MAGNETOACOUSTIC OSCILLATIONS OF A PLASMA CONTAINING TWO SPECIES OF IONS

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ABSTRACT

Numerical calculations of linear magnetoacoustic resonant phenomena in a plasma containing two species of ions have been made for a cylindrical plasma with a model which includes the effects of collisional damping and radial non-uniformities in temperature and number density. At sufficiently high temperatures two frequencies are predicted at which magnetoacoustic resonances for the first radial mode will occur. These are expected from considerations of the effects of the ion-ion hybrid resonance.

I. INTRODUCTION

The R. F. excitation of a long cylindrical axially magnetised plasma column containing one species of ion gives rise to the well known phenomena of magnetoacoustic resonance when the excitation frequency, particle density, steady magnetic field and the geometry are chosen appropriately [1, 2, 3, 4, 5]. The resonance occurs when the radial wave number k_r for magnetoacoustic waves propagating at right angles to the magnetic field B_o, bears a simple relationship to the radius of the plasma column. For the case of a uniform density plasma column without collisional damping, this geometrical resonance would occur for the condition

$$k_r a = \alpha_n \qquad 1.1$$

where a = radius of column
and where α_n is the nth root of the Bessel function J_o. For this case k_r is independent of radial position and given by

$$k_r^{\,2} = \frac{\omega^2 \mu_o \rho_o}{B_o^{\,2}} \qquad 1.2$$

where $\rho_o = n_i m_i + n_e m_e$

Thus, for a given plasma mass density, radius, and steady magnetic field, resonances occur at a series of frequencies corresponding to the roots of J_o. Apart from finite temperature effects (e.g., at twice the ion cyclotron frequency Ω_i) equation 1.2 applies to all frequencies $\omega \ll \Omega_e$. Specifically ω can have values below and above Ω_i.

When two ions species are present, and the driving frequency ω is lower than both ion cyclotron frequencies Ω_1, Ω_2, the phases of the ion velocities in the absence of collisional damping are the same with respect to the driving field, and the same is true if $\Omega_2, \Omega_1 < \omega \ll \Omega_e$. But if $\Omega_2 < \omega < \Omega_1$ where the subscript 2 refers to the heavier ion, the phases of the radial velocity components of the two ion species are opposite, and the resultant net mass motion is less than in the case of a plasma of the same mass density consisting of a single ion species and driven at the same frequency. If the frequency, and the number densities of each species are chosen in a particular way the cancellation of the mass motion can be made complete and we have the well known ion-ion hybrid resonance of Bushsbaum [6] a property of the bulk plasma and not of the bounded geometry. For the case of equal number densities of each species of ion, the ion-ion hybrid resonance frequency here designated ω_{iih} is given by

$$\omega^2_{iih} = \Omega_2 \Omega_1 = \frac{e^2}{m_2 m_1 B_o^{\,2}} \qquad 1.3$$

For this collisionless case with ω chosen close to ω_{iih}, the hybrid resonant behaviour ensures that $k_r^2 \to \infty$ (see eqn. 2.17) and there can be no magnetoacoustic resonance.

The above description of this modification in terms of the relative phases of the ion velocity components can be supplemented by a consideration of the radial wave number k_r. In the one ion species plasma, k_r^2 enters the equations as the coefficient of b_z the z-component of the wave magnetic field in a Bessel's equation governing the behaviour of this component. For the plasma with two species of ion an identical Bessel's equation may be written for b_z, but the k_r^2 coefficient of b_z has a different form from that given in the expression 1.2, and exhibits the bulk plasma resonant behaviour expected from the ion-ion hybrid resonance; but yet reduces to the expression 1.2 when the number density of either species of ion is reduced to zero. The nature of k_r^2 is further discussed in section 2.

The presence of collisions between particles alters the behaviour of the change of phase of the ion motions with frequency; and k_r^2 then no longer goes to infinity near the

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ion-ion hybrid resonance frequency. At low temperatures the two ion species plasma will behave for magnetoacoustic resonance in a way very similar to a one ion species plasma of the same mass density. Nevertheless if the temperature of the plasma is kept sufficiently high, so that the percentage ionization is high signficiant modifications of the magnetoacoustic resonance phenomena can be expected in a two ion species plasma excited near the ion-ion hybrid resonance frequency. The real part of the dispersion relation is bent into a resonant form, but now unlike the collisionless case there is an imaginary part as well.

It is relevant now to draw attention to the work of two sets of authors whose investigations of the ion-ion hybrid resonance bear a close relationship to our work.

Baird and Swanson [7] investigate the bulk plasma property of this hybrid resonance by means of geometric effects, namely the waveguide cut-off for the fast hydromagnetic wave. The condition that the cut off frequency in the waveguide occurs at the hybrid frequency imposes a condition on the density for their experiments which they refer to as the critical density. In our work an analogous density condition exists.

Klima et al. [8,9] consider the propagation of magnetoacoustic waves perpendicular to the magnetic field in tokamaks.

None of these references include the influence of the ion-ion collisions which dominate the behaviour close to the hybrid frequency as we show here.

Below we present the results of calculations for the geometrical resonant behaviour of a plasma column with two ion species in a magnetic field excited near this hybrid resonance frequency. The calculations in principle include the effect of collisions between all species of particles, i.e., two kinds of ion, two kinds of neutrals, and electrons and are made over the range of temperatures from low values, where the collisions dominate, to high temperatures where the modifications due to the hybrid resonance become apparent. In practice even the lowest temperatures of interest here are such that the plasma is fully ionized and the inclusion of collisions with neutrals is not necessary. The effect of collisions between ions of the two different species is however very important.

