Equivalence of light-front and conventional thermal field theory

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It is shown that light-front thermal field theory is equivalent to conventional thermal field theory. The proof is based on the use of spectral representations, and applies to all Lagrangians for which such equivalence has been proven at zero temperature. It is also pointed out that conventional spectral functions can be used to express light-front finite temperature free propagators. As an application of our approach, we derive the light-front finite temperature spin 1/2 fermion propagator in full Dirac space.

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I. INTRODUCTION

The light-front (LF) formulation of quantum field theory, where quantization arises from equal LF ''time'' commutation-anticommutation relations, offers some advantages over the conventional formulation where equal usual times are used for quantization. In this respect, perhaps the three most often quoted advantages of the LF approach are kinematical boosts, a much less complicated vacuum, and much simpler eigenstates. The last two properties are related and may be especially beneficial for thermal field theory where the main object of interest, the ensemble average, is just a sum over eigenstates.

With such considerations in mind, the problem of formulating thermal field theory on the light front has been addressed in a number of recent papers [1–4]. In Ref. [2], Alves, Das, and Perez proposed the LF version of the imaginary time scalar particle propagator, and used this to calculate the self-energy loop diagram. The result of this calculation looked so different from the conventional one that only after Weldon [4] made use of a clever transformation of the integration momentum variable did it become clear that this difference is illusory.

In the present note we show that the *whole* LF approach using the imaginary time scalar particle propagator proposed in Ref. [2], is equivalent to the conventional one. Our method is based on spectral representations of Green functions, and can be applied to the case of any Lagrangian for which such equivalence has been shown at zero temperature [5–12]. In this respect, we note that the spectral function of the Lehmann representation is already well recognized as being very useful for relating different types of Green functions (imaginary time, real time, advanced, retarded, etc.) in the conventional approach. Here we show how to use the conventional spectral function to derive a LF free particle propagator of arbitrary spin in either imaginary or real time formalism. This allows us to derive fermion propagators in the full Dirac index space.

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II. GENERAL PROOF OF EQUIVALENCE OF LIGHT-FRONT AND CONVENTIONAL THERMAL FIELD THEORY

The problem of calculating the ensemble average $\mathrm{Tr}\, e^{-\beta P_L^0}\mathcal{O}_L$ in LF quantized field theory[1] can be reduced to the calculation of the fully dressed LF imaginary time propagator Δ^L given in coordinate and momentum space by[2] (we use units where $\hbar = k_B = 1$)

$$\Delta^L(\tau,\underline{x}) = (\mathrm{Tr}\, e^{-\beta P_L^0})^{-1}\mathrm{Tr}\{e^{-\beta P_L^0}T_\tau[e^{P_L^0\tau}\phi(0,\,\underline{x})\times e^{-P_L^0\tau}\overline{\phi}(0)]\}, \tag{1a}$$

$$\Delta^L(i\omega_n,\underline{p}) = \frac{1}{\sqrt{2}}\int_0^\beta d\tau\, d\underline{x}\, e^{i(\omega_n\tau - \underline{p}\cdot\underline{x})}\Delta^L(\tau,\underline{x}), \tag{1b}$$

where $\omega_n = 2n\pi T$ for bosons, $\omega_n = (2n+1)\pi T$ for fermions, $\underline{x} = (x^-, x^\perp)$, $\underline{p} = (p^+, p^\perp)$, $x^\pm = (1/\sqrt{2})(x^0 \pm x^3)$, $p^\pm = (1/\sqrt{2})(p^0 \pm p^3)$, $\underline{p}\cdot\underline{x} = -p^+x^- + p^\perp x^\perp$, $\phi(x)$ is a noninteracting particle field operator, $\phi(0,\underline{x}) = \phi(x)|_{x^+=0}$, and where the energy operator P_L^0 and the operator of some physical quantity $\mathcal{O}_L$ are defined in the LF quantized theory, i.e., they depend on interacting field operators whose commutation relations are given on the $x^+=0$ hyperplane (see Appendix A for more details). The imaginary time ordering product in Eq. (1a) is defined as

$$\begin{aligned} T_\tau[e^{P_L^0\tau}\phi(0,\underline{x})e^{-P_L^0\tau}\overline{\phi}(0)] &= \theta(\tau)e^{P_L^0\tau}\phi(0,\underline{x})e^{-P_L^0\tau}\overline{\phi}(0) \\ &\pm \theta(-\tau)\overline{\phi}(0)e^{P_L^0\tau}\phi(0,\underline{x})e^{-P_L^0\tau}, \end{aligned} \tag{2}$$

where the upper (lower) sign is for the case of bosons (fermions). Note that although τ is associated with the usual time, in the sense that it is combined with P_L^0 (rather than

[1] Note that the correct definition of ensemble average has energy operator P_L^0 in the exponential $e^{-\beta P_L^0}$ [13] and not the Hamiltonian P_L^- (see the discussion in Ref. [2]).

