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Perturbation theory for bound states and resonances where potentials and propagators have arbitrary energy dependence

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Standard derivations of "time-independent perturbation theory" of quantum mechanics cannot be applied to the general case where potentials are energy dependent or where the inverse free Green function is a nonlinear function of energy. Such derivations cannot be used, for example, in the context of relativistic quantum field theory. Here we solve this problem by providing a new, general formulation of perturbation theory for calculating the changes in the energy spectrum and wave function of bound states and resonances induced by perturbations to the Hamiltonian. Although our derivation is valid for energy-dependent potentials and is not restricted to inverse free Green functions that are linear in the energy, the expressions obtained for the energy and wave function corrections are compact, practical, and maximally similar to the ones of quantum mechanics. For the case of relativistic quantum field theory, our approach provides a direct covariant way of obtaining corrections to wave functions that are not in the center of mass frame.

I. INTRODUCTION

There is a growing interest in calculations, within a covariant quantum field theory framework, of changes in the properties of bound states and resonances induced by small perturbations in the interaction Hamiltonian. The four-dimensional Bethe-Salpeter equation and its various three-dimensional reductions (so-called quasipotential equations) are the most popular tools in this respect. A current example is the Nambu–Jona-Lasinio (NJL) model where the nucleon is described in terms of three relativistic quarks interacting via contact potentials, and where meson exchange provides an important perturbative correction [1]. Another example is provided by relativistic calculations of hadronic atoms where the strong interaction perturbs the Coulomb bound state [2,3], and yet another by various other corrections to relativistic calculations of electromagnetic bound states [4].

The perturbation problem involved in such covariant calculations can be formulated as follows. Denoting the total four-momentum of the system by P, one would like to determine the bound state solution of the equation

$$[G_0^{-1}(P) - K_0(P) - K_1(P)]\Psi = 0, \qquad (1)$$

where $K_1(P)$ is a perturbation to the unperturbed kernel $K_0(P)$, and where it is assumed that the unperturbed Green function $G_u(P)$, defined as the solution to the equation

$$G_u(P) = G_0(P) + G_0(P)K_0(P)G_u(P) \qquad (2)$$

is known completely.[1] Thus we seek the mass M and wave function Ψ such that Eq. (1) with $P^2 = M^2$ is satisfied. A consequence of the complete knowledge of $G_u(P)$ is that the mass spectrum $M_u^n (n = 1,2,3,\dots)$ and corresponding wave functions Φ_n of the unperturbed equation,

$$[G_0^{-1}(P) - K_0(P)]\Phi_n = 0, \qquad (3)$$

where $P^2 = (M_u^n)^2$, are known.

The task of solving Eq. (1) by expressing the mass M and wave function Ψ as a perturbation series with respect to K_1 is a problem whose solution is well known in the corresponding context of nonrelativistic quantum mechanics (given by so-called time-independent perturbation theory). Unfortunately the (textbook) derivation used to obtain the quantum mechanical result is restricted to the case where the inverse free Green function $G_0^{-1}(P)$ is linearly dependent on energy P_0 and where the unperturbed kernel K_0 is an energy-independent Hermitian operator. Although these restrictions lead to the closure and orthonormality conditions

$$\bar{\Phi}_n \Phi_m = \delta_{nm}, \quad \sum_n \Phi_n \bar{\Phi}_n = 1, \qquad (4)$$

which are crucial for the derivation of time-independent perturbation theory, they are not valid in the Bethe-Salpeter case. Indeed none of these restrictions is required in the context of a covariant field theoretic approach. In this paper we

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[1] For simplicity of presentation we generally do not show spin or relative momentum variables; similarly, identical particle factors and all sums and integrals over intermediate state variables are suppressed.

therefore present a new solution to the perturbation problem which is valid for any form of $G_0^{-1}(P)$ and $K_0(P)$; in particular, our solution is valid for the case of covariant field theoretic approaches where $G_0^{-1}(P)$ depends nonlinearly on P_0 and where $K_0(P)$ can be energy (P_0) dependent. Our solution, given in Eq. (24) and Eq. (25) for the nondegenerate case, and in Eq. (43) and Eq. (46) for the degenerate case, expresses the mass M of the bound state or resonance and the corresponding wave function Ψ in terms of compact expressions that take into account the perturbation term K_1 to any order. At the same time, our formulation allows us to write the perturbation series for both M and Ψ, up to any order, in a straightforward way which is maximally close to the analogous quantum mechanical formulation. A further important aspect of our approach is that it is manifestly covariant. This feature enables the direct use of the perturbation series for Ψ also in cases where the bound state or resonance is not at rest. In this way the more involved approach of Lorentz boosting wave functions calculated perturbatively in the rest frame, can be avoided. As such, our approach to the perturbation problem where no restriction is put on the energy dependence of kernels and inverse free Green functions, may provide some important advantages over previous formulations [5,6,2].

