

Convergence under Dynamical Thresholds with Delays

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Abstract—Necessary and sufficient conditions are obtained for the existence of a globally asymptotically stable equilibrium of a class of delay differential equations modeling the action of a neuron with dynamical threshold effects.

Index Terms— Lyapunov functional, omega-limit set, Poincare–Benedixson’s negative criterion, thresholds, time delays.

I. INTRODUCTION

ONE of the mathematical models proposed by Caianiello [3] for the study of the dynamical characteristics of neurons or neuron-like threshold devices consists of a set of “neuronic” and “mnemonic” equations. The neuronic equations describe the dynamics of neurons while the mnemonic equations describe the temporal variation of the synaptical interconnections (or weights). The processes described by the mnemonic equations and in particular the temporal variation of the interneuronal synaptic couplings is much slower with respect to the cellular activities of neurons; we can therefore as a first approximation assume that the weights are constant with respect to time. The neuronic equation (also known as the decision equation) considered by Caianiello and De Luca [5] is of the form

$$u(t + \tau) = H \left[A(t) - \int_{-\infty}^t k(t-s)u(s) ds - c \right] \quad (1)$$

where H denotes the unit step function

$$H(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

and $u(t)$ denotes the neuron response assumed to have the values zero or one, $A(t)$ denotes the external stimulus to the neuron, c denotes the neuronal threshold, $k(t)$ denotes the refractoriness of the neuron after it has fired or responded, i.e., gone from state zero to state one, and τ denotes the delay of the system which indicates the interval of time which must elapse before the system can respond, after the reception of a stimulus of sufficiently strong strength to cause such an action.

The purpose of this paper is to investigate the dynamical characteristics of a model neuron which is capable of firing or responding continuously in time while its firing is modulated by the difference between its current status and a weighted

average of its firing history. Accordingly, we modify (1) and (2) by the following: we replace $u(t + \tau)$ and the nonlinearity in (1) and (2) by

$$u(t + \tau) \approx u(t) + \tau \frac{du(t)}{dt} \quad (3)$$

$$\begin{aligned} & H \left[A(t) - \int_{-\infty}^t k(t-s)u(s) ds - c \right] \\ &= a \tanh \left[u(t) - b \int_{-\infty}^t k(t-s)u(s) ds \right]. \end{aligned} \quad (4)$$

Thus we are led to consider the following integrodifferential equation:

$$\frac{du(t)}{dt} = -\frac{1}{\tau}u(t) + \frac{a}{\tau} \tanh \left[u(t) - b \int_{-\infty}^t k(t-s)u(s) ds \right] \quad (5)$$

in which we can set $\tau = 1$ (otherwise we can rescale the time variable); the constants a and b have the obvious meanings; i.e., a denotes the range of the continuous variable $u(\cdot)$ while b denotes a measure of the inhibitory influence of the past history. An equation of the type in (5) can be considered to represent a case with a local positive feedback with delays. The term $u(t)$ in the argument of the tanh function in (5) denotes a local positive feedback. In biological literature such local positive feedbacks are known as reverberations [4], while in the literature on neural networks these positive feedbacks are known as self excitations. A discussion of the implications of positive feedbacks and delays for temporal processing of information can be found in [7]. Our model will be complete if we specify the delay kernel and this will be done below.

In this article, we examine the stability characteristics of the scalar autonomous delay differential equations of the form

$$\frac{dx(t)}{dt} = -x(t) + a \tanh [x(t) - b x(t - \tau) - c] \quad (6)$$

$$\frac{dx(t)}{dt} = -x(t) + a \tanh \left[x(t) - b \int_0^t k(s)x(t-s) ds - c \right] \quad (7)$$

$$\frac{dx(t)}{dt} = -x(t) + a \tanh \left[x(t) - b \int_0^\infty k(s)x(t-s) ds - c \right] \quad (8)$$

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where a, b, τ , and c are nonnegative numbers and $k : [0, \infty) \rightarrow [0, \infty)$ is a continuous delay kernel satisfying certain restrictions to be specified later. Equations of the above type occur in the study of neural networks with threshold effects (see, for instance, [13]). Equations similar to (6)–(8) appear also in the temporal evolution of sublattice magnetizations [14]. In the study of neural networks, especially concerned with the storage and retrieval of temporal patterns (limit cycles), equations of the above form occur [6], [15].

