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A NEW TECHNIQUE FOR MULTIDIMENSIONAL SIGNAL COMPRESSION

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1. ABSTRACT

The problem of efficiently compressing a large number, L, of $\mathcal{N}$ dimensional signal vectors is considered. The approach suggested here achieves efficiencies over current preprocessing and Karhunen-Loéve techniques when both L and $\mathcal{N}$ are large.

Preprocessing and partitioning techniques are first applied to the $L \times \mathcal{N}$ data matrix $\mathcal{F}$ to reduce the database to a manageable number of subblocks of lower dimension. Within each subblock an iterative chain approximation is proposed that effects a transform at each stage of the iterative scheme. A particularly appealing transform, using prolate spheroidal sequences, is suggested.

To evaluate a reduced dimensionality approximation for the expansion coefficients, the approach used in the orthogonal Procrustes problem solution is combined with an iterative interlacing technique due to Daugavet for factorizing matrices.

2. INTRODUCTION

Efficient compression of a reasonably large number, L, of signals, each represented by an $\mathcal{N}$-vector is an important problem in the area of multidimensional signal processing. In the image processing case, each component of such a signal vector represents an image pixel. Typical applications include handwritten characters and face and object recognition. A novel application is the storage of ionogram back-scatter data from over the horizon radar.

We will deal with two dimensional signals, but the approach we present easily extends to the transmission and storage of signals with abitrary dimension.

Let $\mathcal{F}$, the data array formed by the vectors above, be the $L \times \mathcal{N}$ matrix to be transmitted or stored.

The choice of compression algorithm is determined, essentially, by the following competing criteria: the compression ratio or the transmission cost, the quality of the data reconstruction (QDR), and the computational complexity. The methods commonly used in practice for data (image) compression or coding are based on computing a truncated Singular Value Decomposition (SVD) of the matrix $\mathcal{F}$ to approximately represent the data array $\mathcal{F}$ as the product of two matrices, say, Γ and Υ, $\mathcal{F} \approx \Gamma \times \Upsilon$, with smaller sizes $L \times M$ and $M \times \mathcal{N}$ respectively, $M < \min(L, \mathcal{N})$.

In the case of a large dimensionality of the matrix $\mathcal{F}$ (typically $L, N = O(10^4) \rightarrow O(10^6)$) the processing required to compute a truncated SVD can be prohibitive.

Another difficulty is the data transmission cost. Transmission of two matrices Γ and Υ is cheaper than the initial data array $\mathcal{F}$ transmission. However, for the case of the large L and $\mathcal{N}$ discussed above, the cost of transmitting matrices Γ, Υ can be much greater than the cost of their evaluation, and may become prohibitively high. Thus, it is natural to strive to determine a technique which allows a further reduction in size of the transmitted or stored matrices while maintaining a reasonable QDR.

The motivation of this paper comes from a need to reduce the overall cost of data array transmission.

The method proposed allows a reduction in the total cost of the process of data compression, transmission/storage and reconstruction whilst balancing the criteria discussed earlier.

To achieve this object we propose:

- a new generic scheme of data array compression based on a combination of several preliminary stages and a final stage;
- new discrete transforms for the preliminary stages;
- a new compression algorithm for the final stage.

3. NEW DATA COMPRESSION METHOD

Our approach to the problem solution is illustrated by the block-scheme shown in figure 1.

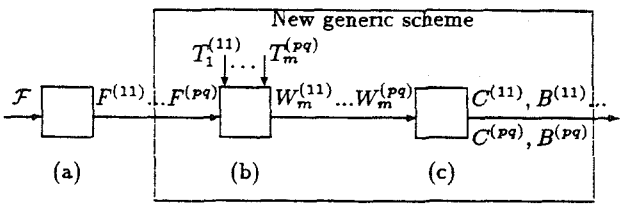

Figure 1. Block scheme of the signal compression method.

The blocks (a), (b) and (c) in this figure mean the following.

