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Hamiltonian Analysis of Modified Gravitational Theories: Towards a Renormalizable Theory of Gravity

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ACADEMIC DISSERTATION

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Abstract

All the fundamental interactions except gravity have been successfully described in the framework of quantum field theory. Construction of a consistent quantum theory of gravity remains a challenge, because the general theory of relativity is not renormalizable. We consider gravitational theories that aim to improve the ultraviolet behavior of general relativity. The main tool of our analysis is the Hamiltonian formulation of theories that possess local (gauge) invariances.

Hořava-Lifshitz gravity achieves power-counting renormalizability by assuming that space and time scale anisotropically at high energies. At long distances the theory flows to an effective theory that is relativistically invariant. We propose a generalization of this theory. Motivated by cosmology, the modified $F(R)$ Hořava-Lifshitz gravity is constructed. It retains the renormalizability of the original Hořava-Lifshitz gravity. The Hamiltonian analysis shows that the theory contains two extra degrees of freedom compared to general relativity: one is associated with the lack of relativistic invariance at high energies and another with the presence of a second-order time derivative of the metric in the Lagrangian due to the nonlinearity of the function $F(R)$. The theory is able to describe inflation and dark energy in a unified manner without extra components. For a certain choice of parameters the theory effectively flows to the relativistic $F(R)$ gravity at long distances.

Hamiltonian analysis of the recently proposed covariant renormalizable gravity is accomplished. The structure of constraints is discovered to be very complicated, especially for the new version of the theory with improved ultraviolet behavior. Moreover, this theory is found to contain a ghost, a degree of freedom with negative energy, which destabilizes the theory.

The Hamiltonian analysis of relativistic higher-derivative gravity is revisited. Conformally invariant Weyl gravity is concluded to be the only theory of this type that could even in principle restrain the existing ghosts, since in all other potentially renormalizable cases the number of ghosts exceeds the number of local invariances.

Lastly, we investigate the idea of deriving a gravitational theory by gauging the twisted Poincaré symmetry of noncommutative spacetime.

Tiivistelmä

Kaikki luonnon perustavanlaatuiset vuorovaikutukset gravitaatiota lukuun ottamatta on kuvattu onnistuneesti kvanttikenttäteorian avulla. Gravitaatiota kuvaavan kvanttiteorian johdonmukainen määrittely on erittäin vaikeaa, koska gravitaatio eroaa sähkömagneettisesta, heikosta ja vahvasta vuorovaikutuksesta olennaisin tavoin. Gravitaation kvanttiominaisuuksien selvittäminen on välttämätöntä, jotta oppisimme ymmärtämään kuinka gravitaatio toimii alkeishiukkasten tasolla ja äärimmäisissä olosuhteissa kuten varhaisessa maailmankaikkeudessa ja mustissa aukoissa. Tässä työssä tutkitaan gravitaatiota kuvaavia teorioita, joilla pyritään muokkaamaan yleistä suhteellisuusteoriaa niin, että teoria voidaan kvantisoida johdonmukaisesti.

Hořava-Lifshitz-gravitaatio on uusi gravitaatiota kuvaava kvanttikenttäteoria. Ehdotamme teorian yleistyksen. Se säilyttää alkuperäisen teorian ominaisuudet erittäin lyhyillä etäisyyksillä, missä kvantti-ilmiöt hallitsevat. Lisäksi se kykenee kuvaamaan koko maailmankaikkeuden kiihtyvän laajenemisen ilman, että teoriaan pitäisi lisätä pimeää energiaa tai muita vastaavia komponentteja.

Toinenkin uusi gravitaatioteoria analysoidaan. Toteamme teorian sisältävän niin kutsutun haamun eli vapausasteen, jonka energia on negatiivinen. Se tekee teoriasta epävakaan. Tämän vuoksi kyseinen teoria ei voi olla oikea gravitaation kuvaus.

Tutkimme myös perinteisiä gravitaatioteorioita, jotka sisältävät korkeamman asteen derivaattoja. Weylin gravitaatioteorian sisältämien haamujen lukumäärä todetaan yhtä suureksi kuin teorian paikallisten symmetrioiden lukumäärä. Tämä saattaa mahdollistaa haamujen vakauden hallinnan ja johdonmukaisen kvanttiteorian määrittelyn.

Lopuksi tutkimme voidaanko gravitaatioteoria johtaa tekemällä epäkommutoivan aika-avaruuden kiertyneestä Poincarén symmetriasta paikallinen symmetria. Tämän todetaan edellyttävän nykyistä syvällisempää ymmärrystä symmetrioiden rakenteesta epäkommutoivassa aika-avaruudessa.