It is also possible to interpret the above system of equations in the following way. $x(t)$ denotes the activation level of a neuron which is capable of self-activation modulated by a dynamic threshold, which depends on the history of its previous activations. The time delays incorporated in the thresholds can lead to wide ranging behavior of the system from relaxation to a stable equilibrium, instability resulting in stable oscillations and “chaos” [15]. There has been recently numerous investigations of Hopfield-type networks with time delays (see, for instance, [1], [2], [9], [10], [16], and the references in these works). Since Hopfield networks with thresholds have not received any attention so far, the system considered in this article is just a beginning in this direction; a multineuron model system with threshold delays will be considered in a future article. A primary purpose of this paper is to obtain sufficient conditions for the systems of the type (1)–(3) to converge to the temporally static equilibrium position; bifurcation to persistent periodic oscillations has been considered in a companion article (see [11]).

II. CONVERGENCE TO EQUILIBRIUM

In this section we derive sufficient conditions for the model systems (1)–(3) to have unique equilibria such that the respective equilibria are globally asymptotically stable. We consider the system (1) with $c = 0$ first and subsequently discuss the behavior of the system when $c \neq 0$; when $c = 0$, we have from (1)

$$\frac{dx(t)}{dt} = -x(t) + a \tanh[x(t) - bx(t - \tau)], \quad t > 0. \quad (9)$$

We note that the delay differential equation (9) is supplemented with an initial condition of the form

$$x(s) = \phi(s), \quad s \in [-\tau, 0]$$

where ϕ is assumed to be a continuous real valued function on $[-\tau, 0]$. We let

$$y(t) \equiv x(t) - bx(t - \tau), \quad t \in [-\tau, \infty) \quad (10)$$

and obtain from (9) that

$$\frac{dy(t)}{dt} = -y(t) + a \tanh[y(t)] - ab \tanh[y(t - \tau)], \quad t > 0. \quad (11)$$

If y^* denotes an equilibrium of (11), then y^* satisfies

$$y^* = a(1 - b) \tanh(y^*), \quad (12)$$

We assume that a, b are such that

$$a > 0, \quad b \geq 0, \quad a(1 - b) < 1. \quad (13)$$

It is now elementary to note that (12) has a unique solution under (13) denoted by y^* . Thus (11) has the trivial resting state $y(0) \equiv 0$ on $[-\tau, \infty)$ as its only equilibrium. Our first result provides sufficient conditions for the global asymptotic stability of $y^* = 0$ of (11).

Proposition 2.1: Suppose that the parameters a, b and τ satisfy

$$a > 0, \quad b \geq 0, \quad a(1 - b) < 1 \quad \text{and} \quad a(1 + b) < 1. \quad (14)$$

Then all solutions of (11) satisfy

$$\lim_{t \rightarrow \infty} y(t) = y^* = 0. \quad (15)$$

Proof: Consider a Lyapunov-like functional $V(y)(t)$ defined by

$$V(y)(t) = |y(t)| + ab \int_{t-\tau}^t |y(s)| ds, \quad t > 0. \quad (16)$$

Calculating the upper right derivative $\frac{DV}{Dt}$ along the solutions of (11),

$$\begin{aligned} \frac{DV}{Dt} &\leq -|y(t)| + a |\tanh[y(t)]| \\ &\quad + ab |\tanh[y(t - \tau)]| + ab|y(t)| - ab|y(t - \tau)| \\ &\leq -\{1 - a(1 + b)\}|y(t)|, \quad t > 0. \end{aligned} \quad (17)$$

It follows from (17) that

$$V(y)(t) + \{1 - a(1 + b)\} \int_0^t |y(s)| ds \leq V(y)(0), \quad t > 0. \quad (18)$$

By hypothesis in (14), $a(1 + b) < 1$ and hence it follows from (18) and (16) that $y(\cdot)$ remains bounded on $[-\tau, \infty)$; consequently, it will follow from (11) that $\frac{dy(\cdot)}{dt}$ remains bounded on $[0, \infty)$. Hence $y(\cdot)$ is uniformly continuous on $[-\tau, \infty)$. Now (18) implies that

$$\int_0^\infty |y(s)| ds < \infty \quad (19)$$

hence we have by Barbalatt's lemma (see [8]) that

$$\lim_{t \rightarrow \infty} y(t) = y^* = 0$$

and this completes the proof. \square

Corollary 2.2: Under the assumptions of Proposition (9), all solutions of (9) satisfy

$$\lim_{t \rightarrow \infty} x(t) = 0. \tag{20}$$

Proof: We have from (9) and (10) that

$$\frac{dx(t)}{dt} = -x(t) + a \tanh[y(t)]; \quad t > 0, \tag{21}$$

An integration of (21) leads to

$$x(t) = x(0)e^{-t} + ae^{-t} \int_0^t e^s \tanh[y(s)] ds; \quad t > 0. \tag{22}$$