[2] To distinguish between LF and usual equal time quantization we work in the operator rather than path integral formalism.

with P_L^-) in the exponent of $e^{P_L^0\tau}$, Eq. (2) nevertheless represents the ordering of the interacting fields with respect to the imaginary LF "time" [see Eq. (8), the discussion below Eq. (8), and the text after Eq. (B5)], in contrast to the usual time ordering in Eq. (3a). It is also important to note that Eqs. (1) define the LF imaginary time formalism exactly, and in the case of scalar particles (only), they define perturbation theory with the free propagator suggested in Ref. [2]. As the perturbation theory for the dressed propagators of Eqs. (1) may not be immediately apparent (the exponents involve the energy operator P_L^0 rather than the Hamiltonian P_L^-), we outline its derivation in the Appendix B.

In this paper we shall prove that the analytic continuation of the fully dressed LF imaginary time propagator $\Delta^L(i\omega_n,\underline{p})$ to real energies, $i\omega_n\to p_0$, is identical to the analytic continuation of the fully dressed conventional imaginary time propagator $\Delta(i\omega_n,\mathbf{p})$, i.e., that $\Delta^L(p_0,\underline{p})=\Delta(p_0,\mathbf{p})$, where

$$\Delta(\tau,\mathbf{x})=(\mathrm{Tr}\,e^{-\beta P^0})^{-1}\mathrm{Tr}\{e^{-\beta P^0}\times T_\tau[e^{P^0\tau}\phi(0,\mathbf{x})e^{-P^0\tau}\bar{\phi}(0)]\},\tag{3a}$$

$$\Delta(i\omega_n,\mathbf{p})=\int_0^\beta d\tau\, d\mathbf{x}\, e^{i(\omega_n\tau-\mathbf{p}\cdot\mathbf{x})}\Delta(\tau,\mathbf{x}),\tag{3b}$$

$\phi(0,\mathbf{x})=\phi(x)|_{x^0=0}$,[3] and P^0 is the conventional energy operator.

We begin our proof by relating $\Delta^L(i\omega_n,\underline{p})$ to the real time LF Green function:

$$\mathcal{D}^L(t,\underline{x})=(\mathrm{Tr}\,e^{-\beta P_L^0})^{-1}\mathrm{Tr}\{e^{-\beta P_L^0}\times T_t[e^{iP_L^0t}\phi(0,\underline{x})e^{-iP_L^0t}\bar{\phi}(0)]\},\tag{4a}$$

$$D^L(p^0,\underline{p})=\frac{1}{\sqrt{2}}\int dt\,d\underline{x}\,e^{i(p^0t-\underline{p}\cdot\underline{x})}\mathcal{D}^L(t,\underline{x}),\tag{4b}$$

where T_t is defined analogously to Eq. (2) as

$$\begin{aligned}T_t[e^{iP_L^0t}\phi(0,\underline{x})e^{-iP_L^0t}\bar{\phi}(0)]&=\theta(t)e^{iP_L^0t}\phi(0,\underline{x})e^{-iP_L^0t}\bar{\phi}(0)\\&\pm\theta(-t)\bar{\phi}(0)e^{iP_L^0t}\phi(0,\underline{x})e^{-iP_L^0t}.\end{aligned}\tag{5}$$

This is done by utilizing the Lehmann representation, which can be derived for LF Green functions in a way similar to that for conventional Green functions (see Ref. [14] for the conventional case):

$$D^L(p^0,\underline{p})=i\int\frac{dp_0'}{2\pi}\frac{\rho^L(p_0',\underline{p})}{p_0-p_0'+i\eta}\pm f(p_0)\rho^L(p_0,\underline{p}),\tag{6a}$$

$$\Delta^L(i\omega_n,\underline{p})=-\int\frac{dp_0'}{2\pi}\frac{\rho^L(p_0',\underline{p})}{i\omega_n-p_0'},\tag{6b}$$

where $f(p_0)=(e^{\beta p_0}\mp 1)^{-1}$ is the distribution function for bosons (upper sign) or fermions (lower sign), and the LF spectral function (defined in Appendix C) is, correspondingly,[4]

$$\rho^L(p_0,\underline{p})=\frac{(2\pi)^4}{\mathrm{Tr}\,e^{-\beta P_L^0}}(1\mp e^{-\beta p_0})\sum_{nm}e^{-\beta E_n}\delta^4(p-P_m+P_n)\times\langle Ln|\phi(0)|Lm\rangle\langle Lm|\bar{\phi}(0)|Ln\rangle.\tag{7}$$

The difference between $\rho^L(p)$ and the conventional spectral function $\rho(p)$ [see Eqs. (13)] is only in the eigenstates $|Lm\rangle$ of the LF four-momentum, $P_L^\mu|Lm\rangle=P_m^\mu|Lm\rangle$,[5] which are different from the conventional ones (denoted by $|m\rangle$ in Ref. [14]). However, in the free case there is no difference between these eigenstates and thus the free LF and conventional spectral functions are identical. The unusual scalar product $p^0t-\underline{p}\cdot\underline{x}$ in the exponent of Eq. (4b) can be written in the invariant form $p^0t-\underline{p}\cdot\underline{x}=p\cdot x'$, where x' is defined by

$$x_0'=\frac{x^-}{\sqrt{2}}+t,\quad (x')^3=-\frac{x^-}{\sqrt{2}},\quad (x')^\perp=x^\perp,$$
$$dt\,d\underline{x}=-\sqrt{2}d^4x'.\tag{8}$$