(a): *Preprocessing.* We first assume that the data base $\mathcal{F}$ of L signals (images):

- has been pre-processed to remove redundancies and similarities in the $\mathcal{F}$ by methods, for example, [6], and after that
- has been partitioned into subblocks $F^{(jk)}, j = 1 : p, k = 1 : q$, according to schemes such as those considered in [7].

After the above initial preprocessing stages the size of the subblock $F^{(jk)} \subseteq \mathcal{F}$ is $l \times n$ and their number is equal to $p \times q$.

(b): *Chain transform.* The next step processes each of the $p \times q$ subblocks $F^{(jk)}$. A new generic scheme for carrying out this processing is proposed.

The key idea of this generic scheme is a sequential reduction of dimensionalities of matrices in a matrix product which approximates each subblock $F^{(jk)}$; these matrices are determined by a discrete transform which is applied at each

step of the reduction scheme. This part can be mathematically represented as a solution of the problem

$$\|F - W_m T_m^T T_{m-1}^T \ldots T_1^T\| \rightarrow \min_{W_m} \quad (1)$$

with $T_m, T_{m-1}, \ldots, T_1$ given where $F = F^{(jk)}, W_m = W_m^{(jk)} \in \mathcal{R}^{l \times q_m}$ and $T_i = T_i^{(jk)} \in \mathcal{R}^{q_{i-1} \times q_i}$, $q_0 = n, i = 1 : m, q_m < \ldots < q_1 < \min(l, n)$.

By using the equality

$$F - W_m T_m^T \ldots T_1^T = (F - W_1 T_1^T) + (W_1 - W_2 T_2^T) T_1^T$$

$$+ \ldots + (W_{m-1} - W_m T_m^T) T_{m-1}^T \ldots T_1^T \quad (2)$$

problem (1) reduces to

$$\|W_{k-1} - W_k T_k^T\| \rightarrow \min_{W_k} \quad (3)$$

for $k = 1, 2, \ldots, m$ and $W_0 = F$.We call this the *iterative scheme of chain approximation.*

Under certain orthonormality conditions, the solution obtained is $W_k = W_{k-1} T_k$ so that $W_m = F T_1 T_2 \ldots T_m$. Diagrammatically this block (a) of the scheme above is represented by a chain as follows

$$F \xrightarrow{T_1^T} W_1 \xrightarrow{T_2^T} W_2 \xrightarrow{T_3^T} \ldots \xrightarrow{T_{m-1}^T} W_{m-1} \xrightarrow{T_m^T} W_m$$

The discrete transforms used in this process could be constructed using prolate spheroidal sequences [1] or Chebyshev polynomials.

We briefly discuss the reason why discrete prolate spheroidal sequences are of interest here. They are defined as follows. Fix an integer $N > 0$ and some positive real number $\gamma \in (0, \frac{1}{2})$. We let $T = T_{\gamma,N}$ be the linear operator on l^2 given by

$$T_{\gamma,N}(\mathbf{v})(k) = \sum_{m=0, m \neq k}^{N-1} \frac{\sin 2\pi\gamma(k-m)}{\pi(k-m)} \mathbf{v}(m) + 2\gamma \mathbf{v}(k), k \in \mathbf{Z},$$

where $\mathbf{v}(k)$ denotes the kth coordinate of $\mathbf{v} \in l^2$. This linear operator is symmetric with finite rank and so [1] it has N real eigenvalues $\lambda_0^{N,\gamma}, \lambda_1^{N,\gamma}, \ldots, \lambda_{N-1}^{N,\gamma} > 0$. Let $\mathbf{v}_k^{N,\gamma}$ $(k = 0, 1, \ldots, N-1)$ be the corresponding eigenvectors normalised so that $\sum_{m=0}^{N-1} \mathbf{v}_k^{N,\gamma}(m)^2 = 1, \sum_{m=0}^{N-1} \mathbf{v}_k^{N,\gamma}(m) \geq 0, \sum_{m=0}^{N-1} (N-1-2m) \mathbf{v}_k^{N,\gamma}(m) \geq 0$. These eigenvectors are called the *discrete prolate spheroidal sequences.* They have many interesting properties (*loc. cit.*), some of which will be of significance to us. Their restrictions to the coordinates $(0, 1, 2, \ldots, N-1)$ are the eigenvectors of the matrix whose (m, n)th component is