We let $f(t) \equiv ae^{-t} \int_0^t e^s \tanh[y(s)] ds$ and note that if

$$\limsup_{t \rightarrow \infty} \left| a \int_0^t e^s \tanh[y(s)] ds \right| < \infty$$

then

$$\lim_{t \rightarrow \infty} x(t) = 0 \tag{23}$$

if

$$\limsup_{t \rightarrow \infty} \left| a \int_0^t e^s \tanh[y(s)] ds \right| = \infty$$

then

$$\lim_{t \rightarrow \infty} f(t) = \lim_{t \rightarrow \infty} \frac{ae^t \tanh[y(t)]}{e^t} = \lim_{t \rightarrow \infty} a(\tanh[y(t)]) = 0 \tag{24}$$

since $y(t) \rightarrow 0$ as $t \rightarrow \infty$. Thus (20) follows from (22)–(24). \square

It will be shown in the next section that the set of sufficient conditions established in corollary (10) for the global asymptotic stability of the trivial solution of (1) when $c = 0$ are also necessary. We consider next models of the form (2) with the assumption that the distributed delay kernel $k : [0, \infty) \mapsto [0, \infty)$ is of the type

$$k(s) = \frac{1}{T} e^{-\frac{1}{T}s}, \quad s \in [0, \infty)$$

where T denotes a characteristic constant representing the decay rate of the weight with which the past effects continue to affect the current dynamics; kernels of the above type denote exponentially fading memory. More general kernels of the type

$$k(s) = \left(\frac{1}{T}\right)^{n+1} \frac{s^n}{n!} e^{-\frac{1}{T}s}, \quad s \in [0, \infty) \quad n = 0, 1, 2, 3, \dots$$

will be considered in future investigations. We consider now the dynamics of the following special case of (2):

$$\begin{aligned} \frac{dx(t)}{dt} \\ = -x(t) + a \tanh \left(x(t) - b \int_0^t \frac{1}{T} e^{-\frac{1}{T}(t-s)} x(s) ds \right), \quad t > 0 \end{aligned} \tag{25}$$

in which a, b and T are positive numbers. We are in particular interested in the conditions on the various parameters a, b and T for the neuron to eventually relax in the sense that $x(t) \rightarrow 0$ as $t \rightarrow \infty$. We simplify the analysis of (25) by letting

$$y(t) = \int_0^t \frac{1}{T} e^{-\frac{1}{T}(t-s)} x(s) ds, \quad t > 0. \tag{26}$$

One can derive from (26) that

$$\frac{dy(t)}{dt} = \frac{1}{T}(x(t) - y(t)), \quad t > 0. \tag{27}$$

It is thus sufficient to consider the dynamics of the system

$$\begin{cases} \frac{dx}{dt} = -x + a \tanh(x - by) \\ \frac{dy}{dt} = \frac{1}{T}(x - y). \end{cases} \tag{28}$$

Since the set of solutions of (28) contain the solutions of (25), the stability of the trivial solution of (28) will imply that of (25). Let (x^*, y^*) be an equilibrium of (28), then

$$x^* = y^* \quad \text{and} \quad x^* = a(\tanh[(1-b)x^*]). \tag{29}$$

We assume in the following that:

$$a(1-b) < 1; \quad a > 0, \quad b \geq 0. \tag{30}$$

When (28) holds, one can verify that (0,0) is the unique equilibrium of (28); note that this is a consequence of the properties of the sigmoid function $f(x) = \tanh(x)$. Under the assumption (30), (28) has a single equilibrium at (0,0). It is natural to inquire, under what conditions is the unique equilibrium globally asymptotically stable? This will then mean that no periodic solutions of (28) are possible. The boundedness of all solutions of (28) can be established easily; for instance, we have from (28) that

$$-x - a \leq \frac{dx}{dt} \leq -x + a \quad \text{for } t > 0 \tag{31}$$

from which it follows that there exist positive numbers t^* and ϵ such that

$$-(a + \epsilon) \leq x(t) \leq (a + \epsilon) \quad \text{for } t \geq t^*. \tag{32}$$