Writing

$$e^{iP_L^0t}\phi(0,\underline{x})e^{-iP_L^0t}=\Phi_L(x'),\tag{9}$$

one finds that Φ_L is a LF Heisenberg field operator; i.e., for any four-vector a,

$$\Phi_L(x+a)=e^{iP_L\cdot a}\Phi_L(x)e^{-iP_L\cdot a},\tag{10}$$

with initial condition

$$\Phi_L(0,\underline{x})=\phi(0,\underline{x}).\tag{11}$$

Given that $\sqrt{2}(x')^+=t$, T_t ordering in Eq. (4a) implies LF time ordering T_+, so that Eqs. (4) can be written in the form

[3] Hopefully no confusion will arise from our not entirely consistent notation for $\phi(0,\mathbf{x})$ and $\phi(0,\underline{x})$; in particular $\phi(0,\mathbf{x})\neq\phi(0,\underline{x})|_{\underline{x}=\mathbf{x}}$.

[4] We do not show explicit spin indices. For particles with spin, one should consider the field ϕ as a column vector, the field $\bar{\phi}$ as a row vector, and quantities such as ρ^L, D^L, Δ^L, etc., as square matrices, in spin index space.

[5] Here P_L^μ is the operator of the four-momentum in the LF approach whereas P_m^μ is its eigenvalue corresponding to the state $|Lm\rangle$.

$$D^L(x)=(\mathrm{Tr}\,e^{-\beta P_L^0})^{-1}\mathrm{Tr}\{e^{-\beta P_L^0}T_+[\Phi_L(x)\bar{\Phi}_L(0)]\}, \tag{12a}$$

$$D^L(p)=\int d^4x e^{ip\cdot x}D^L(x), \tag{12b}$$

with the understanding that $D^L(p_0,\underline{p})=D^L(p)$ and $\mathcal{D}^L(t,\underline{x})=D^L(x')$.

The well-known relations, corresponding to the LF Eqs. (6), connecting conventional real and imaginary time Green functions via the conventional spectral function $\rho(p)$ are [14–16]

$$D(p^0,\mathbf{p})=i\int\frac{dp_0'}{2\pi}\frac{\rho(p_0',\mathbf{p})}{p_0-p_0'+i\eta}\pm f(p_0)\rho(p_0,\mathbf{p}), \tag{13a}$$

$$\Delta(i\omega_n,\mathbf{p})=-\int\frac{dp_0'}{2\pi}\frac{\rho(p_0',\mathbf{p})}{i\omega_n-p_0'}, \tag{13b}$$

where $D(p^0,\mathbf{p})$ is the conventional real time Green function, defined as

$$D(t,\mathbf{x})=(\mathrm{Tr}\,e^{-\beta P^0})^{-1}\mathrm{Tr}\{e^{-\beta P^0}\times T_t[e^{iP^0t}\phi(0,\mathbf{x})e^{-iP^0t}\bar{\phi}(0)]\}, \tag{14a}$$

$$D(p^0,\mathbf{p})=\int dt d\mathbf{x} e^{i(p^0t-\mathbf{p}\cdot\mathbf{x})}D(t,\mathbf{x}). \tag{14b}$$

For a perturbative treatment the propagator of Eq. (14a) should be written in the interaction representation [17], with

$$\Phi(x)\equiv e^{iP^0t}\phi(0,\mathbf{x})e^{-iP^0t}=U(0,t)\phi(x)U(t,0), \tag{15}$$

$$D(x)=(\mathrm{Tr}\,e^{-\beta P^0})^{-1}\mathrm{Tr}\{e^{-\beta P_f^0}S^{-1}T[\phi(x)\bar{\phi}(0)S]\}, \tag{16}$$

where P_f^0 is the free part of P^0, T is the usual time ordering operator, $S=U(\infty,-\infty)$, and

$$U(t_2,t_1)=T\exp\left\{-i\int_{t_1}^{t_2}dt\int_{-\infty}^{\infty}d^3x\mathcal{P}_I^0(x)\right\}, \tag{17}$$

$\mathcal{P}_I^0(x)$ being the interaction part of the Hamiltonian density in the interaction picture. We note the unusual appearance (for zero temperature perturbation theory) of the inverse S matrix, S^{-1}, the source of doubled degrees of freedom [17]. An analogous LF interaction representation can be written for D^L:

$$\Phi_L(x)\equiv e^{iP_L^-x^+}\phi(0,\underline{x})e^{-iP_L^-x^+}=U_L(0,x^+)\phi(x)U_L(x^+,0), \tag{18}$$

$$D^L(x)=(\mathrm{Tr}\,e^{-\beta P_L^0})^{-1}\mathrm{Tr}\{e^{-\beta P_{Lf}^0}S_L^{-1}T_+[\phi(x)\bar{\phi}(0)S_L]\}, \tag{19}$$

where P_{Lf}^0 is the free part of P_L^0, $S_L=U_L(\infty,-\infty)$, P_L^- is the LF Hamiltonian, i.e., the negative component of the four-momentum operator, and

$$U_L(\alpha_2,\alpha_1)=T_+\exp\left\{-i\int_{\alpha_1}^{\alpha_2}dx^+\int_{-\infty}^{\infty}d\underline{x}\mathcal{P}_{LI}^-(x)\right\}, \tag{20}$$

$\mathcal{P}_{LI}^-(x)$ being the interaction part of the LF Hamiltonian density in the interaction representation.