$$\frac{\sin 2\pi\gamma(n-m)}{\pi(n-m)} \quad (n, m = 0, 1, 2, \ldots, N-1)$$

An important property, the orthonormality of this index-limited collection of vectors, stems from the symmetry of the matrix. The unlimited vectors are also orthogonal. Apart from these properties the sequences satisfy various symmetry laws and a difference equation. Of more interest to us is that they (the unlimited sequences) are the most concentrated orthogonal sequences $\mathbf{v} \in l^2$ whose Fourier series $V(f) = \sum_n \mathbf{v}(n) e^{2\pi n f}$ vanishes outside the interval $(-\gamma, \gamma)$ (modulo 1). That is, given the normalization above, they have the smallest l^2 norm. The limited sequences are, on the other hand, those orthogonal sequences of length N whose spectrum is most concentrated in the interval $(-\gamma, \gamma)$ in $L^2(-\frac{1}{2}, \frac{1}{2})$.

The most interesting fact for our purposes concerns the set G_ϵ of sequences $\mathbf{v}$ in l^2 which have $\sum_{n \notin I} |\mathbf{v}(n)|^2 \leq \epsilon$ and

$$\int_{1/2 \geq |f| > \gamma} |V(f)|^2 df \leq \epsilon$$

where $I = \{0, 1, 2, \ldots, N-1\}$. We look for the smallest M such that there is some M dimensional subspace D of l^2 with the property that every member of G_ϵ can be approximated on the coordinate set I to within η by a member $\mathbf{d}$ of D, that is, $\sum_{n \in I} (\mathbf{g}(n) - \mathbf{d}(n))^2 < \eta$. This M depends on N, γ, ϵ and η. For $\eta > \epsilon > 0$ the limit of M/N as $N \rightarrow \infty$ is 2γ and the discrete spheroidal sequences $\mathbf{v}_k^{N,\gamma}$ $(k = 0, 1, 2, \ldots, [2N\gamma(1-\eta)])$ can be used as an orthogonal basis for an optimal space D in this sense. Thus for large N, if the sequences we are interested in are approximately of bandwidth γ and are very small outside the index set I, as described above, then these sequences can be essentially described in terms of the first $[2\gamma N]$ coefficients of their expansions in terms of the discrete prolate spheroidal sequences. In our context if the rows of the matrix which we are trying to approximate have this property it makes sense to use as the basis matrix one whose rows are the discrete prolate spheroidal functions. In effect the property is that there is relatively little high frequency (that is, close to $\pm\frac{1}{2}$) in the sequences.

In the simplest case of using just one "link" in the transform chain above, i.e. $m = 1$ in (1), this approach was initially proposed in [2] and a particular case was considered in [3].

(c): *Interlacing iteration algorithm.* For the final step of this generic scheme, *a new matrix compression algorithm* which combines the approach used in the solution of the orthogonal Procrustes problem [4] and the idea of interlacing iterations [5] is proposed and justified. This algorithm, which we term the Interlacing Iteration Algorithm, IIA, reduces an evaluation of SVD for the matrix of interest to eigen-decomposition evaluations for several special matrices with much lower dimensionalities as follows.

For any matrices $C = C_m^{(jk)} \in \mathcal{R}^{l \times q_{m+1}}$, and $B = B_m^{(jk)} \in \mathcal{R}^{q_m \times q_{m+1}}$ where $q_{m+1} < q_m$, we have

$$F - C_m B_m^T T_m^T \ldots T_1^T = (F - W_1 T_1^T) + (W_1 - W_2 T_2^T) T_1^T + \ldots$$

$$+ (W_{m-1} - W_m T_m^T) T_{m-1}^T \ldots T_1^T + (W_m - C_m B_m^T) T_m^T \ldots T_1^T$$