II. PERTURBATION THEORY

A. Basic equations

In this paper we use the framework of relativistic quantum field theory to illustrate our approach to perturbation theory. Although this is done partly for presentational purposes—it is a particular case where the kernel is energy dependent and where the inverse Green function is nonlinearly dependent on energy—it is also a particularly topical case, as discussed in the Introduction. On the other hand, we emphasize that our approach to perturbation theory does not depend on the particular theoretical framework in which the bound state problem is set—it can be that of nonrelativistic quantum mechanics, relativistic quantum field theory, three-dimensional relativistic quasipotential equations, etc. Similarly, our approach does not depend on the functional form taken by the energy dependence of either the kernel or the inverse free Green function. All we need to assume is the usual overall structure of the dynamical equations involved, as exemplified by Eq. (1) and Eq. (2).

We thus consider the Green function

$$G(P) = G_0(P) + G_0(P)K(P)G(P), \qquad (5)$$

where P is the total four-momentum, G_0 is the fully disconnected part of G, and where the kernel K consists of a part K_0 for which the corresponding Green function is known, and a small part K_1 which can be treated as a perturbation. Thus

$$K(P) = K_0(P) + K_1(P), \qquad (6)$$

and it is assumed that the unperturbed Green function $G_u(P)$ has been previously determined by solving Eq. (2). We are interested in the case where $G_u(P)$ has a pole corresponding to a bound or resonance state. Thus we can write

$$G_u(P) = \frac{i\Phi(P)\bar{\Phi}(P)}{P^2 - M_u^2} + G_u^b(P), \qquad (7)$$

where the wave functions $\Phi(P)$ and $\bar{\Phi}(P)$, the unperturbed bound state mass M_u, and the background term $G_u^b(P)$ are all assumed to be known.[2] In this respect it is worth noting that the pole term of Eq. (7) is separable with respect to initial and final state variables, thus for a two-body system $\bar{\Phi}(P) \equiv \bar{\Phi}(P,p)$ is a function of the initial relative momentum p while $\Phi(P) \equiv \Phi(P,p')$ is a function of the final relative momentum p'. Note also, that as $P \to \bar{P}_u$, where $\bar{P}_u$ is any four-vector such that $\bar{P}_u^2 = M_u^2$, the wave functions $\Phi(P)$ and $\bar{\Phi}(P)$ must reduce to the respective solutions of the bound state equations

$$\Phi(\bar{P}_u) = G_0(\bar{P}_u)K_0(\bar{P}_u)\Phi(\bar{P}_u);$$

$$\bar{\Phi}(\bar{P}_u) = \bar{\Phi}(\bar{P}_u)K_0(\bar{P}_u)G_0(\bar{P}_u). \qquad (8)$$

Although $\Phi(\bar{P}_u)$ and $\bar{\Phi}(\bar{P}_u)$ are therefore specified as the solutions of the above bound state equations, for momenta P not on the mass shell, $P^2 \neq M_u^2$, there is no unique way to define $\Phi(P)$ [and therefore $\bar{\Phi}(P)$] since any definition can be adopted in Eq. (7) with an appropriate redefinition of the background term $G_u^b(P)$. Here we shall choose $\Phi(P)$ to be a Lorentz covariant function of the total momentum P, the relative momenta, and the spinor indices of the constituents [i.e. $\Phi(P)$ is covariant under the simultaneous transformation of all these variables]. The way to construct such a $\Phi(P)$ will be discussed below. Since the full unperturbed Green function $G_u(P)$ is a Lorentz covariant function of its variables from the outset, the Lorentz covariance of the background term $G_u^b(P)$ is therefore assured.

Once the perturbation K_1 is included, the mass M_u will shift to the physical value M and $\Phi(P)$ will modify to the wave function $\Psi(P)$ where

$$G(P) = \frac{i\Psi(P)\bar{\Psi}(P)}{P^2 - M^2} + \hat{G}^b(P). \qquad (9)$$

The wave functions $\Psi(P)$ and $\bar{\Psi}(P)$ are likewise assumed to be covariant functions which reduce in the limit $P \to \bar{P}$, where $\bar{P}^2 = M^2$, to the respective solutions of the bound state equations

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[2]Here, for simplicity, we assume that the bound state is nondegenerate—the degenerate case is considered in detail in the next subsection. Also, here and elsewhere, all references to a "bound state" should be understood to include the case of a "resonance state."

