_j, τ_{ij}, σ_{ij} $(i,j = 1,2,\cdots,n)$ are nonnegative constants and I_i, J_i, a_{ij}, b_{ij} $(i,j = 1,2,\cdots,n)$ are real numbers; for convenience of exposition in the following we choose the signal response function as follows

$$S(x) = \tanh(x), \qquad x \in (-\infty,\infty). \tag{5}$$

The time delays τ_{ij} and σ_{ij} correspond to the finite speed of the axonal transmission of signals; for example τ_{ij} corresponds to the time lag from the time the i-th neuron in the I-layer emits a signal and the moment this signal becomes available for the j-th neuron in the J-layer of (4) (see for instance Domany *et al.* [4]). The constants λ_j, μ_j correspond to the neuronal gains associated with the neuronal activations. We refer to Babcock and Westervelt [1], Marcus and Westervelt [12], [13] and Marcus *et al.* [14] for linear analyses of single layer networks with delays.

One of the problems in the analysis of the dynamics of the delay differential system (4) is the existence of solutions of (4). The initial conditions associated with (4) are assumed to be of the form

$$\left.\begin{aligned}u_i(s) &= \phi_i(s), \quad s \in [-\tau^*,0], \qquad \tau^* = \max_{1\le i,j\le n}\tau_{ij}\\ v_i(s) &= \psi_i(s), \quad s \in [-\sigma^*,0], \qquad \sigma^* = \max_{1\le i,j\le n}\sigma_{ij}\end{aligned}\right\} \quad i = 1,2,\cdots,n \tag{6}$$

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in which ϕ_i, ψ_i are continuous real valued functions defined on their respective domains. One can use the method of steps and continuation (see Elsgolt's and Norkin [5]) and show that solutions of (4)–(6) exist for all $t \geq 0$. The following result provides sufficient conditions for the existence of equilibria associated to each pair of inputs I and J in (4).

Theorem 1: Assume the following:

i) the connection weights a_{ij}, b_{ij} $(i,j = 1,2,\cdots,n)$ are real constants;
ii) the exogenous inputs I_i, J_i $(i = 1,2,\cdots,n)$ are real constants;
iii) the gain parameters μ_j, λ_j $(j = 1,2,\cdots,n)$ and the time-delays τ_{ij} and σ_{ij} $(i,j = 1,2,\cdots,n)$ are nonnegative constants;
iv) there exists a number $c \in (0,1)$ such that the neuronal gains and the connection weights satisfy

$$\left.\begin{aligned}\lambda_i \sum_{j=1}^{n} |a_{ji}| &\leq c < 1\\ \mu_i \sum_{j=1}^{n} |b_{ji}| &\leq c < 1\end{aligned}\right\} \quad i = 1,2,\cdots,n. \tag{7}$$

Then corresponding to each exogenous input pair of vectors $I = (I_1, I_2, \cdots, I_n)$ and $J = (J_1, J_2, \cdots, J_n)$ there exists an unique equilibrium (u^*, v^*) of (4) satisfying

$$\left.\begin{aligned}u_i^* &= \sum_{j=1}^{n} a_{ij} S(\lambda_j v_j^*) + I_i\\ v_i^* &= \sum_{j=1}^{n} b_{ij} S(\mu_j u_j^*) + J_i\end{aligned}\right\} \quad i = 1,2,\cdots,n. \tag{8}$$

Proof: We have from (4) and (5) that an arbitrary solution of (4)–(6) satisfies the following differential inequalities

$$\left.\begin{aligned}-u_i(t) - \alpha_i \leq \frac{du_i(t)}{dt} \leq -u_i(t) + \alpha_i\\ -v_i(t) - \beta_i \leq \frac{dv_i(t)}{dt} \leq -v_i(t) + \beta_i\end{aligned}\right\} \quad i = 1,2,\cdots,n \tag{9}$$

where

$$\left.\begin{aligned}\alpha_i &= \sum_{j=1}^{n} |a_{ij}| + |I_i|\\ \beta_i &= \sum_{j=1}^{n} |b_{ij}| + |J_i|\end{aligned}\right\} \quad i = 1,2,\cdots,n. \tag{10}$$