Then, taking into account (2), (3), the problem

$$\|F - C_m B_m^T T_m^T \ldots T_1^T\| \rightarrow \min_{C,B}$$

subject to

$$B^T B = I$$

is reduced to the problem sequence (3) considered above and the Karhunen-Loéve problem

$$\|W - C B^T\| \rightarrow \min_{C,B} \quad (4)$$

subject to

$$B^T B = I \quad (5)$$

where $W = W_m \in \mathcal{R}^{l \times q_m}$. For its solution, we first consider the related problem for a symmetric matrix $S = W^T W \in \mathcal{R}^{q_m \times q_m}$: find $X \in \mathcal{R}^{q_m \times s}$ and $Y \in \mathcal{R}^{q_m \times s}$ with $s << q_m$ and $r(X) = r(Y) << r(W)$ satisfying the problem

$$\|S - XY^T\| - \min_{X,Y} \tag{6}$$

subject to

$$Y^T Y = I \tag{7}$$

Solution of this problem is given by the following.

Lemma 1. *Let $A = X^T S$ and let the eigenvalue decomposition of the symmetric matrix $\mathcal{A} = AA^T \in \mathcal{R}^{s \times s}$ be*

$$U \Theta U^T = \mathcal{A}, \quad U^T U = UU^T = I.$$

Then X and Y satisfying the equations

$$X = SY$$

$$Y = SX(U\Theta^{-1/2}U^T)$$

solve (6) and (7).

Proof is based on using the Procrustes problem solution [4].

To evaluate X and Y which satisfy these equations we construct an iterative procedure as follows:

Given X_0 and S; for $i = 0, 1, 2, \ldots$ put

$$A_i = X_i^T S \tag{8}$$

$$\mathcal{A}_i = A_i A_i^T \tag{9}$$

$$U_i \Theta_i U_i{}^T = \mathcal{A}_i \tag{10}$$

$$Y_{i+1} = SX_i(U_i \Theta_i^{-1/2} U_i{}^T) \tag{11}$$

$$X_{i+1} = SY_{i+1} \tag{12}$$

We point out that the fact the eigen-decomposition of the small $s \times s$, symmetric matrix $\mathcal{A}_i$ is easy to compute is central to the appeal of IIA.

Choice of the initial iteration X_0 requires special attention. Suppose $rank S = r$ so that $s \leq r \leq q_m$. Put $X_0 = S_s \in \mathcal{R}^{q_m \times s}$ where S_s consists of s linearly independent columns of S. Let the SVD of S_s be

$$S_s = \Phi \Sigma \Psi^T$$

with $\Phi \in \mathcal{R}^{q_m \times r}, \Sigma \in \mathcal{R}^{r \times r}, \Psi \in \mathcal{R}^{s \times r}$. Then

$$X_0 = \Phi P \tag{13}$$

where $P \in \mathcal{R}^{r \times s}$. If the first s rows of the matrix P in (13) are linearly independent then the corresponding matrix X_0 is denoted X_0^*.

Lemma 2. *Let the pair X, Y be a solution of the problem (6),(7). Let $l > q_m$ and let $S = W^T W$ have $rank S = r$. (If $q_m > l$ we define $S = WW^T$). Denote by $\lambda_1, \lambda_2, \ldots, \lambda_r$ the eigenvalues of S which are such that*

$$|\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_r| > 0$$

Suppose that $s < r, X_0 = X_0^*$ and that $\lambda_{s+1} \neq \lambda_s$. Then, the following limits exist

$$X = \lim_{i \to \infty} X_{i+1}, \quad Y = \lim_{i \to \infty} Y_{i+1}$$

and we have

$$\|X - X_{i+1}\| < c_1(\lambda_{s+1}/\lambda_s)^{2(i+1)}$$

$$\|Y - Y_{i+1}\| < c_2(\lambda_{s+1}/\lambda_s)^{2(i+1)}$$

with constants c_1, c_2 which are independent of λ_s, λ_{s+1} and i.

Proof is derived from [5].

The lemmas 1,2 allow us to state the following theorem which includes solutions of the problem (4), (5) and corresponding error estimations.