A. Scalar particles

To compare the dressed real time propagators in the conventional and LF formalisms, given in Eq. (16) and Eq. (19), respectively, we first restrict the discussion to the case of scalar particles. For scalar particles $\mathcal{P}_{LI}^-$ and $\mathcal{P}_I^0$ are the same functions of the free field operators ϕ [7], which means that the perturbation theories for Eq. (16) and Eq. (19) have the same vertices. The inverse S matrix in both Eq. (16) and Eq. (19) leads to free propagators with doubled degrees of freedom, i.e., 2×2 matrices $\hat{D}^f$ and $\hat{D}^{Lf}$ whose (1,1) element is defined by Eq. (16) and Eq. (19) in the no interaction limit:

$$\hat{D}_{11}^f(x)=D^f(x)=(\mathrm{Tr}\,e^{-\beta P_f^0})^{-1}\mathrm{Tr}\{e^{-\beta P_f^0}T_t[\phi(x)\bar{\phi}(0)]\}, \tag{21a}$$

$$\hat{\mathcal{D}}_{11}^{Lf}(t,\underline{x})=\mathcal{D}^{Lf}(\underline{t},\underline{x})=(\mathrm{Tr}\,e^{-\beta P_f^0})^{-1}\mathrm{Tr}\{e^{-\beta P_f^0}\times T_t[e^{iP_f^0t}\phi(0,\underline{x})e^{-iP_f^0t}\bar{\phi}(0)]\}, \tag{21b}$$

$$=D^{Lf}(x')=(\mathrm{Tr}\,e^{-\beta P_f^0})^{-1}\mathrm{Tr}\{e^{-\beta P_f^0}T_+[\phi(x')\bar{\phi}(0)]\}, \tag{21c}$$

where x' is defined in Eq. (8). Similar to the zero-temperature case, straightforward calculation of Eqs. (21) shows that $D^{Lf}(p)=D^f(p)$ [see also Eqs. (24)], therefore the full propagators constructed according to Eq. (16) and Eq. (19) are equal to each other, $D^L(p)=D(p)$, which already means the equivalence of real time LF and conventional thermal field theories for the scalar particle case. As we shall now see, this also leads to the identity between $\Delta^L(p_0,\underline{p})$ and $\Delta(p_0,\mathbf{p})$, the analytic continuations of the LF and conventional imaginary time propagators. To define a unique analytic continuation of $\Delta(i\omega_n,\mathbf{p})$, given for the discrete values $\omega_n=2\pi n/\beta$, only two requirements have been needed in the conventional approach: (i) $|\Delta(z,\mathbf{p})|\to0$ as $|z|\to\infty$, and (ii) that $\Delta(z,\mathbf{p})$ is analytic outside the real axis [14–16]. Placing the same requirements on $\Delta^L(p_0,\underline{p})$, one obtains a unique analytic continuation of the LF imaginary time propagator as well. These analytic continuations are provided by Eqs. (6b) and (13b):

$$\Delta(p_0,\mathbf{p})=-\int\frac{dp_0'}{2\pi}\frac{\rho(p_0',\mathbf{p})}{p_0+i\eta-p_0'}, \tag{22a}$$

$$\Delta^L(p_0,\underline{p}) = -\int \frac{dp_0'}{2\pi}\,\frac{\rho^L(p_0',\underline{p})}{p_0+i\eta-p_0'}. \tag{22b}$$

Equating $D^L(p)$ of Eq. (6a) to $D(p)$ of Eq. (13a) leads to the identity of the LF and conventional spectral functions, as well as to the identity $\Delta^L(p)=\Delta(p)$, as the latter are represented by Eqs. (22).

B. Nonscalar particles

To apply the above proof to the case of nonscalar particles, we note that in the LF approach the components of the nonscalar field become constrained, and as a result of the constraints, the interaction part of the LF Hamiltonian, P^-_{LI}, acquires extra terms, so that perturbation theory for the LF real time propagator $D^L(x)$ [Eq. (19)] has extra vertices compared to that for the conventional propagator $D(x)$ [Eq. (16)]. At the same time, the free nonscalar propagators in the LF and conventional approaches are different [see Eq. (29) for the case of a spinor particle]. Without going into details here, the general strategy to prove the equivalence of LF and conventional thermal field theories for the case of nonscalar particles is similar to the one used in the zero-temperature case [5–11]. For example, in models of spin 1/2 fermions where interactions are given by three-point scalar or pseudoscalar vertices, equivalence at zero temperature is proved by showing that the extra vertices in LF perturbation theory can be taken into account by simply adding the term $i\gamma^+/2p^+$ to the free LF propagator, with the sum being equal to the conventional propagator.[6] For the same models at nonzero temperature one follows a similar procedure: the extra vertices in the LF perturbation theory of Eq. (19) are taken into account by adding the following instantaneous term to the free LF propagator of Eq. (29):

$$i\frac{\gamma^+}{2p^+}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \tag{23}$$