It will follow from (9) that the set $\Omega \subset \mathbf{R}^{2n}$ defined by

$$\Omega = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \middle| \begin{aligned} u &= (u_1, u_2, \cdots, u_n);\ |u_i| \leq \alpha_i,\ i = 1,2,\cdots,n\\ v &= (v_1, v_2, \cdots, v_n);\ |v_i| \leq \beta_i,\ i = 1,2,\cdots,n \end{aligned} \right\} \tag{11}$$

is invariant with respect to the delay differential equations (4). Thus if the system (4) has an equilibrium, then such an equilibrium is a fixed point of the mapping $F : \Omega \to \mathbf{R}^{2n}$ defined by

$$F(x,y) = \begin{Bmatrix} F_1(x,y) \\ F_2(x,y) \end{Bmatrix} \tag{12}$$

where

$$F_1(x,y) = \begin{bmatrix} \sum_{j=1}^{n} a_{1j} S(\lambda_j y_j) + I_1 \\ \sum_{j=1}^{n} a_{2j} S(\lambda_j y_j) + I_2 \\ \cdots \\ \sum_{j=1}^{n} a_{nj} S(\lambda_j y_j) + I_n \end{bmatrix},$$
$$F_2(x,y) = \begin{bmatrix} \sum_{j=1}^{n} b_{1j} S(\mu_j x_j) + J_1 \\ \sum_{j=1}^{n} b_{2j} S(\mu_j x_j) + J_2 \\ \cdots \\ \sum_{j=1}^{n} b_{nj} S(\mu_j x_j) + J_n \end{bmatrix}. \tag{13}$$

We note from (12) and (13) that if (x,y) and (X,Y) are any two points of Ω, then

$$\begin{aligned}
||F(x,y) - F(X,Y)|| &= \left| \sum_{i=1}^{n}\sum_{j=1}^{n} [a_{ij}\{S(\lambda_j y_j) - S_j(\lambda_j Y_j)\} \right.\\
&\qquad \left. + b_{ij}\{S(\mu_j x_j) - S(\mu_j X_j)\}] \right| && (14)\\
&\leq \left| \sum_{i=1}^{n}\sum_{j=1}^{n} [a_{ij}\lambda_j S'(\theta_j^y)(y_j - Y_i) \right.\\
&\qquad \left. + b_{ij}\mu_j S'(\theta_j^x)(x_j - X_j)] \right| && (15)\\
&\leq \sum_{i=1}^{n}\sum_{j=1}^{n} [\lambda_j |a_{ij}||y_j - Y_j| + \mu_j |b_{ij}||x_j - X_j|]\\
&\leq \sum_{i=1}^{n}\sum_{j=1}^{n} [\lambda_i |a_{ji}||y_i - Y_i| + \mu_i |b_{ji}||x_i - X_i|]\\
&= \sum_{i=1}^{n} \left\{ \left(\lambda_i \sum_{j=1}^{n} |a_{ji}|\right) |y_i - Y_i| \right.\\
&\qquad \left. + \left(\mu_i \sum_{j=1}^{n} |b_{ji}|\right) |x_i - X_i| \right\}\\
&\leq c \sum_{i=1}^{n} [|x_i - X_i| + |y_i - Y_i|] && (16)\\
&< ||(x,y) - (X,Y)||; && (17)
\end{aligned}$$

in deriving (15) and subsequent inequalities we have used the facts that θ_j^y lies between $\lambda_j y_j$ and $\lambda_j Y_j$, θ_j^x lies between $\mu_j x_j$ and $\mu_j X_j$ as well as

$$S'(\theta) = 1 - S^2(\theta) \leq 1 \quad \text{for} \quad \theta \in (0, \infty).$$

The mapping F is continuous and $F(\Omega) \subset \Omega$; it follows from (16) and $c < 1$ that F is a contraction on Ω. By the well known contraction mapping principle, we conclude that there exists a unique point say (u^*, v^*) such that

$$F(u^*, v^*) = (u^*, v^*) \tag{18}$$

and this completes the proof.

Thus there exists a unique pattern or memory (or equilibrium) associated with each set of the external inputs I and J when the connection weights a_{ij} and b_{ij} are fixed. In the next section we derive conditions for the global asymptotic stability of the unique equilibrium (u^*, v^*) of (4).