For such $t \geq t^*$, we also have

$$\frac{1}{T}[-(a + \epsilon) - y] \leq \frac{dy}{dt} \leq \frac{1}{T}[(a + \epsilon) - y] \tag{33}$$

which also implies that $y(t)$ remains bounded for $t \in [0, \infty)$. Thus, due to the arbitrary nature of $\epsilon > 0$, the region $\{(x, y) \mid |x| \leq a, |y| \leq a\}$ is an attractor for (28). The next result establishes the global asymptotic stability of the trivial solution of (28).

Lemma 2.3: Assume that a, b, T are positive numbers such that

$$\begin{cases} a(1-b) < 1 \\ a < 1 + \frac{1}{T} \quad \text{or} \quad T < \frac{1}{a-1} \quad \text{when } a > 1. \end{cases} \tag{34}$$

Then all solutions of (28) satisfy

$$\lim_{t \rightarrow \infty} x(t) = 0 = \lim_{t \rightarrow \infty} y(t). \tag{35}$$

Proof: We have seen that all solutions of (28) remain bounded and are eventually attracted to the set $[-a, a] \times [-a, a]$. If we can show that there are no periodic solutions of (28), then the ω -limit set of (28) consists of the single equilibrium (0,0); all solutions of (28) approach their respective ω -limit sets and ω -limit sets are themselves invariant with respect to (28); the ω -limit set of every solution of (28) is the singleton (0,0). Thus we are motivated to look for conditions ensuring the nonexistence of periodic solutions of (28). We

rewrite (28) in the form

$$\begin{cases} \frac{dx}{dt} = f_1(x, y) = -x + a \tanh(x - by) \\ \frac{dy}{dt} = f_2(x, y) = \frac{1}{T}(x - y). \end{cases} \quad (36)$$

It is elementary to note from (36) that the divergence of the vector field (f_1, f_2) satisfies the following:

$$\begin{aligned} \operatorname{div}\{f_1, f_2\} &= \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} \\ &= -1 + a \operatorname{sech}^2(x - by) - \frac{1}{T} \\ &= -1 + a\{1 - \tanh^2(x - by)\} - \frac{1}{T} \\ &\leq -\left(1 + \frac{1}{T}\right) + a \\ &< 0 \quad \text{if } a < 1 + \frac{1}{T} \quad \text{or } a < 1. \end{aligned} \quad (37)$$

By the well-known Poincaré–Bendixson negative criterion (see [12]), it follows that (28) has no nontrivial periodic solutions; now, since solutions of (28) remain bounded for $t > 0$, by the discussion above, the conclusion (35) follows and this completes the proof. \square

In our consideration of (3), we again let $c = 0$ and assume that the delay kernel k is continuous on $[0, \infty)$ with the following properties:

$$\begin{aligned} k : [0, \infty) &\mapsto [0, \infty) \\ \int_0^\infty k(s) ds &= 1 \\ \int_0^\infty sk(s) ds &< \infty. \end{aligned} \quad (38)$$

Note that an initial condition for (3) is of the type

$$x(s) = \phi(s), \quad s \in (-\infty, 0]$$

we assume that ϕ is bounded and piecewise continuous on $(-\infty, 0]$.

Proposition 2.4: Let the delay kernel satisfy (38). Suppose further that

$$a > 0, \quad b \geq 0, \quad a(1 - b) < 1, \quad a(1 + b) < 1. \quad (39)$$

Then all solutions of

$$\begin{aligned} \frac{dx(t)}{dt} &= -x(t) \\ &+ a \tanh\left(x(t) - b \int_0^\infty k(s)x(t-s) ds\right), \quad t > 0 \end{aligned} \quad (40)$$

corresponding to bounded initial conditions on $(-\infty, 0]$ satisfy

$$\lim_{t \rightarrow \infty} x(t) = 0. \quad (41)$$

Proof: We consider a Lyapunov functional $V(x)(t)$ defined by

$$V(x)(t) = |x(t)| + ab \int_0^\infty k(s) \left(\int_{t-s}^t |x(u)| du \right) ds. \quad (42)$$