$$\Psi(\bar{P}) = G_0(\bar{P}) K(\bar{P}) \Psi(\bar{P}), \quad \bar{\Psi}(\bar{P}) = \bar{\Psi}(\bar{P}) K(\bar{P}) G_0(\bar{P}). \tag{10}$$

To write a perturbation series for G, we express G in terms of the known unperturbed Green function G_u through the equation

$$G(P) = G_u(P) + G_u(P) K_1(P) G(P), \tag{11}$$

which follows from the fact that $G^{-1} = G_0^{-1} - K$ and $G_u^{-1} = G_0^{-1} - K_0$. By iterating Eq. (11) we obtain a perturbation series for $G(P)$ with respect to the perturbation $K_1(P)$. What appears more difficult is to find a corresponding perturbation series for the mass M and wave function Ψ. Yet if one closely examines the structure of the above equations, it can be discovered that a mathematically similar problem was solved long ago by Feshbach [7] albeit in the rather different context of nuclear reaction theory. Indeed there are a number of other contexts where analogous problems have been solved, the case of mass and vertex renormalization in pion-nucleon scattering being particularly noteworthy [8]. In the next section we therefore use the method of Feshbach to derive the solution of our covariant perturbation theory problem.

B. Solution

In this subsection we derive expressions for the bound state wave functions Ψ, $\bar{\Psi}$, and the bound state mass M corresponding to the full kernel K of Eq. (6). Although our goal is to formulate the covariant perturbation theory for this problem, we in fact derive expressions for Ψ, $\bar{\Psi}$, and M that are exact with all orders of K_1 being taken into account. Starting from these exact expressions it is then trivial to generate all terms of the perturbation series. To present our solution it will be convenient to discuss the cases of nondegenerate and degenerate states, separately.

1. Nondegenerate case

In the nondegenerate case, to each unperturbed bound state mass M_u there corresponds a unique bound state wave function Φ. The unperturbed Green function $G_u(P)$ then has the "pole plus background" structure, as given in Eq. (7). Having in mind that the full Green function $G(P)$ has a similar structure to that given in Eq. (9), and that our goal is to relate the quantities in these two expressions, we begin by introducing a "background" Green function $G^b(P)$ defined as the solution of the equation

$$G^b(P) = G_u^b(P) + G_u^b(P) K_1(P) G^b(P). \tag{12}$$

Note that $G^b(P) \neq \hat{G}^b(P)$ where $\hat{G}^b(P)$ was defined in Eq. (9). From Eq. (12) it follows that

$$(1 + G^b K_1)^{-1} G^b = G_u^b, \tag{13}$$

where we have dropped the momentum arguments for convenience. Similarly Eq. (11) implies

$$G(1 + K_1 G)^{-1} = G_u. \tag{14}$$

Subtracting the last two equations, we obtain

$$G(1 + K_1 G)^{-1} - (1 + G^b K_1)^{-1} G^b = \frac{i\Phi\bar{\Phi}}{P^2 - M_u^2} \tag{15}$$

and therefore

$$(1 + G^b K_1) G - G^b (1 + K_1 G)$$
$$= (1 + G^b K_1) \frac{i\Phi\bar{\Phi}}{P^2 - M_u^2} (1 + K_1 G). \tag{16}$$

Thus

$$G = G^b + \frac{(1 + G^b K_1) i\Phi\bar{\Phi}(1 + K_1 G)}{P^2 - M_u^2}, \tag{17}$$

which can be solved for $\bar{\Phi}(1 + K_1 G)$ by writing

$$\bar{\Phi}(1 + K_1 G) = \bar{\Phi}(1 + K_1 G^b)$$
$$+ \frac{\bar{\Phi} K_1(1 + G^b K_1) i\Phi\bar{\Phi}(1 + K_1 G)}{P^2 - M_u^2}, \tag{18}$$

and then

$$\bar{\Phi}(1 + K_1 G) = \left[1 - \frac{i\bar{\Phi}(K_1 + K_1 G^b K_1)\Phi}{P^2 - M_u^2} \right]^{-1} \bar{\Phi}(1 + K_1 G^b). \tag{19}$$