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Most importantly, I am ever grateful to my parents for everything, and to my whole family for constant support.

Kotka, June 2013

Markku Oksanen

Contents

List of original publications

This thesis includes the following six original research articles. They are given in a chronological order starting with the eldest publication.

I. *Gauging the twisted Poincaré symmetry as a noncommutative theory of gravitation*,
M. Chaichian, M. Oksanen, A. Tureanu and G. Zet,
Phys. Rev. D **79** (2009) 044016, arXiv:0807.0733 [hep-th].

II. *Modified $F(R)$ Hořava-Lifshitz gravity: a way to accelerating FRW cosmology*,
M. Chaichian, S. Nojiri, S. D. Odintsov, M. Oksanen and A. Tureanu,
Class. Quantum Grav. **27** (2010) 185021 [Erratum-ibid. **29** (2012) 159501], arXiv:1001.4102 [hep-th].

III. *Modified first-order Hořava-Lifshitz gravity: Hamiltonian analysis of the general theory and accelerating FRW cosmology in power-law $F(R)$ model*,
S. Carloni. M. Chaichian, S. Nojiri, S. D. Odintsov, M. Oksanen and A. Tureanu,
Phys. Rev. D **82** (2010) 065020 [Erratum-ibid. D **85** (2012) 129904], arXiv:1003.3925 [hep-th].

IV. *Hamiltonian analysis of non-projectable modified $F(R)$ Hořava-Lifshitz gravity*,
M. Chaichian, M. Oksanen and A. Tureanu,
Phys. Lett. B **693** (2010) 404 [Erratum-ibid. B **713** (2012) 514], arXiv:1006.3235 [hep-th].

V. *Arnowitt-Deser-Misner representation and Hamiltonian analysis of covariant renormalizable gravity*,
M. Chaichian, M. Oksanen and A. Tureanu,
Eur. Phys. J. C **71** (2011) 1657 [Erratum-ibid. C **71** (2011) 1736], arXiv:1101.2843 [gr-qc].

VI. *Higher derivative gravity with spontaneous symmetry breaking: Hamiltonian analysis of new covariant renormalizable gravity*,
M. Chaichian, J. Klusoň, M. Oksanen and A. Tureanu,
Phys. Rev. D **87** (2013) 064032, arXiv:1208.3990 [gr-qc].

The contribution of the author to the included publications

The author of this thesis had a major role in nearly every aspect and stage of the research and creation of each of the included publications. The research was conducted in four collaborations with slightly different compositions. Each of these is described separately.

In the first paper **I**, the author contributed little to the original idea behind the work, but some to the formulation of the research question. The author had a major role in carrying out the research and writing of the paper.

In papers **II** and **III**, the author contributed to the construction and identification of the generalized Hořava-Lifshitz models that were also the most suitable ones for cosmological considerations. In the parts of the papers **II** and **III** that address Hamiltonian analysis, the author contributed greatly to every aspect and stage of the work.

In papers **IV** and **V**, the author recognized and formulated the research problems together with the co-authors. Most of the research and technical derivations leading to the results was performed by the author, as well as the writing of the papers.

In paper **VI**, the author identified and formed the research problem together with the co-authors. The author made the most prominent contributions to the research, technical derivation of the results and writing of the paper. In addition to paper **VI**, some unpublished results of a work in progress with the same collaborators will be presented in Chapter 7. Those results represent the latest evolution of the work in progress, which were derived by the author.

Chapter 1

Introduction

1.1 Motivation for studying alternatives to general relativity

The general theory of relativity (GR) has been the foundation for the research of gravitational phenomena since its publication [1] in 1916. GR has been tested experimentally with great accuracy and remarkably it has passed every test to date [2]. Considering that at its time GR was rather born out of thought experiments than from a need to explain specific new observations, the experimental success of GR is no short of miraculous.[1] Nevertheless, alternatives to GR are still explored and studied actively for several good reasons. Most of these reasons are theoretical in nature, rather than phenomenological, in the sense that no experiment or observation necessarily requests modification of GR. Indeed, the GR based model of the Universe is able to describe gravitational phenomena all the way from the scale of laboratory and everyday life to the scale of the whole Universe very successfully. There are, however, reasons to suspect that GR is incomplete. One is the well known incompatibility of GR with the quantum theory. Construction of a consistent quantum theory of gravity is perhaps the most fundamental problem of current theoretical physics. Another problem of GR is the fact that according to standard cosmology no more than five percent of the energy content of the Universe consists of objects that we are able to observe by means other than gravity. According to present knowledge the newborn Universe underwent an era of exponential expansion, which is called inflation. During inflation the primordial quantum fluctuations of the energy density were produced, which are the origin for structure formation in the Universe. The following cosmological evolution is conventionally described by the ΛCDM model. In addition to usual baryonic matter, it includes dark energy in the form of cosmological constant Λ and cold dark matter (CDM). GR is consistent with observations provided that dark energy accounts for the majority of energy in