Calculating the upper right derivative $\frac{DV}{Dt}$ along the solutions of (40)

$$\begin{aligned} \frac{DV}{Dt} &\leq -x(t) + a \left| \tanh\left(x(t) - b \int_0^\infty k(s)x(t-s) ds\right) \right| \\ &\quad + ab|x(t)| - ab \int_0^\infty k(s)|x(t-s)| ds \\ &\leq -x(t) + a|x(t)| + ab|x(t)| \\ &\leq -\{1 - a(1 + b)\}|x(t)|. \end{aligned} \quad (43)$$

Hence

$$V(x)(t) + [1 - a(1 + b)] \int_0^t |x(s)| ds \leq V(x)(0), \quad t > 0. \quad (44)$$

The remaining details of proof are similar to those of Proposition 2.1 above and hence are omitted. \square

We conclude this section with a brief discussion of the model systems (1)–(3) when $c \neq 0$. For instance, we let

$$y(t) \equiv x(t) - bx(t - \tau) - c \quad (45)$$

in (1) and obtain that y is governed by

$$\frac{dy(t)}{dt} = -[y(t) + c] + a \tanh[y(t)] - ab \tanh[y(t - \tau)]. \quad (46)$$

It is easy to see that an equilibrium y^* of (46) is a solution of

$$y^* + c = a(1 - b) \tanh(y^*). \quad (47)$$

When $c \neq 0$, it is seen from the assumption $a(1 - b) < 1$ that (47) has a unique solution $y^* \neq 0$. Thus including $c \neq 0$, in (1)–(3) corresponds to a translation of the equilibrium on the real line and of course when $c = 0$, y^* becomes the trivial equilibrium. All our stability and convergence results hold for the system (1)–(3) with $c \neq 0$ except that the respective steady states are now defined by (47).

III. NECESSARY CONDITIONS FOR THE STABILITY OF EQUILIBRIA

We have shown that a set of sufficient conditions for the global asymptotic stability of the unique equilibrium of (1) is that the conditions in (14) hold. It is natural to investigate whether these conditions are necessary for the asymptotic stability of the trivial solution of (1); we show now that these conditions are also necessary for the asymptotic stability of the equilibrium of (9).

The linear variational system corresponding to the trivial solution of (9) is given by

$$\frac{dx(t)}{dt} = -x(t) + ax(t) - abx(t - \tau). \quad (48)$$

The characteristic equation associated with (48) is the transcendental equation

$$\lambda = \lambda(\tau) = -(1 - a) - abc^{-\lambda\tau}, \quad (49)$$

The roots of (49) depend continuously on τ and that for $\tau = 0$, the only root of (49) satisfies

$$\lambda(0) = -(1 - a) - ab = a(1 - b) - 1 < 0. \quad (50)$$

By the continuous dependence of λ on τ , it follows from (50) that for small τ , the roots of (49) have negative real parts, which implies the local asymptotic stability of the equilibrium of (9). If there exists a τ^* such that for $\tau = \tau^*$, (49) has a pair of pure imaginary roots, say $\pm i\omega$, $\omega > 0$ then

$$i\omega = -(1 - a) - abc^{-i\omega\tau} \quad (51)$$

which leads to

$$\begin{cases} 1 - a = -ab \cos \omega\tau \\ \omega = ab \sin \omega\tau. \end{cases} \quad (52)$$

If $\omega \neq 0$, then $(1 - a)^2 + \omega^2 = a^2b^2$ or $|1 - a| \leq ab$; thus if

$$a(1 + b) \geq 1 \quad (53)$$

then there exists a τ^* for which (49) has roots with zero real parts and the trivial solution of (9) becomes unstable. We summarize the above analysis as follows.

Proposition 3.1: The sufficient conditions required in (14) are also necessary for the existence of a unique equilibrium for (9) and its asymptotic stability.

Let us now consider the necessity of the conditions (18) for the asymptotic stability of (2) with $c = 0$. We proceed to find conditions necessary for the local asymptotic stability of (0,0) of (11). We consider the variational system

$$\begin{cases} \frac{du}{dt} = (a - 1)u - abv \\ \frac{dv}{dt} = \frac{1}{T}(u - v). \end{cases} \quad (54)$$

The eigenvalues of the coefficient matrix in (54) are the roots of the equation

$$\lambda^2 - \lambda \left[\frac{1}{T} - (a - 1) \right] + \frac{1}{T} [1 - a(1 - b)] = 0. \quad (55)$$

It is easily seen from (55) that when $a(1 - b) < 1$, the roots of (55) have negative real parts whenever

$$a < 1 \quad \text{or} \quad \frac{1}{T} > a - 1 \quad \text{for} \quad a > 1 \quad (56)$$

also when $\frac{1}{T} = a - 1$ or $T = \frac{1}{a-1}$, the roots of (16) become purely imaginary, in which case the asymptotic stability of (0,0) is lost. We summarize these observations in the following.