The calculations in particular show the relative phase of the ion velocities, the real part of $k_r a$ and the normalised on axis magnetic field oscillation amplitude $|b_z(r=0)|/|b_z(r=a)|$, all as functions of the excitation frequency.

In section 2 the plasma model and the equation describing the model are treated analytically for the simplest collisionless uniform density case and we show the behaviour of the phases of the velocities and the resonant behaviour k_r^2. In section 3 details of the numerical calculations are given, and the results of these calculations presented. The numerical treatment extends the analytic work by allowing the introduction of collisions, and by permitting density and temperature variations with radius to exist on the plasma column. In section 4 a discussion of the results is followed by some remarks on the application of these calculations to experimental situations.

II. THE PLASMA MODEL

The plasma is described by a multifluid model in which particle collisions due to finite temperature effects are present, but where gradients of pressure are not included. Small amplitude perturbations only are treated and $k_z = 0$ is assumed throughout.

The equations describing the plasma are taken to be the equation of motion for each particle species, s, together with Maxwell's equations, and the definition of current. We have then five equations of the form:

$$n_s m_s \frac{d\underline{v}_s}{dt} = n_s \varepsilon_s e(\underline{E} + \underline{v}_s \times \underline{B}) - \sum_{s' \neq s} \underline{P}_{ss'} \qquad 2.1 - 2.5$$

where

$\epsilon_s = +1$ for each type of ion. (We consider only singly charged ions.)

$\epsilon_s = 0$ for each type of neutral

$\epsilon_s = -1$ for electrons,

where the momentum transfer $\underline{P}_{ss'}$ to the species s from the species s' will be introduced to the calculations through temperature dependent collision frequencies $\nu_{ss'}$ and cross sections $Q_{ss'}$, , and relative velocities; and where the other symbols have their usual meaning.

Maxwell's equations:

$$\nabla \times \underline{E} = - \frac{\partial \underline{B}}{\partial t} \qquad 2.6$$

$$\nabla \times \underline{B} = \mu_0 \underline{J} \qquad 2.7$$

may be written without displacement current since we are concerned with relatively low frequencies of the order of the ion cyclotron frequency.

The definition of current is conveniently written:

$$\underline{J} = \sum_s n_s \varepsilon_s e\underline{v}_s \qquad 2.8$$

When the plasma is excited by r. f. current flowing in a solenoid wrapped around the column, the field quantities are perturbed from their equilibirum values by small first order wave fields. The following conditions are assumed:

$\underline{B} = \underline{B}_0 + \underline{b}$ where $\underline{B}_0$ is the steady axial magnetic field $\underline{B}_0 = B_0 \hat{z}$ and $\underline{b} = b_z(r)\, e^{i\omega t}\, \hat{z}$ is the first order wave field

$\underline{E} = \underline{E}(r)\, e^{i\omega t}$ there are no zero order electric fields

$\underline{J} = \underline{J}(r)\, e^{i\omega t}$ there are no zero order currents or velocities

$\underline{v}_s = \underline{v}_s(r)\, e^{i\omega t}$

The small first order wave fields permit us to linearise the equations of motion which with the $\underline{P}_{ss'}$ expressed as noted above become

$$i\omega\, m_s\, \underline{v}_s = \varepsilon_s\, e[\underline{E} + \underline{v}_s \times \underline{B}_o] - m_s \sum_{s' \neq s} \nu_{ss'}\, (\underline{v}_s - \underline{v}_{s'}) \qquad 2.13$$

In our calculations equations 2.9 - 2.13 are treated as a set of 15 algebraic equations in 15 velocity components and are solved numerically for the velocity components in terms of the $\underline{E}$-field components and the other parameters.

These solutions are inserted in the definition of current 2.8 which is used in 2.7. The differential equations 2.6 and 2.7 then become a set of six differential equations in the six components of the wave fields $\underline{E}$ and $\underline{b}_z$.

The excitation by a simple long solenoid inposes azimuthal symmetry and no changes with axial position. Thus the operators $\partial/\partial\theta$ and $\partial/\partial z$ in the Maxwell's equations produce zeros, and two first order differential equations remain. These are combined to yield one second order differential equation in b_z which is solved numerically by

The calculations described in the next section and their results can be made more transparent by a brief analytic treatment of the uniform collisionless plasma column. For this special case, the 15 algebraic equations relating the velocity components reduce to nine. Those governing the z-components are decoupled from those for r and θ components of the six equations of interest here have colutions

$$v_{sr} = \frac{\dfrac{E_\theta}{B_o} + \dfrac{E_r}{B_o}\dfrac{i\omega}{\varepsilon_s \Omega_s}}{1 - \dfrac{\omega^2}{\Omega_s^{\,2}}} \qquad 2.14$$

and

$$v_{s\theta} = \frac{\dfrac{E_\theta}{B_o}\dfrac{i\omega}{\varepsilon_s \Omega_s} - \dfrac{E_r}{B_o}}{1 - \dfrac{\omega^2}{\Omega_s^{\,2}}} \qquad 2.15$$

where

$$s = 1, 2 \text{ or } e$$

The relative phase between the ion velocity components as ω is varied is obvious from these equations.