[1] Albeit the ability to correctly explain the anomalous perihelion shift of the planet Mercury served as an important qualification for the field equations of GR.

the Universe and the amount of dark matter is over five times greater than that of usual luminous matter. Since the observed effects of dark energy and dark matter have been purely gravitational so far, it is natural to ask whether their effects could be explained by a modification of GR.

Modifications of GR can be most conveniently classified according to the aspects of gravity that a given modified gravitational theory aims to improve on. Each of those aspects of gravity can be identified with a distance scale, where the associated gravitational phenomena appear. Thus it is convenient to classify modified gravitational theories according to the nature of the modification in question and the distance scales it affects. In this work we are mostly interested in the behavior of gravity in very short distances, where gravitational quantum effects are expected to become significant. But we shall also address theories which modify gravity significantly at the cosmological scale, both in early and late stages of the Universe. A modification of gravity that is relevant mostly at the scale of galaxies and galaxy systems is mentioned briefly. Since the number of known modifications of GR is quite large, the selection of these theories discussed in this work is naturally a limited one. Before we introduce those modifications of GR, we shall outline the essential role of Hamiltonian formulation of dynamics in the study of constrained dynamical systems like gravity, and physics in general.

1.2 Hamiltonian formalism

In classical dynamics, Hamiltonian formalism provides an alternative to the Lagrangian formulation of dynamics, which has vastly enriched and enlarged the applicability and scope of classical analysis of dynamical system. In modern physics, the Hamiltonian formalism is the foundation of the canonical quantization of dynamical systems. Thus the quantum theory is fundamentally rooted in the Hamiltonian formulation of dynamical systems. The full power of Hamiltonian formalism is realized in the analysis and quantization of dynamical systems that possess constraints and continuous symmetries. Such constrained systems are usually referred to as gauge theories, and the continuous symmetry associated with such a theory is called gauge invariance. All the current fundamental theories of physics are field theories that exhibit local (gauge) invariances. The Hamiltonian formalism provides a reliable and well founded formulation of gauge theories.

1.3 Quantum gravity — the shortest distance scales

All fundamental interactions except gravity are successfully described in the framework of quantum field theory (QFT). The description of the electromagnetic, weak and strong interactions is provided by the standard model of particle physics. It has been experimentally verified with great accuracy, especially by particle accelerator experiments. The recently discovered candidate for the Higgs boson will likely be the last missing piece of the standard model. More precisely the standard model is a gauge field theory, i.e., it possesses a local symmetry under *internal* gauge transformations of its fields. Its internal gauge symmetry is based on the gauge group $SU(3)_{\mathrm{C}} \times SU(2)_{\mathrm{L}} \times U(1)_{\mathrm{Y}}$, where "C" refers to the color symmetry of quarks in quantum chromodynamics, "L" refers to the doublets of left-handed fermions in the electroweak theory and "Y" refers to the weak hypercharge. The deepest understanding of the structure of such gauge theories is provided by the Hamiltonian formalism discussed in Chapter 2. Gravity as understood in GR differs greatly from the other interactions. First of all gravity is universal. In other words, it affects particles of all kinds. The dynamical quantity of GR is the metric of spacetime. Unlike for other interactions, the local symmetry associated with gravity is *external.* Namely the symmetry under diffeomorphisms of spacetime and under local Lorentz transformations of local reference frames involves the coordinates of spacetime. Therefore incorporating gravity into the standard model has not been successful.

There exist several approaches to the quantization of gravity, and to its possible integration with particle physics. Loop quantum gravity is based on the canonical formulation of GR. It replaces space with a quantized structure, a spin network, which consists of finite loops whose size is comparable to the Planck length. In string theory, a massless spin-2 particle called graviton is included in the spectrum of closed strings. It is the quantum that mediates gravity. No approach has been able to provide a fully satisfying formulation of quantum gravity to date. In this work, we approach gravity from the viewpoint of QFT and canonical formulation of modified gravitational theories.