Theorem 1. *Let $Z = X^T Y$ where X, Y is a solution of (6),(7) and let the eigen-decomposition of S be*

$$S = G\Lambda G^T, \quad G^T G = GG^T = I$$

with $rank S = r$ and $\Lambda = \Lambda_r \in \mathcal{R}^{r \times r}$. Let $s < r$, and let $Z = V \Omega V^T, V \in \mathcal{R}^{s \times s}$ orthogonal, be the eigen-decomposition of the symmetric matrix Z. Then, $\Omega = \Lambda_s$ and the solution required is defined by the relations

$$C = WB,$$

$$B = YV$$

Moreover, $C^T C = \Lambda_{q_{m+1}}$.

Theorem 2. *Let the conditions of Theorem 1 be fulfilled and let V_{i+1} be defined by the eigen-decomposition $V_{i+1}\ \Omega_{i+1} V_{i+1}^T = Z_{i+1}$ where $Z_{i+1} = X_{i+1}^T Y_{i+1}$.*

If $\lambda_{s+1} \neq \lambda_s$, then for

$$C_{i+1} = WB_{i+1}, \tag{14}$$

$$B_{i+1} = Y_{i+1} V_{i+1} \tag{15}$$

we have

$$\|C - C_{i+1}\| < c_3(\lambda_{s+1}/\lambda_s)^{2(i+1)}$$

$$\|B - B_{i+1}\| < c_4(\lambda_{s+1}/\lambda_s)^{2(i+1)}$$

with constants c_3, c_4 which are independent of λ_s, λ_{s+1} and i.

Proofs of these theorems can be established drawing on the results [5].

On the basis of the above, *the interlacing iteration algorithm*, IIA, for solution of the problem (4), (5) is as follows:

1. given W, S and X_0; compute Y_{i+1} and X_{i+1} for $i = 0, 1, 2, \ldots$ in accordance with (8)-(12);
2. when X_{i+1} and Y_{i+1} satisfy the stopping criteria $\|X_{i+1} - X_i\| \leq \epsilon', \|Y_{i+1} - Y_i\| \leq \epsilon''$ with ϵ', ϵ'' given, compute the small $(s \times s)$ dimension symmetric matrix $Z_i = X_i^T Y_i$ for this i and the eigen-decomposition

$$Z_i = V_i \Omega_i V_i^T \tag{16}$$

   where $\Omega_i = diag\{\lambda_1, \lambda_2, \ldots, \lambda_s\}$ as in Theorem 1.
3. Finally, compute the matrices

$$B_i = Y_i V_i, \qquad C_i = WB_i \tag{17}$$

   Note that only C_i and B_i, of much reduced dimensionalities, are transmitted and this leads to a corresponding reduction of the transmission costs.

Let us consider the *computational complexity* of IIA. Suppose, $l > q_m$ in (4), (5). It follows from (8)-(12) and (14), (15), that each iteration of IIA takes $O(sq_m)$ flops to compute the matrices A_i, Y_{i+1} and X_{i+1}. It also requires $O(lq_m q_{m+1})$ flops to evaluate the matrix C_i. These counts determine the dominant part of the computational complexity of IIA. By comparison, using a QR-type algorithm to solve the problem (4), (5) for C and B requires $O(4l^3/3)$ flops per iteration.

Also note that IIA has the high degree of convergence.

4. NUMERICAL EXAMPLES

To demonstrate the viability of the IIA method we took some electro encephalographic (EEG) data in which there are 512 rows, each with 167 readings taken from one EEG sensor. This data was normalized to have maximum modulus unity. The rows are grouped in sets of 32, each group representing one test with 32 simultaneously sampled sensors. We applied the compression technique above to the first $j = 32$ rows, the first $j = 128$ rows and then to all $j = 512$ rows. For each choice of row dimension we ran IIA with $s = 4, 12, 16$ to give a range of compression ratios. On each test, we took the first s columns of the matrix S as a starting matrix X_0 and ran the iteration (8)-(12) until the relative difference, between successive approximations, of Frobenius norm $\|S - XY^T\|_F$ was smaller than 10^{-2}. All the calculations were done using IEEE double precision arithmetic with a machine epsilon of about 2×10^{-16} and the Frobenius norm is used throughout.