This turns the LF propagator into the conventional 2×2 real time propagator. In this way one can prove the equivalence of LF and conventional real time thermal field theories for nonscalar particles described by Lagrangians for which such an equivalence has been shown at zero temperature, as done for example in Refs. [5–11].[7] The rest of the proof, showing equivalence of the LF and conventional imaginary time thermal field theories, is based on spectral representations and follows the same procedure presented above for the scalar particle case.

[6] This applies only to internal propagators (those not corresponding to an external leg).

[7] In this respect it should be noted that vector particles need a more sophisticated treatment [8,9], and equivalence may not mean that the dressed LF propagator is equal to the conventional one as in the scalar particle case, or even effectively equal to the conventional one as in the spinor case discussed above.

III. FREE PROPAGATORS

With the help of spectral functions, we will derive the free propagators of LF thermal field theory, including the spin 1/2 fermion (spinor) propagator in the entire Dirac index space. In Ref. [2] it was suggested that the fermion propagator be expressed in the spinor subspace projected by $P^+=\gamma^-\gamma^+/2$. Yet there is clear need for the fermion propagator in the entire spinor space, for example, to keep under control the compensation between differences in vertices and propagators with respect to the conventional approach.

Using the eigenstates of the noninteracting system in Eq. (7), one gets the spectral functions of the free scalar and spinor particles:

$$\rho_0^{Lf}(p)=\rho_0^f(p)=2\pi\epsilon(p_0)\,\delta(p^2-m^2)\quad(\text{scalar}), \tag{24a}$$

$$\rho_{1/2}^{Lf}(p)=\rho_{1/2}^f(p) = 2\pi\epsilon(p_0)(\not p+m)\,\delta(p^2-m^2)\quad(\text{spinor}), \tag{24b}$$

where $\epsilon(p_0)=p_0/|p_0|$. Equations (24) can also be obtained from Eq. (C4) by using the well-known free field commutators or anticommutators. Although the free spectral functions are the same in the LF and conventional approaches, the fermion bare propagators are not:

$$D_{1/2}^{Lf}(p)=(\overline{\not p}+m)\left[\frac{i}{p^2-m^2+i\eta} -2\pi f(|p_0|)\,\delta(p^2-m^2)\right], \tag{25a}$$

$$D_{1/2}^{f}(p)=(\not p+m)\left[\frac{i}{p^2-m^2+i\eta} -2\pi f(|p_0|)\,\delta(p^2-m^2)\right], \tag{25b}$$

where $\overline{p}$ in Eq. (25a) is the on mass shell momentum with components $\overline{p}^-=(p_\perp^2+m^2)/2p^+$, and $\underline{\overline{p}}=\underline{p}$, which depend only on p^+ and $p^\perp$. This difference arises because different components of the four-momentum p are fixed in the integrals of Eqs. (6) and (13). A similar difference arises in the imaginary time formalism where the spinor propagator can be derived using the same spectral function, given by Eq. (24b), and the representation of Eq. (6b):

$$\Delta_{1/2}^{Lf}(i\omega_n,\underline{p}) = -\frac{\overline{\not p}+m}{2p_n^-p^+-m^2-p_\perp^2} \tag{26}$$

where $p_n^-=\sqrt{2}i(2n+1)\pi T-p^+=\sqrt{2}p_n^0-p^+$. When fermions are involved the interaction part of the LF Hamiltonian has an extra term resulting from resolving constraints, whose effect is an instantaneous addition of $\gamma^+/2p^+$ to the propagator $\Delta_{1/2}^{Lf}$:

$$\frac{\overline{\not p}+m}{2p_n^- p^+ - m^2 - p_\perp^2} + \frac{\gamma^+}{2p^+} = \frac{2p^+(\gamma^+ p_n^- + \gamma^- p^+ - \gamma_\perp p_\perp + m)}{2p^+(2p_n^- p^+ - m^2 - p_\perp^2)} = \frac{\not p_n + m}{p_n^2 - m^2}, \tag{27}$$

where $\underline{p}_n = \underline{p}$. Equation (27) should be compared with the conventional imaginary time fermion propagator

$$\Delta_{1/2}^f(i\omega_n, \mathbf{p}) = \left.\frac{\not p + m}{p^2 - m^2}\right|_{p_0 = i(2n+1)\pi T}^{(\mathbf{p}\text{ fixed})} = -\frac{i(2n+1)\pi T\gamma_0 - \mathbf{p}\cdot\boldsymbol{\gamma} + m}{[(2n+1)\pi T]^2 + \mathbf{p}^2 + m^2}. \tag{28}$$

The difference between Eq. (27) and Eq. (28) is similar to the scalar particle case in that the replacement $p^0 \to i\pi(2n+1)T$ in the zero-temperature propagator is carried out when different remaining variables $\underline{p}$ and $\mathbf{p}$ are fixed, respectively.