Every consistent theory formulated in the framework of QFT has to be renormalizable in order to consistently remove the divergences that arise when physically relevant quantities are computed. Unfortunately, GR is not a renormalizable theory, when considered as a QFT in the weak-field approximation on a fixed background. This is an expected property since the gravitational coupling constant κ – related to Newton's gravitational constant G_{N} as $\kappa = 8\pi G_{\mathrm{N}}$ – has a negative mass dimension, $[\kappa] = (\mathrm{mass})^{-2}$.[2] The gravitational coupling appears in the Einstein-Hilbert (EH) action

$$S_{\mathrm{EH}}[g_{\mu\nu}] = \frac{1}{2\kappa} \int_{\mathcal{M}} \mathrm{d}^4 x \sqrt{-g}\,(R - 2\Lambda)\,, \tag{1.1}$$

where the dynamical variable is the metric $g_{\mu\nu}$ of the spacetime manifold $\mathcal{M}$, the scalar curvature defined by the metric is denoted by R, and the cosmological constant Λ is optional. Matter is described by adding an action $S_{\mathrm{mat}}[g_{\mu\nu}, \psi]$ which couples the matter fields ψ minimally to the spacetime. The field equations for the metric $g_{\mu\nu}$ are obtained as

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu}\Lambda = \kappa T_{\mu\nu}\,, \qquad T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_{\mathrm{mat}}[g, \psi]}{\delta g^{\mu\nu}}\,, \tag{1.2}$$

where $R_{\mu\nu}$ is the Ricci tensor and $T_{\mu\nu}$ is the energy-momentum tensor for the matter fields. It is indeed well known that perturbative renormalization of GR

[2]We assume units $\hbar = c = 1$, unless otherwise stated.

using dimensional regularization and the background-field method gives rise to one-loop divergences which are quadratic in the Riemann curvature tensor [3–5].[3] Pure GR without matter happens to be free of divergences at one-loop order [3], but that is merely accidental since the divergences appear for two-loops [7]. These divergences are known to arise for couplings to different kinds of physically relevant matter components: scalars, photons and spin-$\frac{1}{2}$ fermions. The appearance of divergences requires us to include invariants quadratic in the Riemann curvature tensor into the Lagrangian as counterterms. Higher-order curvature terms appear at higher loop orders.

On the other hand, the generation of curvature invariants of all possible orders into a gravitational action can be seen as the consequence of quantum fluctuations of vacuum, whenever spacetime is allowed to be curved [8].

These considerations motivate us to consider a gravitational Lagrangian consisting of the EH action and the two independent quadratic curvature invariants, chosen here as $R_{\mu\nu}R^{\mu\nu}$ and R^2,

$$S_{R^2}[g_{\mu\nu}] = \int_{\mathcal{M}} \mathrm{d}^4x\sqrt{-g}\left(\Lambda + \frac{R}{2\kappa} + \alpha R_{\mu\nu}R^{\mu\nu} + \beta R^2\right). \tag{1.3}$$

The Riemann tensor squared term can be excluded due to the Gauss-Bonnet-Chern theorem, which states that $\int_{\mathcal{M}} \mathrm{d}^4x\sqrt{-g}\left(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2\right)$ is a topological invariant. The quadratic curvature terms in the action (1.3) are known to render the theory renormalizable [9], provided the cosmological constant Λ is absent.[4] The theory is also known to be asymptotically free [10]. Renormalizability can be attained thanks to the fourth-order spacetime derivatives in the Lagrangian, which imply that the graviton propagator behaves as k^{-4} in the momentum space for high momenta k. For couplings $\alpha \neq 0$ and $\beta \neq -\alpha/3$, the action (1.3) contains eight local degrees of freedom [9, 11]: two are associated with the usual massless spin-2 graviton, one with a massive scalar, and five are associated with a massive spin-2 excitation. A major problem is that the massive spin-2 field carries a negative energy. Such a field is often called a ghost.[5] The presence of a ghost can be seen by considering the momentum-dependence of the propagator,

$$\frac{1}{k^2 + k^4/m_\alpha^2} = \frac{1}{k^2} - \frac{1}{k^2 + m_\alpha^2}, \tag{1.4}$$

where the first term represents the massless graviton, and the second term with a wrong sign is a ghost of mass squared $m_\alpha^2 \sim (2\kappa\alpha)^{-1}$. This means that the theory is unstable due to the interactions between the positive and negative energy degrees of freedom. The problem with unstable ghosts hampers field theories with higher-order time derivatives generally, unless a local symmetry prevents it. Alternatively, the presence of such interacting ghosts can be regarded to destroy

[3] The covariant Feynman rules for GR were worked out in [6].