These velocity components used in the two non-varnishing curl $\underline{B}$ and component equations give the $\underline{E}$ components in terms of b_z. Then the curl $\underline{E}$ equation yields a Bessel's equation for b_z

$$\frac{d^2 b_z}{dr^2} + \frac{1}{r}\frac{db_z}{dr} + k_r^{\,2}\, b_z = 0 \qquad 2.16$$

where $\pi_s^2 = n_s e^2/\epsilon_o m_s$ = the square of the plasma frequency for the species s. The resonant behaviour of k_r^2 as ω is varied near ω_{iih} is contained in equation 2.17 (see below).

This ion-ion hybrid resonance conditions is dependent on the relative number densities of the two ion species but not on the absolute number densities. It follows then that for ω near ω_{iih} $k_r a$ is very large for any plasma density and equation 1.1 cannot be satisfied and the magnetic acoustic resonance will not occur if $\omega \simeq \omega_{iih}$. Again, in a collisionless plasma column with a radial density gradient where k_r is a function of radial position, as ω approaches ω_{iih} k_r will become very large at <u>all</u> r.

The presence of collisions modifies these remarks but in the case of radial density gradient and collisions the values of the plasma parameters particularly the density, for which magnetoacoustic resonance occurs may be very different in the two ion species plasma compared with the one ion species plasma.

III. NUMERICAL CALCULATIONS AND RESULTS

3a. Uniform density and temperature

In the numerical calculations the spatial components of the equations of motion 2.1 - 2.5 for the particle species are rewritten with all terms in the velocity on the left hand sides and used to construct a matrix equation of the form:

$$A\,v = E \qquad 3.1$$

where A is a 15 by 15 square matrix containing terms only in the species mass and charge, the collision frequencies which contain the effect of temperature, the magnetic field strength and excitation frequency. The vector v consists of

$$k_r^{\,2} = \frac{\left(\displaystyle\sum_s \frac{\pi_s^{\,2}}{\Omega_s^{\,2}}\, \frac{\omega}{\left(1 - \dfrac{\omega^2}{\Omega_s^{\,2}}\right)}\right)^2 - \left(\displaystyle\sum_s \frac{\pi_s^{\,2}}{\varepsilon_s \Omega_s}\, \frac{1}{\left(1 - \dfrac{\omega^2}{\Omega_s^{\,2}}\right)}\right)^2}{c^2 \displaystyle\sum_s \frac{\pi_s^{\,2}}{\Omega_s^{\,2}}\, \frac{1}{\left(1 - \dfrac{\omega^2}{\Omega_s^{\,2}}\right)}} \qquad 2.17$$

the species velocity components. The resultant vector E consists of components of the associated electric field. The coefficient of B_z in the Bessel's equation, the species velocity components and the associated electric fields are numerically computed from the complex matrix A.

We have chosen to make calculations for a hydrogen, deuterium plasma. The choice of this mixture eleminates the need to consider multiple ionization effects. Following Buchsbaum [6] we make calculations for this mixture using a number density ratio of 2:3 for hydrogen and deuterium respectively, i.e., close to that which optimises the magnitude of the ion-ion hybrid resonance. We have not chosen the precise optimum value because in setting up experiments this is unlikely to be achieved exactly or maintained throughout a discharge; it is desirable to show that the predicted effects are not critically dependent on maintaining optimal conditions. For the calculations the total number density is chosen such that the magneto-acoustic resonance for an equivalent single species plasma of the same mass density and the ion-ion hybrid resonance would occur within the frequency range over which the calculations are made. The actual value chosen corresponds to a total filling pressure of 2.5 m Torr. The steady field B_0 = 0.78 Tesla and plasma radius = 0.052 m used in the calculations are typical values for small laboratory experiments, and are actual values used in some earlier magneto-acoustic resonance experiments in this laboratory [11]. For this choice of parameters $\omega_{iih}/2\pi$ = 8.978 MH_z in all calculations presented here.

The choice of the total number density such that these two resonances occur at the same frequency defines a critical density (Baird III and Swanson [6]). Corresponding to this critical density is the minimum plasma temperature at which the ion-ion hybrid resonance becomes observable. If the number density is varied from this critical density higher plasma temperatures are required before the effects of the ion-ion hybrid resonance may be detected. The values of atomic ion density, neutral density, and plasma temperature are given in the figure captions.

The real part of $k_r a$ plotted as a function of excitation frequency for various temperatures is shown in figure 1 for the chosen set of plasma parameters given above.