Theorem 3.2: Let $a, b, T \in (0, \infty)$; suppose that $a(1 - b) < 1$. Then a set of necessary and sufficient conditions for the global asymptotic stability of (0,0) is that either $a \in (0, 1]$ or $T < \frac{1}{a-1}$.

We consider the necessity of the conditions (39) for the convergence of all solutions of (3) to its equilibrium. The corresponding linear variational system associated with (3) is

$$\frac{du(t)}{dt} = -u(t) + au(t) - ab \int_0^\infty k(s)u(t - s) ds, \quad t > 0. \quad (57)$$

The characteristic equation corresponding to (57) is

$$\lambda = -(1 - a) - ab \int_0^\infty k(s)e^{-\lambda s} ds. \quad (58)$$

We shall show that $a(1 + b) < 1$ implies that all the roots of (58) have negative real parts while $a(1 + b) = 1$ implies the existence of a root of (58) with zero real part. For instance let

$$\lambda = \alpha + i\omega \quad (59)$$

in (58) to obtain

$$\alpha = -(1 - a) - ab \int_0^\infty k(s)e^{-\alpha s} \cos \omega s ds \quad (60)$$

if $Re(\lambda) = \alpha \geq 0$ and $a(1 + b) < 1$, then

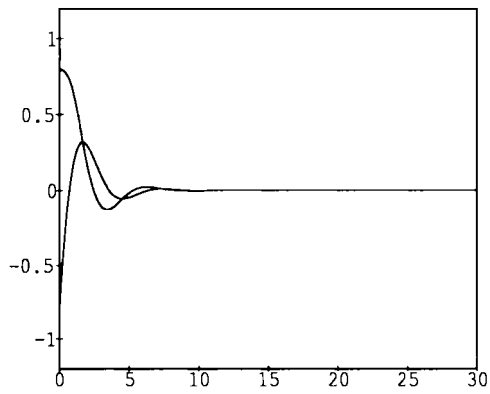
$$0 \leq \alpha \leq -(1 - a) + ab = -[1 - a(1 + b)] < 0 \quad (61)$$

which is impossible. Also we note that when $a(1 + b) = 1$, (58) has a root with zero real part. Hence the condition $a(1 + b) < 1$ is also necessary for the asymptotic stability of the trivial solution of (3) with $c = 0$; the same conclusion holds for (3) when $c \neq 0$ with a different equilibrium.

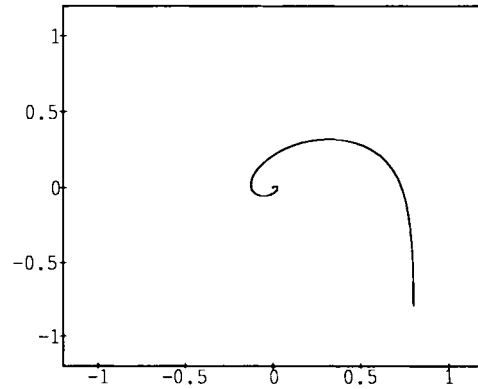
IV. CONCLUDING REMARKS

We conclude with the following interpretation of the results obtained in this article. If the model systems (1)–(3) possess unique equilibria (known as static memories or patterns in neural network literature), then all the solutions of model systems converge to the equilibria (or recall the patterns) whenever the neuron gain is small in comparison with the threshold parameter; this associative recall is independent of the size of the time delays in threshold dynamics. Unlike many results of the neural network dynamics, all our convergence to equilibria are global in the sense that solutions corresponding to initial values far from the equilibria converge to the equilibria of the respective models.

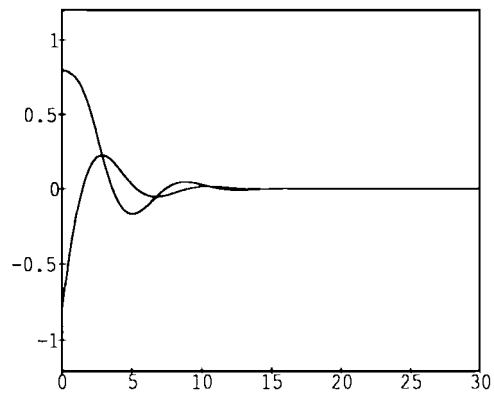
The numerical simulations of the system (34) for different values of the various parameters are graphically (trajectory and phase plots) displayed in Figs. 1–7.