Using this result in Eq. (17) we obtain the result we are seeking:

$$G(P) = \frac{i\psi(P)\bar{\psi}(P)}{P^2 - M_u^2 - i\bar{\Phi}(P)[K_1(P) + K_1(P) G^b(P) K_1(P)]\Phi(P)} + G^b(P), \tag{20}$$

where the functions $\psi(P)$ and $\bar{\psi}(P)$ are defined by

$$\psi(P) = [1 + G^b(P) K_1(P)]\Phi(P), \tag{21}$$

$$\bar{\psi}(P) = \bar{\Phi}(P)[1 + K_1(P)G^b(P)]. \qquad (22)$$

A comparison of Eq. (20) with Eq. (9) shows that $\Psi(\bar{P}) = \sqrt{Z}\psi(\bar{P})$, and $\bar{\Psi}(\bar{P}) = \sqrt{Z}\bar{\psi}(\bar{P})$, where

$$Z = \frac{1}{1 - i\{\bar{\Phi}(P)[K_1(P) + K_1(P)G^b(P)K_1(P)]\Phi(P)\}'}\Bigg|_{P^2 = \bar{P}^2 = M^2}, \qquad (23)$$

with the prime indicating a derivative with respect to P^2, and

$$M^2 = M_u^2 + i\bar{\Phi}(\bar{P})[K_1(\bar{P}) + K_1(\bar{P})G^b(\bar{P})K_1(\bar{P})]\Phi(\bar{P}). \qquad (24)$$

In this respect it is worth noting that because all our wave functions and Green functions are Lorentz covariant, the quantity in the curly brackets of Eq. (23) [which also appears in Eq. (24)], is a Lorentz scalar depending only on P^2.

Thus, in the nondegenerate case, the properly normalized wave functions for the full perturbation theory are

$$\Psi(\bar{P}) = (1 - i\{\bar{\Phi}(\bar{P})[K_1(\bar{P}) + K_1(\bar{P})$$
$$\times G^b(\bar{P})K_1(\bar{P})]\Phi(\bar{P})\}')^{-1/2}$$
$$\times [1 + G^b(\bar{P})K_1(\bar{P})]\Phi(\bar{P}), \qquad (25)$$

$$\bar{\Psi}(\bar{P}) = \bar{\Phi}(\bar{P})[1 + K_1(\bar{P})G^b(\bar{P})]$$
$$\times (1 - i\{\bar{\Phi}(\bar{P})[K_1(\bar{P}) + K_1(\bar{P})$$
$$\times G^b(\bar{P})K_1(\bar{P})]\Phi(\bar{P})\}')^{-1/2}. \qquad (26)$$

We note that these wave functions satisfy the normalization condition

$$i\bar{\Psi}(P) \frac{\partial G^{-1}(P)}{\partial P^2} \Psi(P)\Bigg|_{P=\bar{P}} = 1. \qquad (27)$$

2. Reference frame dependence of the wave functions

As far as we know, all previous attempts at developing perturbation theory for relativistic systems have considered bound states only at rest (see e.g. Ref. [5]). On the other hand, for observables involving scattering off the bound state (e.g. electromagnetic form factors) taking into account the total momentum dependence of the bound state wave function is important. In the relativistic case there are some subtleties in the determination of this dependence perturbatively and at the same time in a manifestly covariant way. One possible way to do this is to derive the wave function to the needed order in the rest reference frame, and then to boost it in order to give it the desired momentum. There are two disadvantages to this approach: one is that it involves two separate steps—the perturbation expansion and the boosting. The second disadvantage is that the unit vector $n = \bar{P}/M$, which determines the boost [14], itself may need to

be calculated perturbatively. To illustrate this, we consider the determination of a scalar bound state wave function $\Psi(\bar{P})$ to first order in the perturbation. Showing explicitly one relative momentum p in addition to the total on-shell momentum $\bar{P}$, we first write the perturbed wave function as a boosted wave function at rest:

$$\Psi(\bar{P},p) = S_{L_n}\Psi(L_n\bar{P},L_np) = S_{L_n}\Psi_0(L_np), \qquad (28)$$

where L_n is the boost Lorentz transformation, $L_n\bar{P} = (M,\mathbf{0})$, S_{L_n} is the associated transformation matrix acting on the spin indices of the constituents, and $\Psi_0(q)$ is the bound state wave function at rest. The next step is to calculate $\Psi_0(q)$ to first order in the perturbation: $\Psi_0(q) = (1 + \eta_1)\Phi_0(q)$, where the first-order correction factor η_1 is given explicitly in Eq. (57). Thus

$$\Psi(\bar{P},p) = S_{L_n}(1 + \eta_1)\Phi_0(L_np). \qquad (29)$$

Since L_n is a function of the unit vector $n = \bar{P}/M = (\sqrt{\mathbf{P}^2 + M^2},\mathbf{P})/M$, and therefore of M, and because we need $\Psi(\bar{P},p)$ up to first order, the mass M should be approximated up to first order in the perturbation. Denoting the first-order perturbation correction to M^2 by δ_1 [given explicitly in Eq. (53)], the approximation $n(M) \approx n(M_u + \delta_1/2M_u)$ should thus be used in Eq. (29) with a subsequent expansion of the resulting $\Psi(\bar{P},p)$ up to first order in δ_1. If admixtures of higher-order corrections were acceptable, then this last expansion could be neglected.