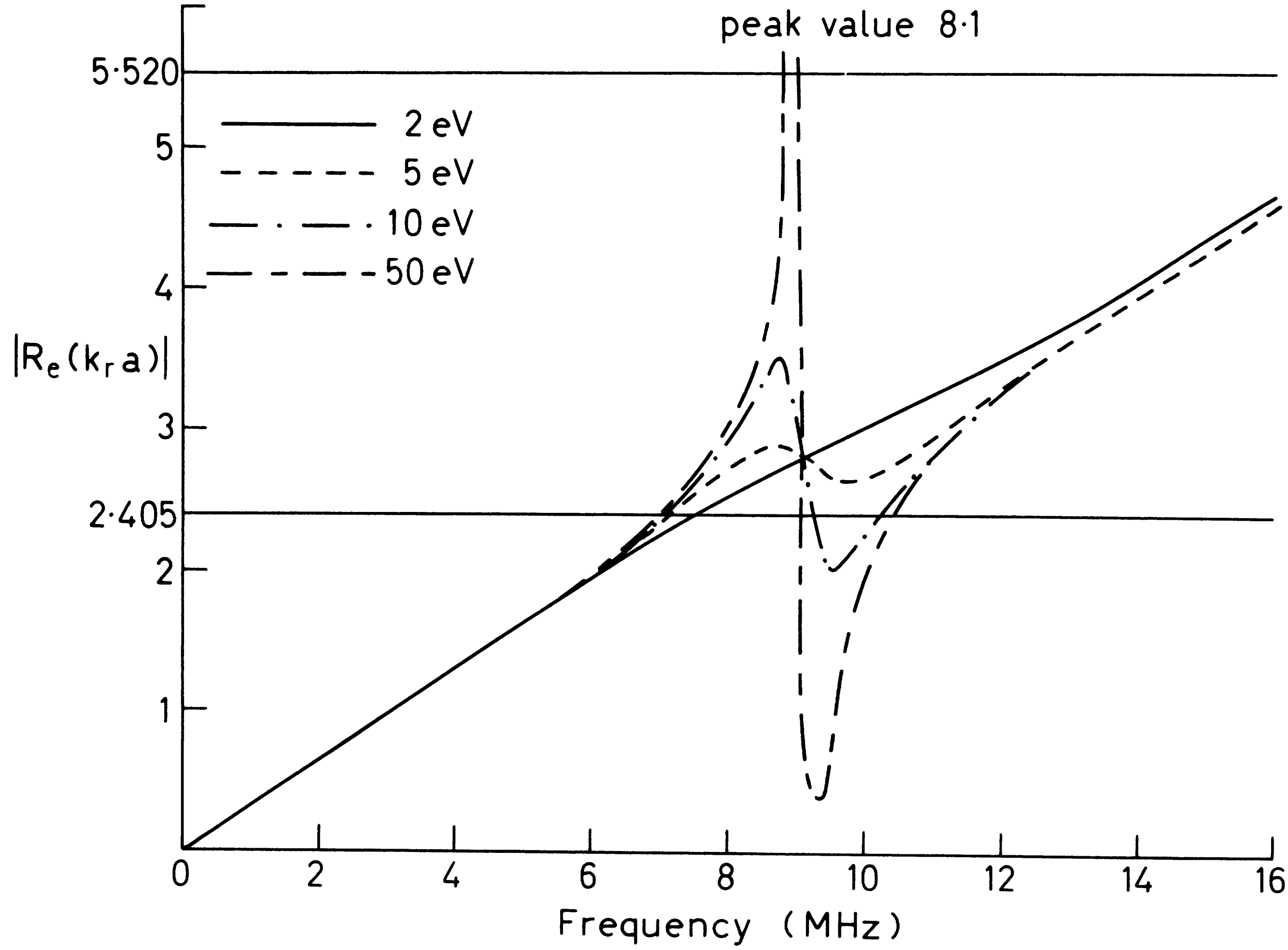

Figure 1. Variations of $Re(k_r a)$ with frequency in a uniform fully ionised hydrogen-deuterium plasma with number densities in the ratio 2:3 and total filling pressure of neutral gas 2.5 millitorr, at plasma temperatures of 2 eV, 5 eV, 10 eV and 50 eV. $\omega_{iih}/2\pi$ = 8.978 MHz.

If collisional damping effects were low one might expect three frequencies at which a magnetoacoustic resonance would occur satisfying the condition $Re(k_r a) = \alpha_1 = 2.405$. But in fact the imaginary part of $k_r a$ is a strong function of the forcing frequency ω, and no resonance occurs at the central frequency. The $Im(k_r a)$ has a maximum value at ω_{iih} the ion-ion hybrid frequency, and the proper way to consider the existence of a magnetoacoustic resonance is to find the condition under which the ratio $|b_z(o)/b_z(a)|$ is a maximum. This maximisation procedure takes into account both $Re(k_r a)$ and $Im(k_r a)$ and a plot of $|b_z(o)/b_z(a)|$ against frequency shows only two frequencies for which any particular order radial mode resonances occur. For the first radial mode these are shown in figure 2.

At higher temperatures (e.g., 10 eV and above) the curve for $|b_z(o)/b_z(a)|$ begins to show evidence of structure between the two resonance peaks belonging to the first radial mode, which develops to a third sharply defined resonance peak by the time the temperature has reached 50 eV. This feature is the resonance for the 2nd order radial mode. This may be understood by reference to figure 3 which shows both the real and imaginary parts of $k_r a$ and $|b_z(o)/b_z(a)|$ on the same graph. It is noted that $Re(k_r a)$ intersects the line at 3.405 representing the first root of the Bessel function J_0 at three frequencies. For the lowest and highest of these, $Im(k_r a)$ is very small and we expect the resonant behaviour. The intersection at about 9 MHz is heavily damped and does not correspond to a resonance. Although the intersection of $Re(k_r a)$ with the line 5.520 representing the second root of the Bessel function occurs at a frequency of 8.8 MHz, near to the hybrid resonance frequency, $Im(k_r a)$ is still small compared