It is not difficult to derive a spectral representation for the 2×2 propagators of the real time formalism and to write down the analog of Eq. (7) for the corresponding 2×2 spectral function. Then again we will see that the spectral function of a free particle is identical for the LF and conventional approach. As a result we will obtain the following expression for the LF real time fermion propagator:

$$\hat{D}_{1/2}^{Lf}(p) = (\overline{\not p}+m)\begin{pmatrix} \dfrac{i}{p^2-m^2+i\eta} & 0 \\ 0 & \dfrac{-i}{p^2-m^2-i\eta} \end{pmatrix} - 2\pi(\not p + m)\begin{pmatrix} f(|p_0|) & f(|p_0|)-\theta(-p_0) \\ f(|p_0|)-\theta(p_0) & f(|p_0|) \end{pmatrix} \times \delta(p^2-m^2), \tag{29}$$

which differs from the conventional one in having $\overline{p}$ as the on mass shell momentum in the numerator of the first term on the right-hand side of the equation.

APPENDIX A: LIGHT-FRONT QUANTIZATION

As the precise meaning of "LF quantization" appears to vary in the current literature, here we give the exact sense in which this term is used in the present paper. It is sufficient to consider only the scalar particle case. We follow the traditional formulation of LF quantization. Starting with a Lagrangian density $\mathcal{L}[\Phi] \equiv \mathcal{L}(\Phi(x), \partial_\mu \Phi(x))$ and the equations of motion

$$\frac{\partial}{\partial x^\mu}\frac{\partial \mathcal{L}}{\partial(\partial\Phi/\partial x^\mu)} - \frac{\partial \mathcal{L}}{\partial \Phi} = 0, \tag{A1}$$

quantization enters through the specification of the initial conditions for these equations. In the case of LF quantization, we specify the initial conditions as

$$\Pi_L(x) = \frac{\partial \mathcal{L}}{\partial(\partial\Phi_L/\partial x^+)}, \quad [\Phi_L(0), \Pi_L(x)]\delta(x^+) = \frac{i}{2}\delta^4(x), \tag{A2}$$

where $\Pi_L(x)$ is the momentum conjugate to the field $\Phi_L(x)$, and the commutation relation between Π_L and Φ_L is specified on the LF surface $x^+=0$. Note that we have added a subscript L to those solutions of Eq. (A1) whose initial conditions are given by Eq. (A2). The LF initial conditions should be contrasted with those of usual equal time quantization:

$$\Pi(x) = \frac{\partial \mathcal{L}}{\partial(\partial\Phi/\partial x^0)}, \quad [\Phi(0), \Pi(x)]\delta(x^0) = i\delta^4(x). \tag{A3}$$

The LF four-momentum P_L^μ is given as an integral on the same surface as that specifying the LF commutation relations:

$$P_L^\mu = \int \delta(x^+) T^{+\mu}[\Phi_L] d^4x = \int_{x^+=0} T^{+\mu}[\Phi_L] d\underline{x}, \tag{A4}$$

where the energy-momentum tensor $T^{\mu\nu}$ is connected to the Lagrangian in a way that is independent of quantization:

$$T^{\mu\nu}[\Phi] = \frac{\partial \mathcal{L}}{\partial(\partial\Phi/\partial x^\mu)}\frac{\partial\Phi}{\partial x_\nu} - g^{\mu\nu}\mathcal{L}[\Phi]. \tag{A5}$$

Similarly, the conventional four-momentum is defined by

$$P^\mu = \int \delta(x^0) T^{0\mu}[\Phi] d^4x = \int_{x^0=0} T^{0\mu}[\Phi] d\mathbf{x}. \tag{A6}$$

In the free case (no interactions), one can show that the LF and conventional free fields are identical, as are the LF and conventional free momenta:

$$\phi_L(x) = \phi(x), \quad P_{Lf}^\mu = P_f^\mu. \tag{A7}$$

Constructed in this way, both the LF and conventional four-momenta act as generators of space-time translations:

$$[P_L^\mu, \Phi_L(x)] = -i\frac{\partial \Phi_L(x)}{\partial x_\mu}, \tag{A8a}$$

$$[P^\mu, \Phi(x)] = -i\frac{\partial \Phi(x)}{\partial x_\mu}. \tag{A8b}$$

We note the universal nature of these commutation relations—they do not depend on the type of quantization used. For the LF energy operator, P_L^0, and the LF Hamiltonian, $P_L^- = (1/\sqrt{2})(P_L^0 - P_L^3)$, we have

$$[P_L^0, \Phi_L(x)] = -i \left.\frac{\partial \Phi_L(x)}{\partial x^0}\right|_{x^3, x^\perp \text{ fixed}} = \frac{-i}{\sqrt{2}} \left.\frac{\partial \Phi_L(x)}{\partial x^+}\right|_{x^3, x^\perp \text{ fixed}}, \tag{A9}$$

$$[P_L^-, \Phi_L(x)] = -i \left.\frac{\partial \Phi_L(x)}{\partial x^+}\right|_{x^-, x^\perp \text{ fixed}} = \frac{-i}{\sqrt{2}} \left.\frac{\partial \Phi_L(x)}{\partial x^0}\right|_{x^-, x^\perp \text{ fixed}}. \tag{A10}$$

It is seen that both P_L^0 and P_L^- determine evolution in LF time x^+, the only difference being in the variables that are kept constant.