In what follows we show a more straightforward way to obtain the perturbed wave function $\Psi(\bar{P})$ when $\mathbf{P} \neq 0$. For this purpose we shall require $G^b(P)$, which determines the wave function via Eq. (25), to be Lorentz covariant; that is, we would like it to transform kinematically under any Lorentz transformation L of the momenta involved, as $G^b(P;p',p) = S_L G^b(LP;Lp',Lp)S_L^\dagger$ where p and p' are the initial and final relative momenta. In order for this to be satisfied, $G_u^b(P;p',p)$ should also be Lorentz covariant in view of Eq. (12). Using the definition (7) for $G_u^b(P;p',p)$ one can see that the unperturbed wave function $\Phi(P,p)$ should also be a Lorentz covariant function under any Lorentz transformation L of P and p.

Thus the essential problem is a practical one: how to construct a wave function $\Phi(P,p)$ that is Lorentz covariant, and which satisfies the bound state equation [first of Eqs. (8)] for any P such that $P^2 = M_u^2$. For this purpose it is useful to have

a separate notation for the bound state wave functions, so to this end we denote by $\widetilde{\Phi}(\bar{P}_u,p)$ all the solutions of the bound state equation [first of Eqs. (8)] for which the total momentum $\bar{P}_u$ has the property $\bar{P}_u^2 = M_u^2$. We then note that one cannot simply define $\Phi(P,p) = \widetilde{\Phi}(\bar{P}_u,p)$ where $P = (P_0,\mathbf{P})$ and $\bar{P}_u = (\sqrt{\mathbf{P}^2 + M_u^2},\mathbf{P})$, so that $\Phi(P,p)$ does not depend on P_0—such a $\Phi(P,p)$ cannot be Lorentz covariant since a Lorentz transformation will change this function to $S_L\Phi(LP,Lp)$ which will necessarily depend on the (arbitrary) value of P_0 (the three-vector part of LP depends on P_0).

To make progress, we note that the bound state wave function $\widetilde{\Phi}(\bar{P}_u,p)$ is covariant under the transformation $\bar{P}_u \to L\bar{P}_u$, $p \to Lp$:

$$\widetilde{\Phi}(\bar{P}_u,p) = S_L\widetilde{\Phi}(L\bar{P}_u,Lp). \tag{30}$$

Since this is true for *any* four-vector $\bar{P}_u$ satisfying $\bar{P}_u^2 = M_u^2$, it will certainly be true for the four-vector $M_u P/\sqrt{P^2}$ where P is arbitrary. Thus, if we define wave function $\Phi(P,p)$ as

$$\Phi(P,p) = \widetilde{\Phi}\left(\frac{M_u P}{\sqrt{P^2}},p\right), \tag{31}$$

it immediately follows that

$$\Phi(P,p) = S_L\Phi(LP,Lp), \tag{32}$$

which is the statement that wave function $\Phi(P,p)$ is Lorentz covariant in the way we need. In this way we have constructed a wave function $\Phi(P,p)$ that satisfies the sought-after Lorentz covariance, while at the same time reducing to the bound state wave function $\widetilde{\Phi}(\bar{P}_u,p)$ as $P \to \bar{P}_u$ [in fact $\Phi(P,p)$, as defined by Eq. (31), is the bound state wave function with total momentum $M_u P/\sqrt{P^2}$]. By choosing the form of $\Phi(P)$ given in Eq. (31), we guarantee that Eq. (7) is expressed in a manifestly covariant way. The immediate consequence of this is that the exact wave function $\Psi(P)$ is given, up to a scalar normalization, in a manifestly covariant way by Eq. (21), and so is each term in Eq. (56) corresponding to any order of perturbation theory for Ψ. The same is valid for the denominator of Eq. (20), Eq. (24) for the mass, and the expression for the renormalization constant (23). If instead we had chosen $\Phi(P)$ to transform differently from Eq. (32), even the fact that the solution of Eq. (24) does not depend on $\mathbf{P}$ would be hidden.