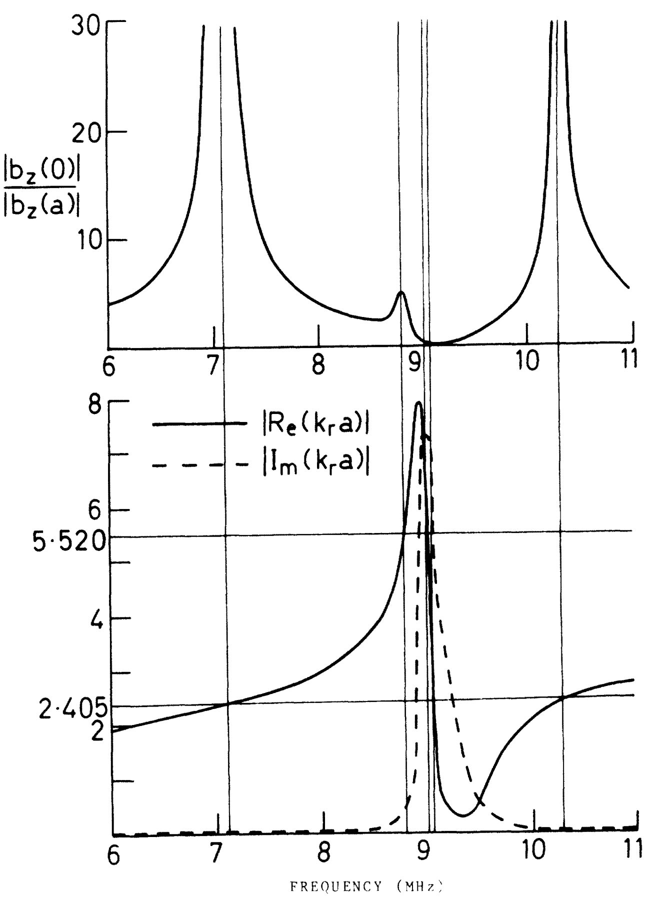

Figure 3. Variation of $Re(k_r a)$, $Im(k_r a)$ and $|b_z(o)|/|b_z(a)|$ with frequency for the plasma as specified in the caption of figure 1 at a temperature of 50 eV.

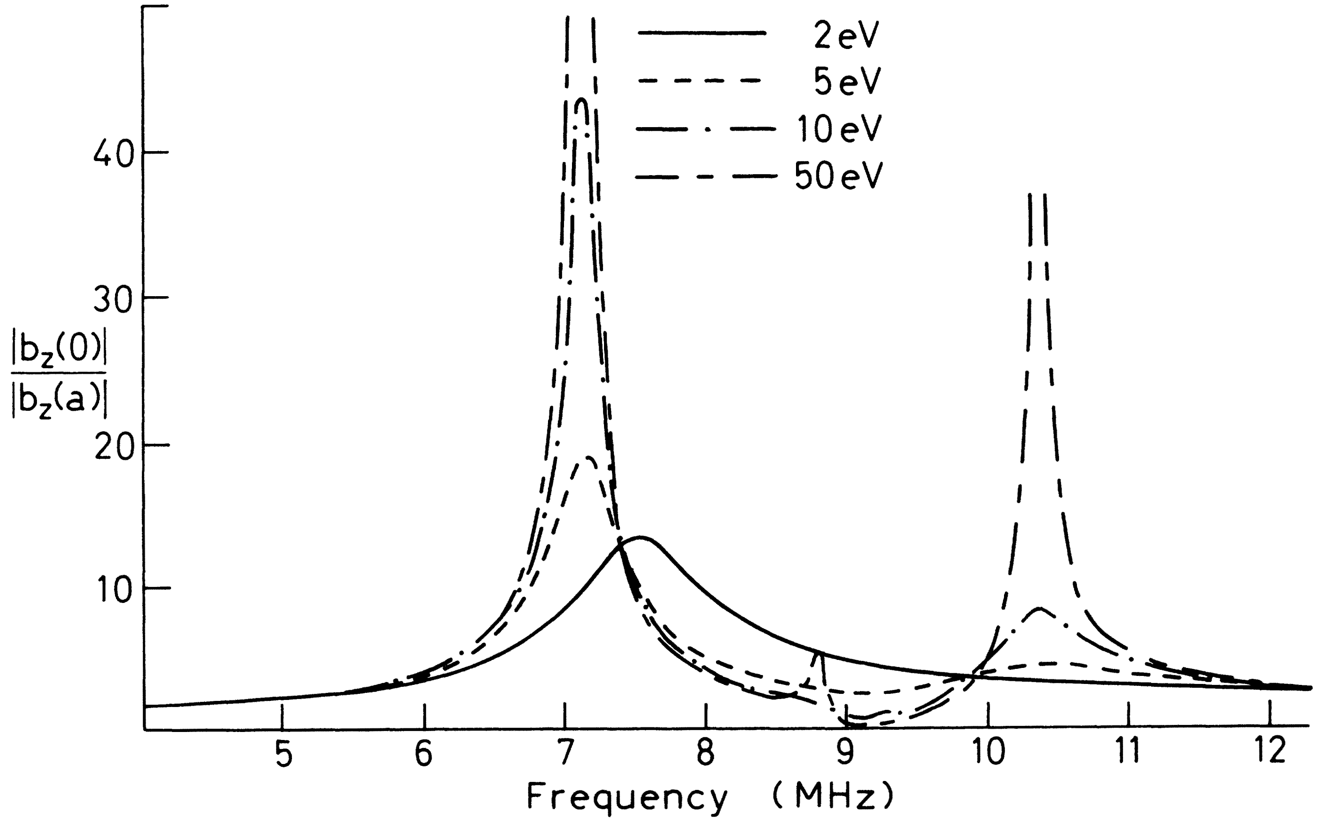

Figure 2. Variation of the wave magnetic field $|b_z(o)|/|b_z(a)|$ with frequency for plasma conditions as specified on the caption of figure 1.

to $Re(k_r a)$ and we get the resonance for this second radial mode. The second radial mode nature of this resonance at 8.8 MHz is confirmed by a calculated $|b_z(r)/b_z(a)|$ profile.