The results of these illustrative examples (not meant to be a thorough numerical investigation) are summarized in the Table 1 below. Column 1 of this table shows the number of rows in the sample and Column 2, the number s. The next two columns, 3 and 4, show the ratios

$$N_W = \frac{\|W - W^{(p)}_{App}\|}{\|W - W^{(0)}_{App}\|}, \quad N_{S-XY} = \frac{\|S - X_p Y_p^T\|}{\|S - X_0 Y_0^T\|} \qquad (18)$$

respectively, where p is the number of iterations required for convergence and W_{App} is the approximation to W given by IIA. Column 5 shows the compression, $Cp = (< C > + < B >)/ < W >$, where $< \cdot >$ denotes the number of elements in the argument matrix, and Column 6 shows the number p of iterations for convergence.

We observed that, universally, the major reduction in the norms (18) occurs in the first iteration: subsequent iterations only produce a small decrease in the accuracy of the approximation. This is illustrated in Table 2, where the error at each of the four iterations needed for convergence, is shown. The compression for this example has reduced the size of the approximation to 16.5% of the original size. Clearly, the improvement obtained depends on the choice of initial matrices X_0 and Y_0. However, even for arbitrarily chosen X_0 and Y_0 we saw very similar convergence behaviour. In Figure 2, we show the corresponding EEG trace and its approximation (i.e. the reconstruction after compression by IIA) on the same axes.

j	s	N_W	N_{S-XY}	Cp	p
32	4	$8.2E-02$	$2.4E-03$	$1.5E-01$	4
32	12	$3.3E-02$	$3.7E-04$	$4.5E-01$	6
32	16	$2.2E-02$	$1.7E-04$	$6.0E-01$	9
128	4	$2.0E-01$	$1.7E-02$	$5.5E-02$	3
128	12	$1.1E-01$	$3.3E-03$	$1.7E-01$	4
128	16	$8.8E-02$	$2.1E-03$	$2.2E-01$	6
512	4	$3.6E-01$	$4.8E-02$	$3.2E-02$	3
512	12	$2.2E-01$	$1.5E-02$	$9.5E-02$	4
512	16	$1.9E-01$	$1.0E-02$	$1.3E-01$	4

Table 1. Numerical examples: error ratios, compression and number of iterations

i	$\|S - X_i Y_i^T\|$
0	$9.454E+02$
1	$3.656E+00$
2	$3.199E+00$
3	$3.142E+00$
4	$3.124E+00$

Table 2. Error reduction for 128 rows with s=12

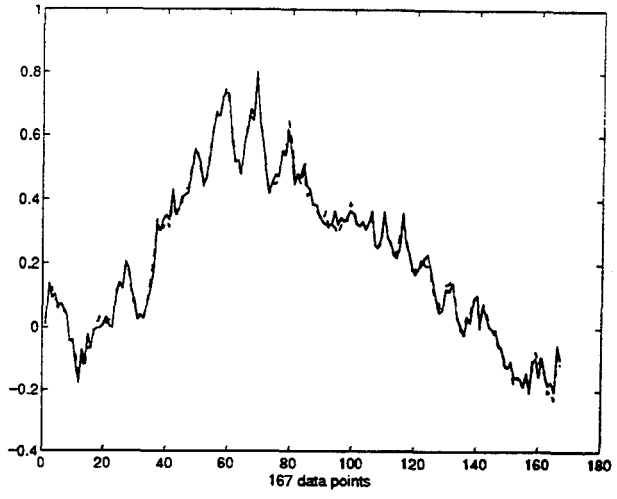

Figure 2. A typical EEG trace and its reconstruction

5. ADVANTAGES AND SIGNIFICANCE

In comparison with the known methods [1-5]

- The specific generic scheme of the signal compression proposed makes it possible to considerably cut the transmission cost;
- The new discrete transforms suggested lead to a high quality of the data compresed reconsruction;
- The new approach of uncorrelated compression based on a specific combination of the ideas [4], p. 582, and [5] has a high speed of convergence and consequently allows us to economize the computational expenditures needed to reach the desired accuracy.

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