3. Degenerate case

In the degenerate case there is more than one solution Φ of the unperturbed bound state equation, Eq. (8), for a single unperturbed bound state mass M_u. Assuming an r-fold degeneracy, we denote such wave function solutions as Φ_j where $j = 1,2,3,\ldots,r$. In this case the pole structure of the unperturbed Green function $G_u(P)$ is easily seen to be

$$G_u(P) = \frac{i\sum_j \Phi_j(P)\bar{\Phi}_j(P)}{P^2 - M_u^2} + G_u^b(P). \tag{33}$$

As for the nondegenerate case, we shall assume our wave functions to be covariant but not dependent on P^2. The wave functions Φ_j are, by the assumption of r-fold degeneracy, linearly independent. Applying this fact to the pole structure of the identity $G_u G_u^{-1} G_u = G_u$, we obtain the normalization condition for these wave functions:

$$i\bar{\Phi}_i \frac{\partial G_u^{-1}(P)}{\partial P^2}\Phi_j\bigg|_{P = \bar{P}_u} = \delta_{ij}. \tag{34}$$

Equation (33) can be written exactly as Eq. (7) with Φ now defined to be a row matrix whose elements are the Φ_j:

$$\Phi \equiv \begin{pmatrix} \Phi_1 & \Phi_2 & \Phi_3 & \cdots & \Phi_r \end{pmatrix}, \tag{35}$$

with $\bar{\Phi}$ being the corresponding column matrix with elements $\bar{\Phi}_j$. With this redefinition of Φ and $\bar{\Phi}$, the above derivation for the nondegenerate case remains valid up to and including Eq. (22). In this way we obtain, for the degenerate case, that

$$G(P) = i\psi(P)A^{-1}(P)\bar{\psi}(P) + G^b(P), \tag{36}$$

where ψ and $\bar{\psi}$ are row and column matrices defined by elements

$$\psi_j(P) = [1 + G^b(P)K_1(P)]\Phi_j(P),$$

$$\bar{\psi}_j(P) = \bar{\Phi}_j(P)[1 + K_1(P)G^b(P)], \tag{37}$$

respectively, and A is an $r \times r$ matrix whose elements are

$$A_{ij}(P) = (P^2 - M_u^2)\delta_{ij} - i\bar{\Phi}_i(P)$$
$$\times [K_1(P) + K_1(P)G^b(P)K_1(P)]\Phi_j(P). \tag{38}$$

We are interested in the masses M for which the Green function $G(P)$ of Eq. (36) develops a bound state or resonance pole. This will happen when the determinant of matrix $A(P)$ becomes zero. This, in turn, can be determined by finding the matrix $S(P)$ which diagonalizes $A(P)$. With $S(P)$ determined, we have that

$$D(P) \equiv S^{-1}(P)A(P)S(P)$$

$$= \begin{pmatrix} D_1(P) & 0 & 0 & \cdots & 0 \\ 0 & D_2(P) & 0 & \cdots & 0 \\ 0 & 0 & D_3(P) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & D_r(P) \end{pmatrix},$$
$$\tag{39}$$

$$G(P) = i\psi^S(P)D^{-1}(P)\bar{\psi}^S(P) + G^b(P), \qquad (40)$$

where

$$D_{ij}(P) = (P^2 - M_u^2)\delta_{ij} - i\bar{\Phi}_i^S(P)$$
$$\times [K_1(P) + K_1(P)G^b(P)K_1(P)]\Phi_j^S(P), \qquad (41)$$

$$\psi^S(P) \equiv \psi(P)S(P), \quad \bar{\psi}^S(P) \equiv S^{-1}(P)\bar{\psi}(P), \qquad (42)$$

with similar definitions holding for $\Phi^S(P)$ and $\bar{\Phi}^S(P)$. Since $\det D(P) = \Pi_j D_j(P) = 0$, the Green function $G(P)$ will have poles at $P^2 = M_j^2$, $j = 1,2,3,\ldots,r$, where M_j is the solution of the equation

$$M_j^2 = M_u^2 + i\bar{\Phi}_j^S(P_j)$$
$$\times [K_1(P_j) + K_1(P_j)G^b(P_j)K_1(P_j)]\Phi_j^S(P_j), \qquad (43)$$

with P_j being any momentum satisfying $P_j^2 = M_j^2$, and the functions $\bar{\Phi}_j^S(P)$ and $\Phi_j^S(P)$ being the jth elements of $\bar{\Phi}^S(P)$ and $\Phi^S(P)$, respectively.

Taking into account the diagonal nature of $D(P)$, Eq. (40) can be written as

$$G(P) = i\sum_j \psi_j^S(P)D_j^{-1}(P)\bar{\psi}_j^S(P) + G^b(P). \qquad (44)$$

Thus, assuming that the perturbed bound state mass M_j is itself nondegenerate, we can find its corresponding wave function Ψ_j as in the nondegenerate case above: $\Psi_j = \sqrt{Z_j}\psi_j^S(P_j)$, where

$$Z_j = \frac{1}{1 - i\{\bar{\Phi}_j^S(P_j)[K_1(P_j) + K_1(P_j)G^b(P_j)K_1(P_j)]\Phi_j^S(P_j)\}'}. \qquad (45)$$