The change of the bulk plasma properties at the ion-ion hybrid resonance are most readily understood by computing the relative phase difference ϕ between the radial ion velocity components. For a high plasma temperature it is observed that these velocity components are approximately 180 degrees out of phase as shown in figure 4. The result is in agreement with S. J. Buchsbaum [6] and V. L. Yakimenko [10] who reported for a cold, uniform collisionless plasma the two ion clouds oscillate transversely to the static magnetic field and 180 degrees out of phase with each other.

The effect on the magnetoacoustic resonance of changing the density near the critical density is shown in figure 5. It is noted that for each of the conditions chosen, two magnetoacoustic resonant frequencies occur, but that the best situation to observe the resonances occurs when the choice of filling density is made nearest the ciritical density.

3b. Calculations where the density and temperature vary with radius

Now the elements of the matrix A in equation 3.1 become radially dependent functions. We have examined separately the effect of the temperature gradient and the ion density gradient, then made calculations for a realistic plasma with both temperature and density gradients. The results of these calculations are presented in graphical form in figure 6 and 7 only for these last realistic cases.

In considering the effect of temperature gradient we did not attempt to compare plasma columns with the same average temperature, but compared plasma columns with the same on-axis temperature with quadratic radial profiles $T(r) = T_0(1 - \epsilon r^2)$. By introducing this form of temperature gradient we have lowered the radially averaged plasma temperature. The effect of this lowered average temperature was to decrease the amplitude of the magnetic field associated with the magnetoacoustic resonance. This is partly due to increased effective damping from ion electron collisions and partly due to the shifting of the geometric resonances closer to the ion-ion hybrid resonance