As the LF four-momentum P_L^μ is defined on the LF hyperplane $x^+ = 0$, as indicated by Eq. (A4), it follows that it is a functional of $\Phi_L(0,\underline{x}) = \phi(0,\underline{x})$. With the same being true of the free momentum operator P_f^μ, it follows that $P_{LI}^0(\tau) \equiv e^{\tau P_f^0}(P_L^0 - P_f^0)e^{-\tau P_f^0}$ depends on $\phi_\tau(\underline{x}) = e^{P_f^0 \tau}\phi(0,\underline{x})e^{-P_f^0 \tau}$. This observation is at the heart of LF imaginary time perturbation theory, discussed in Appendix B.

APPENDIX B: LF IMAGINARY TIME PERTURBATION THEORY

Here we give a brief derivation of perturbation theory for the fully dressed LF imaginary time propagator Δ^L given in Eqs. (1). The perturbation theory is based on the following operator exponent expansion:

$$\mathcal{U}(\tau_1, \tau_2) \equiv e^{P_f^0 \tau_1} e^{P_L^0(\tau_2 - \tau_1)} e^{-P_f^0 \tau_2} = T_\tau \exp\left[\int_{\tau_1}^{\tau_2} P_{LI}^0(\tau) d\tau\right], \tag{B1}$$

where P_L^0 and P_f^0 are the zero components of the LF four-momentum operator in the interacting and free case, respectively, $P_{LI}^0(\tau) = e^{P_f^0 \tau}(P_L^0 - P_f^0)e^{-P_f^0 \tau}$, and T_τ is the ordering operator, as in Eq. (2), which orders τ-dependent quantities with respect to τ, so that

$$T_\tau[P_{LI}^0(\tau) P_{LI}^0(\tau')] = \theta(\tau - \tau') P_{LI}^0(\tau) P_{LI}^0(\tau') + \theta(\tau' - \tau) P_{LI}^0(\tau') P_{LI}^0(\tau). \tag{B2}$$

Note that both P_L^0 and P_f^0 are expressed in terms of the free field operators $\phi(0,\underline{x})$ on the LF hyperplane $x^+ = 0$. Then by analogy with textbook derivations of perturbation theory in the imaginary time formalism [14–16], the Green function Δ^L of Eq. (1a) can be written in the form

$$(\mathrm{Tr}\, e^{-\beta P_L^0}) \Delta^L(\tau, \underline{x}) = \mathrm{Tr}\{e^{-\beta P_L^0} T_\tau[e^{P_L^0 \tau}\phi(0,\underline{x})e^{-P_L^0 \tau}\overline{\phi}(0)]\}$$
$$= \mathrm{Tr}\left\{ e^{-\beta P_f^0} T_\tau\left[\exp\left(\int_0^\beta P_{LI}^0(\tau') d\tau'\right) \times e^{P_f^0 \tau}\phi(0,\underline{x})e^{-P_f^0 \tau}\overline{\phi}(0)\right]\right\}, \tag{B3}$$

where T_τ orders operators $P_{LI}^0(\tau')$, $\phi_\tau(\underline{x}) = e^{P_f^0 \tau}\phi(0,\underline{x})e^{-P_f^0 \tau}$, and $\overline{\phi}_0(\underline{0}) = \overline{\phi}(0)$ with respect to τ', τ, and 0 according Eq. (2). It is important to note that as a result of the definition of Eq. (1a), the interaction part of the energy operator, $P_{LI}^0(\tau')$, depends on $\phi_{\tau'}(\underline{x}) = e^{P_f^0 \tau'}\phi(0,\underline{x})e^{-P_f^0 \tau'}$, the free field operators on the $x^+ = 0$ LF hyperplane $\phi_0(\underline{x}) = \phi(0,\underline{x})$, shifted by imaginary usual time. In the analogous conventional expression given in Refs. [14–16], the interaction part of the energy depends on the free field operators on the hyperplane $t = 0$ (not $x^+ = 0$) shifted by imaginary usual time. This difference is reflected in the difference of the imaginary time propagators of the corresponding perturbation theories. Indeed using Wick's theorem in Eq. (B3) one ends up with imaginary time perturbation theory with propagators

$$\Delta^{Lf}(\tau, \underline{x}) = (\mathrm{Tr}\, e^{-\beta P_f^0})^{-1} \times \mathrm{Tr}\{e^{-\beta P_f^0} T_\tau[e^{P_f^0 \tau}\phi(0,\underline{x})e^{-P_f^0 \tau}\overline{\phi}(0)]\}, \tag{B4}$$

$$\Delta^{Lf}(i\omega_n, \underline{p}) = \frac{1}{\sqrt{2}}\int_0^\beta d\tau\, d\underline{x}\, e^{i(\omega_n \tau - \underline{p}\cdot\underline{x})}\Delta^{Lf}(\tau, \underline{x}). \tag{B5}$$