II. Recall of Associative Memory

The equilibrium or the pattern (u^*, v^*) of (4) associated with a given (I, J) is said to be globally asymptotically stable independent of the delays if every solution of (4) corresponding to an arbitrarily given set of initial values (6) satisfy

$$\lim_{t\to\infty} u_i(t) = u_i^* \text{ and } \lim_{t\to\infty} v_i(t) = v_i^* \quad (i = 1, 2, \cdots, n). \tag{19}$$

We remark that when $a_{ij} \neq b_{ij}$, $\lambda_j \neq 1$, $\mu_j \neq 1$, $\tau_{ij} \equiv 0$, $\sigma \equiv 0$, the method of Lyapunov functions exploited by Hopfield [8], Cohen and Grossberg [3], Kosko [9] is not applicable; the existence of a Lyapunov function is dependent on the symmetry of the synaptic connection weights in [8], [9] and $a_{ij} = b_{ij}$ in the case of Kosko [9].

Theorem 2 Suppose the assumptions of Theorem 1 hold. Then the equilibrium (u^*, v^*) of (4) is globally asymptotically stable.

Proof: Define the variables x_i and y_i by the following

$$\left.\begin{array}{ll} x_i : [-\tau_i, 0] \to \mathbf{R}, & x_i(t) \equiv u_i(t) - u_i^* \\ y_i : [-\sigma_i, 0] \to \mathbf{R}, & y_i(t) \equiv v_i(t) - v_i^* \end{array}\right\}. \tag{20}$$

One can derivative that the deviations x_i, y_i are governed by

$$\left.\begin{array}{l} \frac{dx_i(t)}{dt} = -x_i(t) + \sum_{j=1}^{n} a_{ij}\lambda_j S_j'(\theta_{ij}^y(t)) y_j(t - \sigma_{ij}) \\ \frac{dy_i(t)}{dt} = -y_i(t) + \sum_{j=1}^{n} b_{ij}\mu_j S_j'(\theta_{ij}^x(t)) x_j(t - \tau_{ij}) \end{array}\right\} \quad i = 1, 2, \cdots, n \tag{21}$$

in which $\theta_{ij}^y(t)$ lies between $\lambda_j v_j^*$ and $\lambda_j v_j(t - \sigma_{ij})$ and $\theta_{ij}^x(t)$ lies between $\mu_j u_j^*$ and $\mu_j u_j(t - \tau_{ij})$ for $j = 1, 2, \cdots, n$. We note that

$$S'(s) = 1 - S^2(s) \leq 1, \qquad s \in \mathbf{R}$$

and hence we have from (21) that

$$\left.\begin{array}{l} \frac{dx_i(t)}{dt} \leq -x_i(t) + \sum_{j=1}^{n} |a_{ij}|\lambda_j |y_j(t - \sigma_{ij})| \\ \frac{dy_i(t)}{dt} \leq -y_i(t) + \sum_{j=1}^{n} |b_{ij}|\mu_j |x_j(t - \tau_{ij})| \end{array}\right\} \quad i = 1, 2, \cdots, n. \tag{22}$$

Consider a Lyapunov functional $V(t) = V(x, y)(t)$ defined by

$$V(t) = \sum_{i=1}^{n}\left[|x_i(t)| + |y_i(t)| + \sum_{j=1}^{n} |a_{ij}|\lambda_j \int_{t-\sigma_{ij}}^{t} |y_j(s)| ds + \sum_{j=1}^{n} |b_{ij}|\mu_j \int_{t-\tau_{ij}}^{t} |x_j(s)|\, ds\right]. \tag{23}$$

Calculating the upper right derivative D^+V of V along the solutions of (21)

$$\begin{aligned} D^+V(t) \leq \sum_{i=1}^{n}\Bigg\{ &-|x_i(t)| + \sum_{j=1}^{n} |a_{ij}|\lambda_j |y_j(t - \sigma_{ij})| \\ &- |y_i(t)| + \sum_{i=1}^{n} |b_{ij}|\mu_j |x_j(t - \tau_{ij})| \\ &+ \sum_{j=1}^{n} |a_{ij}|\lambda_j \big[|y_j(t)| - |y_j(t - \sigma_{ij})|\big] \\ &+ \sum_{j=1}^{n} |b_{ij}|\mu_j \big[|x_j(t)| - |x_j(t - \tau_{ij})|\big] \Bigg\} \end{aligned} \tag{24}$$

$$\begin{aligned} &= \sum_{i=1}^{n}\left[\left(-1 + \mu_i \sum_{j=1}^{n} |b_{ji}|\right)|x_i(t)| + \left(-1 + \lambda_i \sum_{j=1}^{n} |a_{ji}|\right)|y_i(t)|\right] \\ &\leq -(1 - c)\sum_{i=1}^{n}\big[|x_i(t)| + |y_i(t)|\big] \end{aligned} \tag{25}$$

it is consequence of (25) that

$$V(x, y)(t) + (1 - c)\int_{t_o}^{t}\sum_{i=1}^{n}\Big[|x_i(s)| + |y_i(s)|\Big]\, ds \leq V(x, y)(t_o). \tag{26}$$