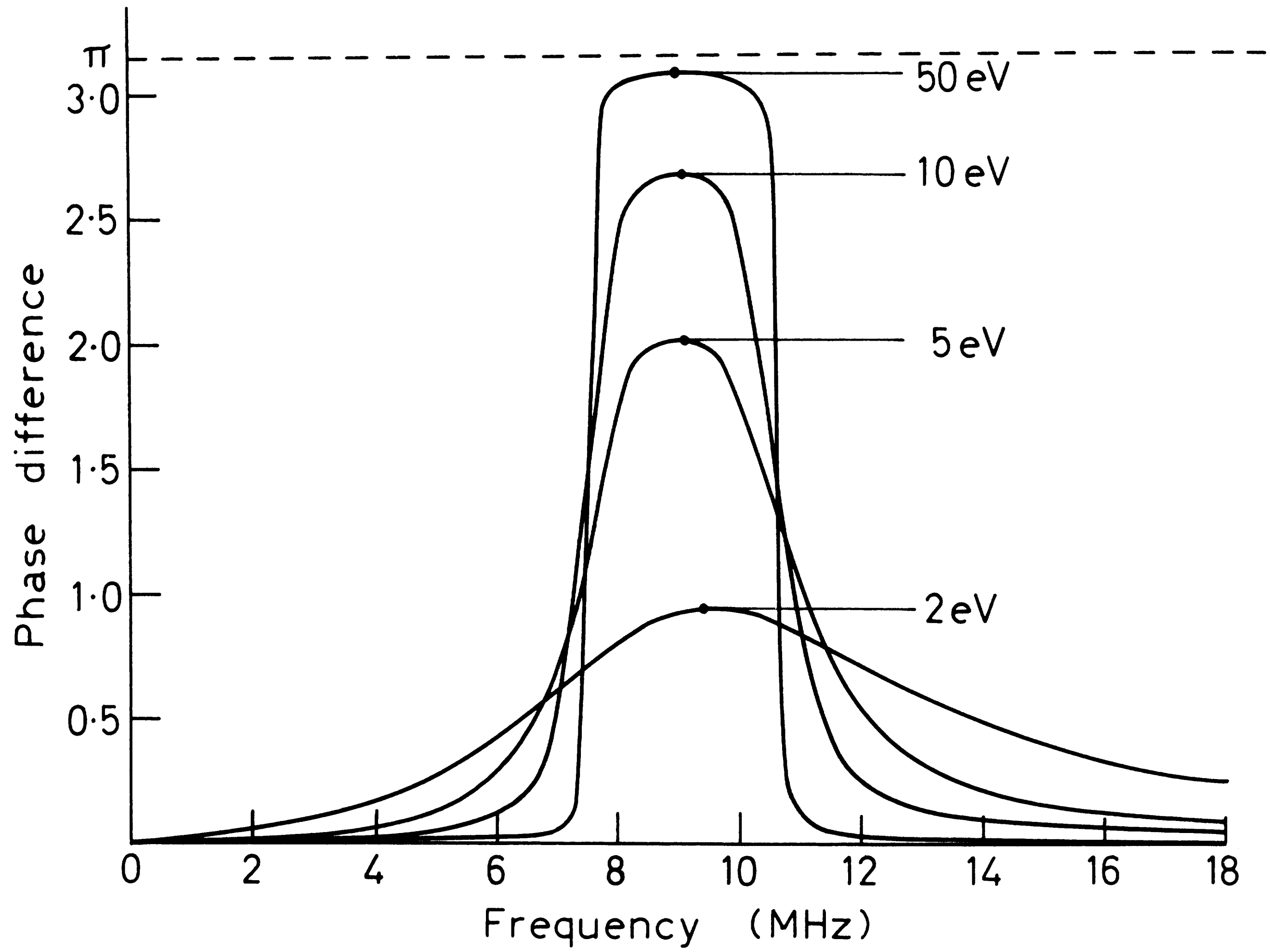

Figure 4. Plot of the variation of phase difference between the radial component of velocity of the hydrogen ions and that of the deuterium ions for the plasma conditions specified in the caption of figure 1.

where stronger damping is experienced due to ion-ion collisions. The results of these calculations showed a tendency to suppress the resonances.

The inhomogeneity in the ion number density was introduced with a quadratic profile, but here we changed the on-axis density for such cases in order that total mass of the plasma column was the same as in the uniform density case. The results of the calculation showed that as the inhomogeneity increased the resonance curves were displaced in the same way as they would have been for a slightly decreased number density in a uniform plasma column.

We note that Vaclavik [4] makes calculations of magnetoacoustic resonance phenomena for a single ion species plasma column with a radial density gradient. In his work the total mass of the column has not been kept constant as we have done here and the large shifts in frequency and magnitude of the resonant wave fields reports in that paper are largely due to these mass changes.

The combined effect of temperature gradients and number density gradients have been calculated for plasma columns of the critical density (figure 6) and for one higher density (figure 7). These results indicate that temperature gradients are very important suggesting that it would be necessary to work at densities near the critical density if the effects predicted here are to be observed.

IV. DISCUSSION

We summarise the results of the calculations by observing that, provided the temperature is sufficiently high, the excitation of a plasma column of defined properties containing two species of ion will exhibit two frequencies at which a magnetoacoustic resonance with $|b_z(r)|/|b_z(a)|$ radial profiles characteristic of the first radial mode will occur rather than one which is the case for an one ion species plasma. For a plasma column with realistic density and temperature gradients this feature persists, though the sharpness of the resonances is reduced.

These calculations have been made with a particular experimental apparatus in mind. That is, the plasma source FPS-2 in which plasma is prepared by passing an axial discharge current through an approximately 10 cm diameter gas column. This source has been used for many magnetoacoustic experiments in argon and hydrogen [3,11]. This method of plasma preparation fails to produce sufficiently high temperatures in the appropriate density hydrogen-deuterium plasma for us to expect to see the effects predicted in the work presented in this paper.

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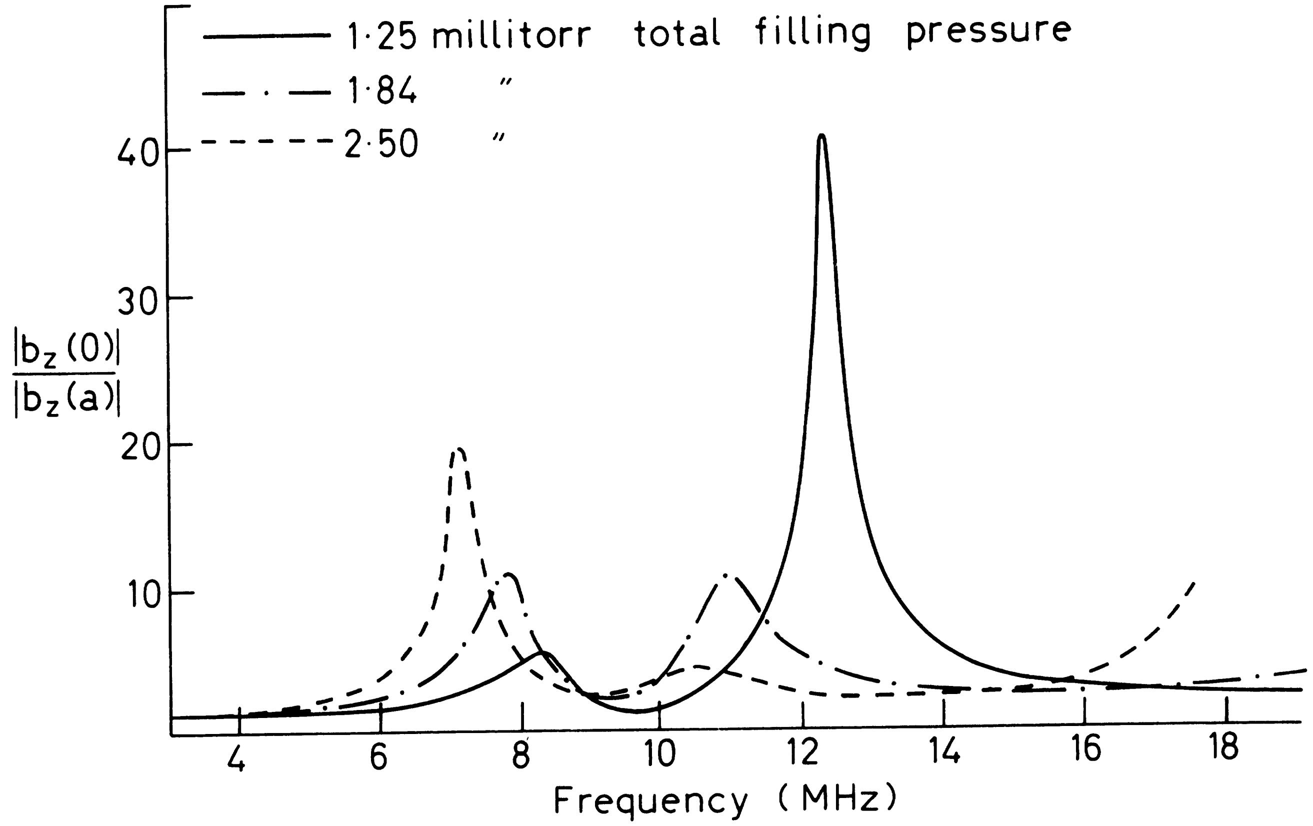