Thus, in the degenerate case of the unperturbed theory, the properly normalized wave functions corresponding to the (nondegenerate) bound state mass M_j of the full perturbation theory are

$$\Psi_j = \sqrt{Z_j}[1 + G^b(P_j)K_1(P_j)]\Phi_j^S(P_j), \qquad (46)$$

$$\bar{\Psi}_j = \sqrt{Z_j}\bar{\Phi}_j^S(P_j)[1 + K_1(P_j)G^b(P_j)]. \qquad (47)$$

4. Comments

The main results of this subsection are the expressions for M^2 and Ψ given in the nondegenerate case by Eq. (24) and Eq. (25), and in the degenerate case by Eq. (43) and Eq. (46), respectively. Not only are these expressions exact and compact, but they can also be easily used to write down the explicit perturbation series for these quantities. For this purpose it is most convenient to treat all functions of P as functions of P^2 and the unit four-vector $n = P/\sqrt{P^2}$, and at the same time to use the covariant form for the unperturbed wave function given by Eq. (31), since then Φ will not depend on P^2. For example, in the nondegenerate case, to generate the perturbation series for M^2 we use Eq. (12) to write Eq. (24) as an infinite series

$$M^2 = M_u^2 + i\bar{\Phi}[\tilde{K}_1 + \tilde{K}_1\tilde{G}_u^b\tilde{K}_1 + \tilde{K}_1\tilde{G}_u^b\tilde{K}_1\tilde{G}_u^b\tilde{K}_1$$
$$+ \tilde{K}_1\tilde{G}_u^b\tilde{K}_1\tilde{G}_u^b\tilde{K}_1\tilde{G}_u^b\tilde{K}_1 + \ldots]\Phi, \qquad (48)$$

where a tilde over K_1 or G_u^b indicates that this quantity is evaluated at $P^2 = M^2$. By making Taylor series expansions

$$\tilde{K}_1 = K_1 + \delta K_1' + \frac{\delta^2}{2!}K_1'' + \ldots \qquad (49)$$

$$\tilde{G}_u^b = G_u^b + \delta G_u^{b\prime} + \frac{\delta^2}{2!}G_u^{b\prime\prime} + \ldots, \qquad (50)$$

where

$$\delta \equiv M^2 - M_u^2 \qquad (51)$$

and each term without a tilde is evaluated at $P^2 = M_u^2$, we can immediately write M^2 as a perturbation series with respect to orders of $K_1 \equiv K_1(M_u)$:

$$M^2 = M_u^2 + \delta_1 + \delta_2 + \delta_3 + \ldots, \qquad (52)$$

where

$$\delta_1 = i\bar{\Phi}K_1\Phi \qquad (53)$$

$$\delta_2 = i\bar{\Phi}[\delta_1 K_1' + K_1 G_u^b K_1]\Phi \qquad (54)$$

$$\delta_3 = i\bar{\Phi}\left[\delta_2 K_1' + \frac{\delta_1^2}{2} K_1'' + \delta_1 (K_1 G_u^b K_1)'\right.$$

$$\left. + K_1 G_u^b K_1 G_u^b K_1\right]\Phi \tag{55}$$

etc.

Similarly, the wave function of Eq. (25) can be written as a perturbation series in orders of K_1:

$$\Psi = (1 + \eta_1 + \eta_2 + \eta_3 + \ldots)\Phi, \tag{56}$$

where

$$\eta_1 = \frac{1}{2}\Delta_1 + G_u^b K_1 \tag{57}$$

$$\eta_2 = \frac{1}{2}\Delta_2 + \frac{3}{8}\Delta_1^2 + \delta_1 (G_u^b K_1)' + \eta_1 G_u^b K_1 \tag{58}$$

$$\eta_3 = \frac{1}{2}\Delta_3 + \frac{3}{4}\Delta_1\Delta_2 + \frac{15}{48}\Delta_1^3 + (\delta_2 + \delta_1\eta_1)(G_u^b K_1)'$$

$$+ \frac{1}{2}\delta_1^2 (G_u^b K_1)'' + \eta_2 G_u^b K_1 \tag{59}$$

etc., where Δ_i is derived from δ_i by putting an extra derivative on each K_1 and G_u^b; that is,

$$\Delta_1 = i\bar{\Phi} K_1' \Phi \tag{60}$$

$$\Delta_2 = i\bar{\Phi}[\delta_1 K_1'' + (K_1 G_u^b K_1)']\Phi \tag{61}$$

$$\Delta_3 = i\bar{\Phi}\left[\delta_2 K_1'' + \frac{\delta_1^2}{2} K_1''' + \delta_1 (K_1 G_u^b K_1)''\right.$$

$$\left. + (K_1 G_u^b K_1 G_u^b K_1)'\right]\Phi \tag{62}$$

etc.