It follows from (26) and (23) that V is bounded on $(0, \infty)$; the boundedness of V on $(0, \infty)$ implies that x_i, y_i are bounded on $(0, \infty)$. We note from (22) that $\dot{x}_i$, $\dot{y}_i$ are bounded on $(0, \infty)$ and this implies that x_i, y_i are uniformly continuous on $(0, \infty)$. A consequence of (26) is that

$$\sum_{i=1}^{n}[|x_i(t)| + |y_i(t)|] \in L_1(0, \infty). \tag{27}$$

The uniform continuity of $\sum_{i=1}^{n}\big[|x_i(t)| + |y_i(t)|\big]$ on $(0, \infty)$ together with (27) implies by Barbalat's Lemma (see Barbalat [2] or Gopalsamy [6]) that

$$\sum_{i=1}^{n}[|x_i(t)| + |y_i(t)|] \to 0 \qquad \text{as} \qquad t \to \infty. \tag{28}$$

Thus it follows $x_i(t) \to 0, y_i(t) \to 0$ as $t \to \infty$, $i = 1, 2, \cdots, n$ and hence

$$u_i(t) \to u_i^*, \qquad v_i(t) \to v_i^* \qquad \text{as} \qquad t \to \infty.$$

This completes the proof.

III. Distributed Delays

The use of constant fixed delays in models of delayed feedback provides of a good approximation in simple circuits consisting of a small number of cells. Neural networks usually have a spatial extent due to the presence of a multitude of parallel pathways with a variety of axon sizes and lengths. Thus there will be a distribution of conduction velocities along these pathways and a distribution of propagation delays. In these circumstances the signal propagation is not instantaneous and cannot be modeled with discrete delays and a more appropriate way is to incorporate continuously distributed delays. The extent to which the values of the state variable in the past affect its present dynamics is determined by a delay kernel; the case of constant discrete delays corresponds to a choice of the delay kernel to be a Dirac delta function. Kernels of the form

$$K(t) = \frac{\alpha^{m+1}}{m!} t^m e^{-\alpha t}, \qquad t \geq 0, \qquad \alpha, m \in [0, \infty)$$

have been found mathematically tractable in many models of dynamic systems in mathematical ecology especially in population dynamics.

We shall now consider a class of bidirectional associative memory networks with continuously distributed delays described by

$$\left.\begin{aligned} \frac{du_i(t)}{dt} &= -u_i(t) + \sum_{j=1}^{n} a_{ij} \int_{-\infty}^{t} k_{ij}^{(1)}(t-s) S(\lambda_j v_j(s))\, ds + I_i \\ \frac{dv_i(t)}{dt} &= -v_i(t) + \sum_{j=1}^{n} b_{ij} \int_{-\infty}^{t} k_{ij}^{(2)}(t-s) S(\mu_j v_j(s))\, ds + J_i \end{aligned}\right\} \quad i = 1, 2, \cdots, n \tag{29}$$

in which the external inputs I_i, J_i and the connection weights a_{ij}, b_{ij} are constants; the neuronal gains λ_j and μ_j are positive constants; the delay kernels $k_{ij}^{(1)}$, $k_{ij}^{(2)}$ are nonnegative valued continuous functions defined on $[0, \infty)$ satisfying

$$\left.\begin{aligned} &\int_0^\infty k_{ij}^{(1)}(s) ds = 1, \qquad \int_0^\infty k_{ij}^{(2)}(s)\, ds = 1 \\ &\int_0^\infty s k_{ij}^{(1)}(s) ds < \infty, \qquad \int_0^\infty s k_{ij}^{(2)}(s) ds < \infty \end{aligned}\right\} \quad i, j = 1, 2, \cdots, n. \tag{30}$$

The initial values associated with (29) are of the form

$$u_i(s) = \phi_i(s), \qquad y_i(s) = \psi_i(s), \qquad s \in (-\infty, 0], \quad i = 1, 2, \cdots, n \tag{31}$$

where ϕ_i, ψ_i are assumed to be bounded continuous functions defined on $(-\infty, 0]$. For an extensive discussion of the stability and asymptotic behavior of integro-differential equations such as (29) and their applications, we refer to the recent monograph by Gopalsamy [6]. For applications of integro-differential equations with continuously distributed delays such as those in (29), we refer to Tank and Hopfield [15].