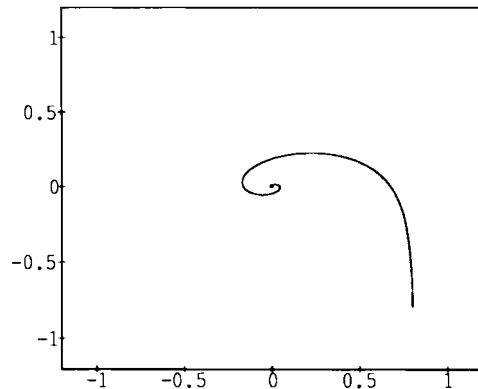
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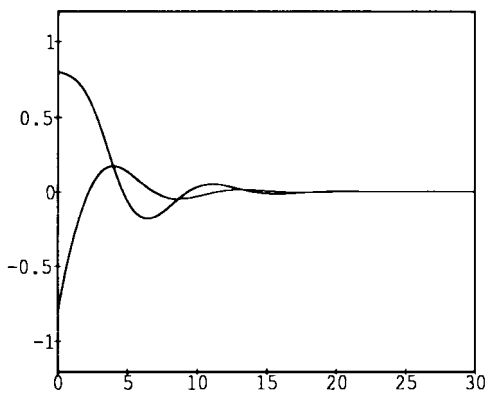
(b)

Fig. 1. (a) and (b). $a = 0.8$, $b = 1.75$, $T = 1$.

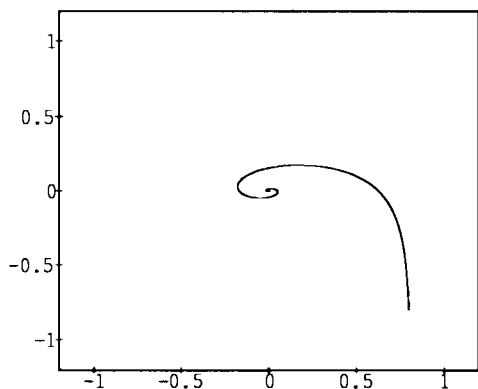
(a)



(b)

Fig. 2. (a) and (b). $a = 0.8$, $b = 1.75$, $T = 2$.

(a)



(b)

Fig. 3. (a) and (b). $a = 0.8$, $b = 1.75$, $T = 3$.

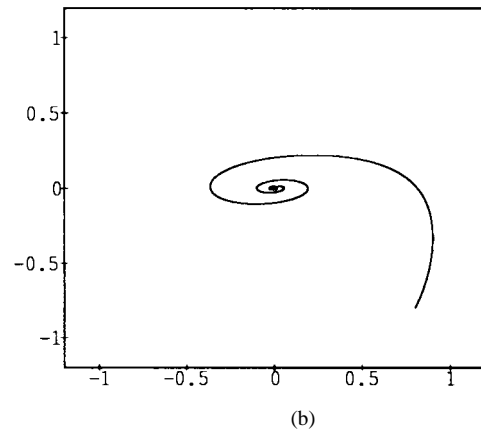
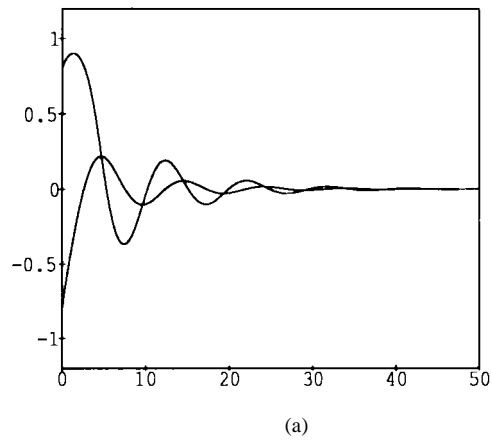


Fig. 4. (a) and (b). $a = 1$, $b = 1.75$, $T = 4$.