A similar procedure can be used to generate the perturbation series for the degenerate case.

It is worth noting that the perturbative corrections to the bound state wave function, as derived here, are particularly important to take into account when calculating corrections to vertices (electromagnetic, axial, etc.) within constituent models. It is only by taking into account the appropriate order of wave function perturbation exactly will symmetry properties, such as, for example, gauge invariance, be preserved at each order in the vertex correction—for a concrete example, see Ref. [10] where Eq. (57) was used to determine the full lowest order pionic correction to the nucleon vertex function in the NJL model.

It is also worth pointing out that in the case where the perturbation K_1 is too large for a perturbative treatment, our expressions of Eq. (24), Eq. (25), Eq. (43), and Eq. (46) may still be useful for performing practical nonperturbative calculations of M^2 and Ψ. Indeed, in both the degenerate and

nondegenerate cases, the main calculational effort would be in solving Eq. (12) for the "background" Green function G^b. Yet this is an especially simple equation, of standard Lippmann-Schwinger form, where G^b has no pole at $P^2 = M^2$ and G_u^b has no pole at $P^2 = M_u^2$ (since they have been subtracted), and where there are no singularities in the integration over momenta. Even in the unlikely event that G_u^b happens to have an unsubtracted pole close to $P^2 = M^2$, this case can be easily handled numerically. Finally, it is useful to note that G_u^b has already been constructed for the important case of the nonrelativistic Coulomb problem by Schwinger [11]—a result that can be easily adapted to the relativistic Coulomb case [2].

III. DISCUSSION AND SUMMARY

In this work we have presented a general formulation of perturbation theory applicable to bound states and resonances where the bound state equations involve kernels and inverse free Green functions that have an arbitrary energy dependence. Our formulation is thus directly applicable to the important case of relativistic quantum field theory. One can consider our results as extending the well-known time-independent perturbation theory of quantum mechanics to the case where the kernels are energy dependent and where the inverse propagators are nonlinear in the energy.

In particular, we have derived expressions for the bound state (or resonance) mass M and wave function Ψ of a system whose interaction kernel K consists of a part K_0 for which the corresponding Green function G_u is known, and a part K_1 which plays the role of a perturbation. Our results for M and Ψ are contained in Eq. (24) and Eq. (25) for the nondegenerate case, and in Eq. (43) and Eq. (46) for the degenerate case, and have the feature that they are exact, with the perturbation K_1 taken into account to all orders. The key element in these expressions is the Green function G^b which needs to be found by solving Eq. (12). For sufficiently small K_1, Eq. (12) can be solved simply by iteration, in this way generating a perturbation expansion in K_1 that is the analogue of the time-independent perturbation theory of quantum mechanics. On the other hand, if K_1 is not small enough to generate a convergent perturbation series, Eq. (12) could still be solved by standard numerical techniques for integral equations.

As far as we know, our formulation of the perturbation theory problem is new. However, there are a few alternative formulations available in the literature, all presented for the particular case of relativistic quantum field theory. The first of these is a method where the perturbation series for M^2 and Ψ are expressed in terms of contour integrals. Originally developed by Kato [9] and described in Messiah's standard text [12] for the case of quantum mechanics, the contour method was extended to the covariant case by Lepage [5] and used, for example, by Murato [13]. Another method, of Bodwin and Yennie [6], is closest in spirit to our approach, but does not have the feature of having closed expressions for the perturbed mass and wave function. A third approach is the recent formulation of Ivanov *et al.* [2] whose perturbative expansion is expressed in terms of a certain "relativ-

istic generalization of a projection operator.'' In this approach the second derivative of the inverse free propagator, $\partial^2 G_0^{-1}/\partial E^2$, looks very much like a genuine and necessary relativistic feature, yet it does not appear in our formulation at all and is thus just an artifact of the particular derivation used. Similarly, the expression for the lowest-order wave function correction derived directly from Eq. (9) of Ref. [2] contains four terms against our only one.

In each of the above three alternative approaches, perturbative corrections to the bound state wave function were derived only for the special case where the bound state is at rest. Thus, in order to describe a scattering process where the bound state has nonzero total momentum, such wave function corrections need to be modified by the appropriate Lorentz boost (that itself depends on the order of perturbation being considered). By contrast, our approach has enabled us to write expressions for the bound state wave function corrections that are Lorentz covariant at each order of the perturbation, thus avoiding the step of boosting from the rest frame. Although all perturbation expansions must mathematically be identical, it is evident that the expressions provided by our Eq. (24), Eq. (25), Eq. (43), and Eq. (46) are the simplest both practically and conceptually.

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