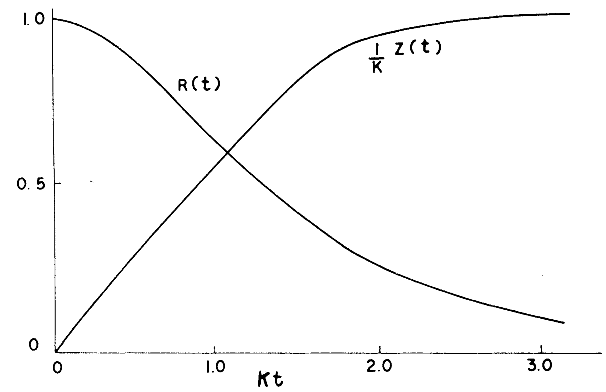

Fig. 1. The reliability function $R(t) = \text{sech } kt$ and the normalized hazard rate $1/k\, Z(t) = \tanh kt$.

$R(t)$ and a normalized $Z(t)$ are sketched in Figure 1. $Z(t)$ evolves from zero and grows to the constant k; it reaches $0.9k$ for $kt = 1.472$.

To evaluate the reliability of statistically independent series components, consider the joint probability of n failures in time t. For analytic simplicity, consider two such components. Since they are independent their joint probability $P_{II}(n, t)$ will be the convolution of the individual failure probabilities:

$$P_{II}(n,t) = \frac{(\text{sech } k_1 t)}{(2n)!} (\text{sech } k_2 t) \sum_{j=0}^{n} \binom{2n}{2j} (k_2 t)^{2j} (k_1 t)^{2(n-j)}. \tag{8}$$

The reliability for two units in series is

$$R_{II}(t) = \text{sech } k_1 t \text{ sech } k_2 t. \tag{9}$$

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S. C. Kak was born in Srinagar, India on March 26, 1947. He received his B.E. and Ph.D degrees in Electrical Engineering from Kashmir University and Indian Institute of Technology, Delhi in 1967 and 1970 respectively. He presently teaches at the Electrical Engineering Department of I.I.T. Delhi.

He has worked on various problems of signal and system representation and classification. He is presently engaged in research on modelling of socio-economic and biological systems. He has published about twenty papers in various international journals in these areas.

Use of Dynamic Programming for Reliability Engineers

D. K. KULSHRESTHA AND M. C. GUPTA

Abstract—**This paper aims to obtain the optimum cost allocation to a number of components connected in series (no redundancy) with a view to maximize the system reliability subject to a given total cost of the system. The reliability of each component is a function of its cost. The technique of Dynamic Programming has been employed to achieve the results.**

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D. K. Kalshrestha is with the School of Mathematical Sciences, Flinders University of South Australia, Bedford Park, South Australia.

M. C. Gupta is with the Statistics and Information Division, Ministry of Education and Youth Services, New Delhi, India.

Reader Aids:
Purpose: Tutorial
Special math needed for explanations: Programming
Special math needed for results: Dynamic programming
Results useful to: Reliability analysts

INTRODUCTION

The purpose of this note is to select the optimum cost for each component of the system with a view to maximizing system reliability subject to a given total cost. There are a number of alternate components with varying costs and capable of performing the same specific purpose, but each has different reliability; the reliability of each alternate component is a function of its cost. Bellman and Dreyfus [1] have discussed the reliability R of a system containing several subsystems, each of which has redundancy; they maximize the reliability subject to a cost constraint. Kettle [2] has described an approach of least cost allocation with a view to minimizing the sum of the subsystem unreliabilities instead of maximizing system reliability. We assume here that the reliability of a component is a monotonically increasing function of its cost.

NOTATION

$R_n(C_n)$ reliability of component n.
C_n cost of component n.
R system reliability.
N number of components in the system.
The system reliability is

$$R(C_1, C_2, \cdots, C_N) = \prod_{n=1}^{N} R_n(C_n).$$

Next we obtain a suitable expression for $R_n(C_n)$. Breiphol [3] has expressed the cost of a component as a function of its reliability:

$$C_n = \frac{K_{1n}}{1 - R_n(C_n)} \exp\left[-K_{2n}\{1 - R_n(C_n)\}\right] \tag{1}$$

where K_{1n} and K_{2n} are constants. Equation 1 cannot be solved for $R_n(C_n)$. Therefore, (2) has been assumed and leads to approximately the same sets of values of reliability and cost as does (1) for $0.9 < R_n(C_n) < 1$.

$$R_n(C_n) = a_n + b_n \log C_n, \tag{2}$$

$0 < a_n < 1$; $b_n > 0$, but small enough to allow some terms to be neglected later on.

Sandler [4] has assumed a similar expression for $R_n(C_n)$. The problem discussed in this paper is

$$\underset{C_1, C_2, \cdots, C_N}{\text{Maximize}} \left\{ \prod_{n=1}^{N} (a_n + b_n \log C_n) \right\}$$

$$\text{Subject to } \sum_{n=1}^{N} C_n = C, C_n > 0. \tag{3}$$

DYNAMIC PROGRAMMING FORMULATION

With a view to maximizing the system reliability in N stages, define stage j ($j = 1, 2, \cdots, N$) to consist of j components counted from the last, i.e. N to $(N - j + 1)$.

Define:

$F(j, C_{N-j+1}, C)$ = Reliability of stage j when C_{N-j+1} is individually allocated to component $N - j + 1$; $C - C_{N-j+1}$ is optimally allocated to the remaining $j - 1$ components contained in stage j; $j = 1, 2, \cdots, N$.