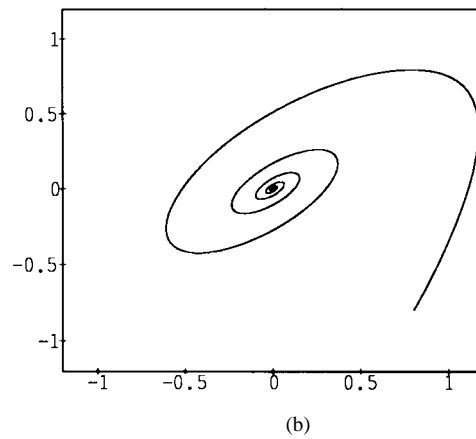
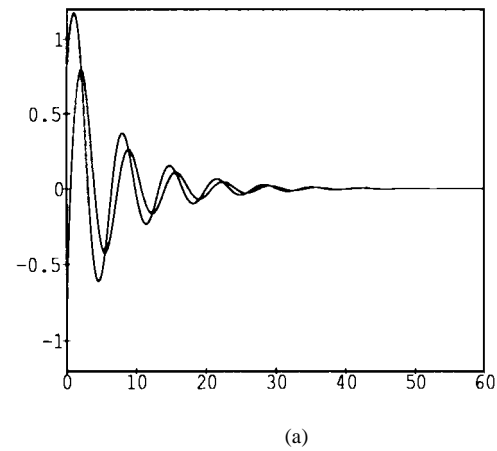


Fig. 5. (a) and (b). $a = 1.75$, $b = 0.93$, $T = 1$.

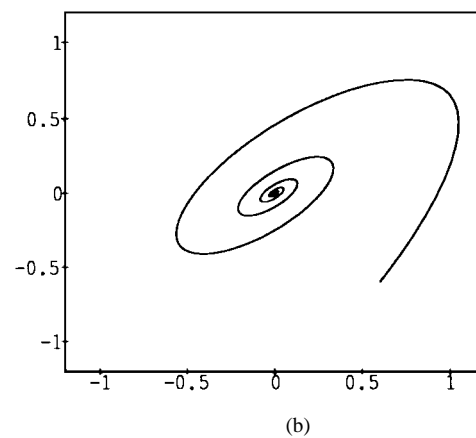
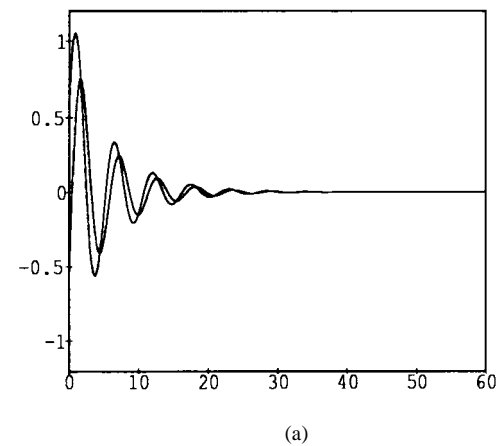
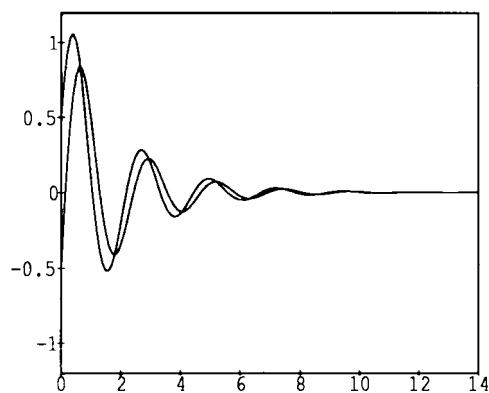
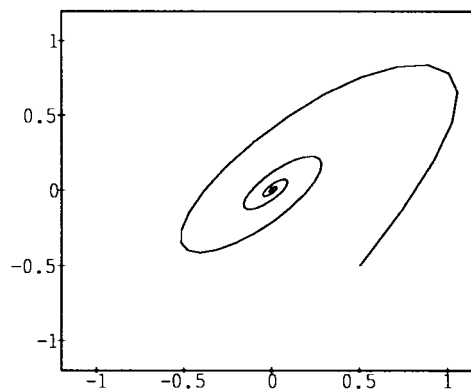


Fig. 6. (a) and (b). $a = 2$, $b = 1$, $T = 0.75$.



(a)



(b)

Fig. 7. (a) and (b). $a = 4$, $b = 1.25$, $T = 0.23$.

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