Applying the arguments leading to Eq. (8) to the imaginary time $t = -i\tau$, one gets $e^{P_f^0 \tau}\phi(0,\underline{x})e^{-P_f^0 \tau} = \phi(z)$, where z is a complex coordinate space four-vector with purely imaginary LF time $z^+ = -i\tau/\sqrt{2}$ and real spatial components $z^3 = -x^-/\sqrt{2}$ and $z^\perp = x^\perp$. This suggests that the propagator of Eq. (B4) corresponds to imaginary LF time ordering, in contrast to the usual time ordering in the conventional approach. Direct calculation leads to the scalar particle imaginary time propagator suggested in Ref. [2],

$$\Delta_0^{Lf}(i\omega_n, \underline{p}) = -\frac{1}{2p_n^- p^+ - m^2 - p_\perp^2}, \tag{B6}$$

with $p_n^- = \sqrt{2} i\pi 2nT - p^+ = \sqrt{2}p_n^0 - p^+$, and the spinor particle propagator of Eq. (26) derived above. Even in the scalar particle case, the difference between the imaginary time formalisms in the two quantizations is more than a simple change of variables: although the energy variable is purely imaginary in both Eq. (B6) and the conventional propagator, p^3 is real in the conventional approach but complex in the LF one. Note also that in the LF dynamics the interaction does not affect P_L^+. Only the operator P_L^- acquires an interaction part; therefore the interaction part of the energy operator is $P_{LI}^0 = P_{LI}^-/\sqrt{2}$ and in the case of scalar particles (only) it is related to the interaction Lagrangian (without coupling with derivatives) by a factor of $\sqrt{2}$: $P^0_{LI}=-L_I/\sqrt{2}$.

APPENDIX C: LIGHT-FRONT SPECTRAL FUNCTION

Here we introduce the LF spectral function ρ^L used in Eqs. (6), and specify some of its properties by exploiting the analogy with the well known spectral function ρ of conventional thermal field theory [14]. In coordinate space, one can use Eq. (12a), defining the real time LF propagator $D^L(x)$, to write

$$\begin{aligned} D^L(x) &= \theta(x^+)D^{L>}(x)\pm\theta(-x^+)D^{L<}(x) \\ &= \theta(x^+)\rho^L(x)\pm D^{L<}(x), \end{aligned} \tag{C1}$$

where

$$D^{L>}(x)=(\mathrm{Tr}\, e^{-\beta P^0_L})^{-1}\mathrm{Tr}\{e^{-\beta P^0_L}\Phi_L(x)\overline{\Phi}_L(0)\}, \tag{C2}$$

$$D^{L<}(x)=(\mathrm{Tr}\, e^{-\beta P^0_L})^{-1}\mathrm{Tr}\{e^{-\beta P^0_L}\overline{\Phi}_L(0)\Phi_L(x)\}, \tag{C3}$$

and

$$\begin{aligned} \rho^L(x) &= D^{L>}(x)\mp D^{L<}(x) \\ &= (\mathrm{Tr}\, e^{-\beta P^0_L})^{-1}\mathrm{Tr}\{e^{-\beta P^0_L}[\Phi_L(x),\overline{\Phi}_L(0)]_{\mp}\}, \end{aligned} \tag{C4}$$

is the LF spectral function defined as the ensemble average of the commutator-anticommutator $[\Phi_L(x),\overline{\Phi}_L(0)]_{\mp}$. The Kubo-Martin-Schwinger relation is the same as in Ref. [14],

$$D^{L<}(x)=D^{L>}(x^0-i\beta,\mathbf{x}). \tag{C5}$$

It is easy to express the above equations in momentum space, thereby obtaining the following relations:

$$\rho^L(p)=D^{L>}(p)\mp D^{L<}(p), \tag{C6}$$

$$D^{L<}(p)=\pm e^{-\beta p^0}D^{L>}(p), \tag{C7}$$

$$D^{L>}(p)=[1\pm f(p^0)]\rho^L(p), \tag{C8}$$

$$D^{L<}(p)=\pm f(p^0)\rho^L(p), \tag{C9}$$

which lead to Eq. (6a). The x^+ ordering in Eq. (C1) (instead of conventional x^0 ordering) is reflected in that $\underline{p}$ is fixed (not $\mathbf{p}$) in $\rho^L(p)$ in the spectral energy integral of Eq. (6a). To derive Eq. (6b), one can easily verify that

$$\Delta^L(\tau,\underline{x})=\mathcal{D}^{L>}(-i\tau,\underline{x})=D^{L>}(z), \tag{C10}$$

with $z^+=-i(\tau/\sqrt{2})$, $z^3=-x^-/\sqrt{2}$, and $z^\perp=x^\perp$, if τ is in the interval $[0,\beta]$.

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