$F^*(j, C)$ = Maximum of $F(j, C_{N-j+1}, C)$ with respect to

$$C_{N-j+1}; 0 < C_{N-j+1} < C, \quad j = 1, 2, \cdots, N$$
$$= 1 \quad j = 0. \tag{4}$$

Evidently $F^*(N, C) = \max \{R(C_1, \cdots, C_N)\}$ with respect to $C_1, \cdots, C_N$ subject to the restriction, $C_1 + \cdots + C_N = C$. Bellman's principle of optimality results in the following functional equations:

$$F^*(j, C) = \max \{R_{N-j+1}(C_{N-j+1}) F^*(j-1, C - C_{N-j+1})\}$$
with respect to C_{N-j+1},
$$0 < C_{N-j+1} < C, \quad j = 2, \cdots, N.$$
$$F^*(1, C) = \max \{R(N, C_N)\} \text{ with respect to } C_N,$$
$$0 < C_N < C. \tag{5}$$

The optimum values of $C_1, \cdots, C_N$ can be obtained by solving the above mentioned functional equations. This formulation is similar to that discussed in [1, 6]. If standard Dynamic Programming procedures were followed the accurate solution of (5) would be voluminous and cumbersome. Therefore it is desirable to achieve approximate solutions which are almost as good. Misra [5] has pointed out that, as compared to the "product form", the "sum form" of functional equations provides faster but approximate solutions; but his approach is not applicable here. Through an example a simple approach is described to obtain approximate but explicit expressions for optimum costs.

Example. $N = 3$

$$F^*(1, C) = a_3 + b_3 \log C \tag{6a}$$
$$F(2, C_2, C) = (a_2 + b_2 \log C_2)(a_3 + b_3 \log (C - C_2)). \tag{6b}$$

Let

$$k \equiv C_2/C, \quad (0 < k < 1)$$
$$g_i \equiv a_i + b_i \log C, \quad i = 1, 2, 3;$$

and substitute in (6b). Remember that g_i is a function of C. Assume that b_2 and b_3 are small enough to neglect the term $b_2 b_3 \log k \cdot \log(1 - k)$ in (6b); we have,

$$F(2, Ck, C) = g_2 g_3 + b_3 g_2 \log (1 - k)$$
$$+ b_2 g_3 \log k$$
$$\equiv \phi(k).$$

The optimum value of k, denoted by k^*, is defined by

$$\frac{d\phi(k)}{dk} = 0, \quad \text{for } k = k^*.$$

Hence

$$k^* = \frac{b_2 g_3}{b_3 g_2 + b_2 g_3}$$

and

$$F^*(2, C) = \phi(k^*) = F(2, Ck^*, C).$$

On similar lines for the third stage

$$F(3, C_1, C) = R(1, C_1) F^*(2, C - C_1).$$

Let $\theta(l) \equiv F(3, Cl, C)$ where $l \equiv C_1/C$, $0 < l < 1$. Assume that all b_i are small enough to neglect terms containing $b_1 b_2$, $b_2 b_3$, $b_1 b_3$, and $b_1 b_2 b_3$; then the optimum value of l, denoted by l^*, is defined by

$$\frac{d\theta(l)}{dl} = 0, \quad \text{for } l = l^*.$$

Hence

$$l^* = \frac{b_1 h_2 h_3}{b_1 h_2 h_3 + b_2 h_1 h_3 + b_3 h_1 h_2}$$

where

$$h_1 \equiv g_1$$
$$h_2 \equiv g_2 + b_2 \log k^*$$
$$h_3 \equiv g_3 + b_3 \log (1 - k^*).$$

Optimum values of C_1, C_2, C_3, denoted by the *, are obtained as follows:

$$C_1^* = Cl^*, C_2^* = Ck^*(1 - l^*), C_3^* = C(1 - l^*)(1 - k^*). \tag{7}$$

Similar techniques can be used for higher values of N, if the values of b_i turn out to be small enough. No research on values of b_i for actual components was made to see if the approach is a practical one.

Illustration. Let $N = 3$, $C = 1{,}500$. Assume the following data taken from [3]:

R_1	C_1	R_2	C_2	R_3	C_3
.999	1,820	.999	1,030	.999	6,310
.995	100	.995	100	.995	200.

We obtain

$$a_1 = .97910 \quad a_2 = .97530 \quad a_3 = .98086$$
$$b_1 = .00138 \quad b_2 = .00171 \quad b_3 = .00116$$
$$k^* = .5957 \quad l^* = .3246$$
$$C_1^* = 486.9 \quad C_2^* = 603.5 \quad C_3^* = 409.6$$

Maximum reliability = .99111.

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D. K. Kulshrestha was born on 6th July, 1935, in India. He completed his Master of Science degree in mathematics in 1954, from Aligarh Muslim University, Aligarh, U.P. India. After serving as a lecturer in mathematics for four years he joined the Research and Development Organization, Ministry of Defence, India, where he actively participated in different types of practical projects, mostly based on the application of Operations Research. During his stay in the Ministry of Defence he published more than a dozen research papers on disposal decision problems and theory of reliability, which have appeared in standard journals of Operations Research. He submitted the thesis entitled "Maintenance Problems in Operations Research" to the University of Delhi, India, and was awarded the degree of Doctor of Philosophy in 1968. Since 1971 he is teaching and developing graduate and undergraduate Operations Research courses at the School of Mathematical Sciences, The Flinders University of South Australia. His current research interests are application of operations research techniques in general and Dynamic Programming and reliability in particular.

M. C. Gupta was born in India on 27th July, 1946. After completing his Master of Science degree in Mathematics in 1968, from the University of Delhi, he has served as a lecturer in Mathematics in a public school. He joined the Ministry of Education and Youth Services as statistical assistant in 1970. At present he is working for the degree of Doctor of Philosophy in reliability theory from the University of Delhi and his main research interest includes application of cost structures to reliability problems.