One can see that the equilibrium (u^*, v^*) defined by (15) is again an equilibrium of (29) due to (30).

Theorem 3: Suppose the hypotheses of Theorem 1 hold; suppose further (30) holds. Then the equilibrium (u^*, v^*) of (29) is globally asymptotically stable in the sense that all solutions of (29)–(31) satisfy

$$\lim_{t \to \infty} u_i(t) = u_i^*, \qquad \lim_{t \to \infty} v_i(t) = v_i^*, \qquad i = 1, 2, \cdots, n. \tag{32}$$

Proof: Details of proof are similar to those of Theorem 2 and hence we shall to brief. As before we define x_i, y_i by (22) and derive from (29)–(30) that x_i and y_i satisfy the system

$$\left.\begin{aligned} \frac{dx_i(t)}{dt} &= -x_i(t) + \sum_{j=1}^{n} a_{ij}\lambda_j \int_0^\infty k_{ij}^{(1)}(s) S'(\theta_j^y(s)) y_j(t-s) ds \\ \frac{dy_i(t)}{dt} &= -y_i(t) + \sum_{j=1}^{n} b_{ij}\mu_j \int_0^\infty k_{ij}^{(2)}(s) S'(\theta_j^x(s)) x_j(t-s) ds \end{aligned}\right\} \quad i = 1, 2, \cdots, n \tag{33}$$

which as before leads to

$$\left.\begin{aligned} \frac{dx_i(t)}{dt} &\leq -x_i(t) + \sum_{j=1}^{n} |a_{ij}|\lambda_j \int_0^\infty k_{ij}^{(1)}(s) |y_j(t-s)| ds \\ \frac{dy_i(t)}{dt} &\leq -y_i(t) + \sum_{j=1}^{n} |b_{ij}|\mu_j \int_0^\infty k_{ij}^{(2)}(s) |x_j(t-s)| ds \end{aligned}\right\} \quad i = 1, 2, \cdots, n. \tag{34}$$

We consider a Lyapunov type functional $V(t) = V(x, y)(t)$ defined by

$$\begin{aligned} V(t) = \sum_{i=1}^{n} \Bigg[&|x_i(t)| + |y_i(t)| \\ &+ \sum_{j=1}^{n} |a_{ij}|\lambda_j \int_0^\infty k_{ij}^{(1)}(s) \left(\int_{t-s}^{t} |y_j(r)| dr \right) ds \\ &+ \sum_{j=1}^{n} |b_{ij}|\mu_j \int_0^\infty k_{ij}^{(2)}(s) \left(\int_{t-s}^{t} |x_j(r)|\, dr \right) ds \Bigg]. \end{aligned} \tag{35}$$

By our assumption on the initial values on $(-\infty, 0]$ and the hypotheses on the delay kernels, one can verify that V is defined on $(0, \infty)$ and V is bounded on $(0, \infty)$. Also the upper right derivative D^+V of V along the solutions of (33) can be calculated so that

$$D^+V(t) \leq -(1-c) \sum_{i=1}^{n} [|x_i(t)| + |y_i(t)|]. \tag{36}$$

The remaining details of proof are identical to these of Theorem 2 and hence are omitted. The proof is complete.

IV. Remarks

Recall of memories is one of the processes by which the brain returns in some sense from a current state to another state in which it has been before. In neural network models, memory corresponds to a temporally stationary or nonstationary equilibrium and recall is modeled by the convergence of neuronal activations in the neuronal activation space to the equilibrium. The trigger provided for the system to recall the memory may come from outside as external inputs. Thus the patterns or equilibria associated with external inputs are recalled by the convergence of system dynamics; global asymptotic stability of an equilibrium means that the recall is "perfect" in the sense no hints or guesses are needed as in the case of local stability analyses; that is when the external inputs are provided to the system, irrespective of the initial values, the system converges to the equilibrium associated with the inputs. Recall with the help of hints and guesses correspond to local stability of equilibria; since the initial values have to be in a suitable neighborhood of the corresponding equilibrium.

It is known (see [12]) that time delays in response or transmission can induce sustained oscillations and "chaos". We have discussed the stability characteristics of an already trained bidirectional not necessarily symmetric associative network with transmission delays and obtained sufficient conditions for the absence of delay induced persistent oscillations or chaos. In a forthcoming article we discuss the dynamics of associative recall in a bidirectional adaptive network with delays in the learning dynamics.

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