Figure 5. Variation of the wave magnetic field $|b_z(o)|/|b_z(a)|$ with frequency in a fully ionized hydrogen-deuterium plasma with number densities in the ratio of 2:3 and total filling pressure of neutral gas having the values of 1.25 millitorr, 1.84 millitorr and 2.50 millitorr; all with plasma temperature of 5 eV.

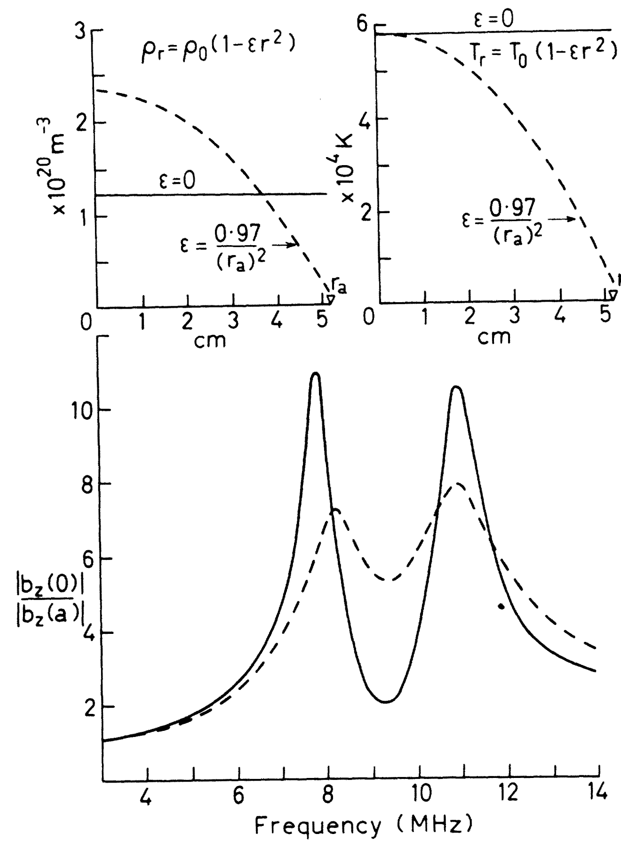

Figure 6. Variation of the wave magnetic field $|b_z(o)|/|b_z(a)|$ with frequency in a hydrogen-deuterium plasma with number densities in the ratio 2:3 and total filling pressure of neutral gas 1.84 millitorr, corresponding to the critical density, and with number density and temperature radial profiles as shown in the curve with the borken line and compared with variations of the wave fields in a uniform plasma at a temperature of 5 eV shown in the solid curve.

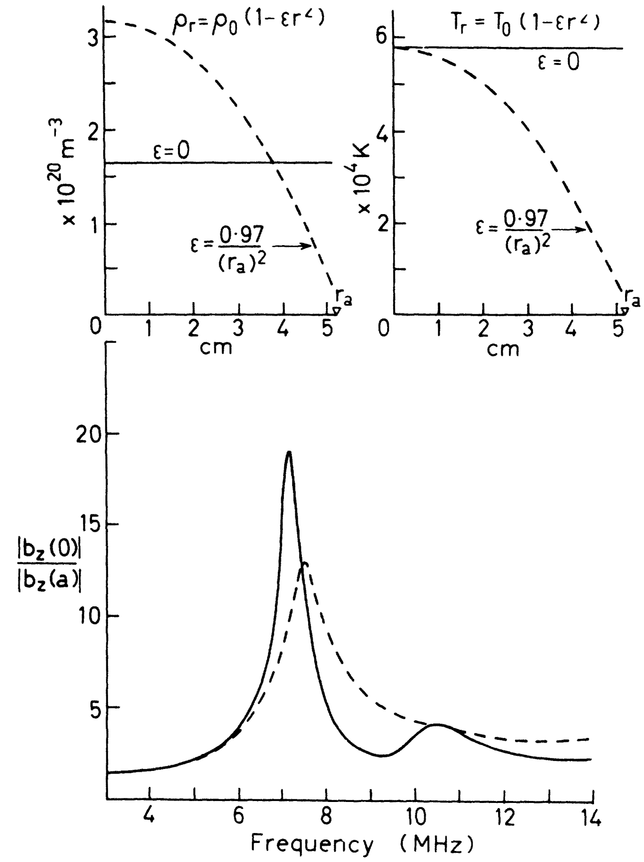

Figure 7. The variation of wave magnetic fields with frequency, for the conditions specified in the caption of figure 6 except the total filling pressure is now 2.50 millitorr.

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