[4] In (1.3), Λ has a different dimension and sign compared to the EH action (1.1).

[5] These ghost fields should not be confused with the friendly ghosts encountered in BRST quantization of gauge theories, nor with their predecessor the famous Faddeev-Popov ghosts. Quantization of gauge theories is discussed in Sec. 2.3.

unitarity, if the negative energy states are interpreted as positive energy states with indefinite norm. Note that such interpretation overthrows the postulate of quantum theory that states have to be normalizable. However, choosing α to be sufficiently small, the violation of unitarity can be chosen to appear at arbitrarily high energy scale. Thus the theory can be considered as an effective theory of quantum gravity.

An interesting case of curvature-squared gravity (1.3) is the conformally invariant Weyl gravity [12], whose action is the square of the Weyl tensor. (For a recent review, see [13] and references therein.) We consider Hamiltonian formulation of Weyl gravity and other renormalizable curvature-squared gravitational theories in Chapter 7.

In supersymmetric theories of gravity the problems with divergences are milder than in GR. One-loop and two-loop divergences are absent in four-dimensional supergravity. Explicit three-loop calculations have not been performed in GR nor supergravity due their extremely complicated nature. Three-loop divergences might vanish in maximally supersymmetric supergravity.

Recently, some interesting new approaches to quantum gravity have emerged. Those theories will play a central role in this work. Hořava-Lifshitz (HL) gravity [14] is a novel attempt to construct a consistent QFT of gravity. It is based on the idea that space and time scale anisotropically at very high energies, while at long distances the conventional relativistic structure of spacetime emerges. At high energies the theory is deeply nonrelativistic. This enables one to improve the ultraviolet (UV) behavior of the graviton propagator. According to the power-counting argument the Lagrangian of HL gravity possesses dimensional properties that suggest the theory is renormalizable. In order to ensure that the theory truly is renormalizable, the presence of pathologies such as unstable ghosts or strong coupling at low energies has to be ruled out. HL gravity and its Hamiltonian dynamics are presented in Chapter 4. We proposed the modified $F(R)$ HL gravity in papers **II** [15] and **III** [16]. It combines the favorable UV behavior of HL gravity and the interesting cosmological aspects of relativistic $f(R)$ gravity (see Sec. 1.4). Our modified HL theory and its Hamiltonian formulation are discussed in Chapter 5.

Covariant renormalizable gravity (CRG) [17] aims to achieve a similar UV behavior as HL gravity, but preserving relativistic invariance at the fundamental level. Lorentz invariance of the graviton propagator of CRG is, however, broken spontaneously at high energies. The advantage of CRG compared to HL gravity is the spontaneous breaking of relativistic invariance. On the other hand, Lorentz invariance could equally well be broken explicitly at high energies as long as it is restored at sufficiently low energies. A new version of CRG has been proposed [18], where a perturbative analysis around Minkowski spacetime showed that the theory is free of propagating extra degrees of freedom. However, the CRG action contains higher-order derivatives, which suggests it should exhibit extra degrees of freedom. We have studied the Hamiltonian structure and degrees of freedom of the CRG theories in papers **V** [19] and **VI** [20]. These works will be outlined in Chapter 6.

1.3.1 Noncommutative spacetime

Convincing arguments based on GR, quantum theory and string theory imply that the continuous nature of the spacetime manifold breaks down at the Planck scale around 10^{-35} metres. These arguments also indicate that the quantized structure of spacetime can be described in terms of noncommutative coordinates. This can be accomplished by defining the coordinate operators $\hat{x}^\mu$ of noncommutative spacetime to satisfy the commutation relations

$$\left[\hat{x}^\mu, \hat{x}^\nu\right] = i\theta^{\mu\nu} \,. \tag{1.5}$$

In the simplest case, $\theta^{\mu\nu}$ is an antisymmetric constant matrix of dimension length squared. The noncommutative algebra of operators generated by (1.5) can be represented on the algebra of ordinary functions on commutative spacetime via Weyl quantization. Noncommutative field theory (for reviews, see [21, 22]) provides an alternative approach for describing the quantum structure of spacetime. The discovery of the twisted Poincaré symmetry has provided a substitute for the concept of relativistic invariance to noncommutative field theory [23, 24]. Formulation of a fully consistent and viable theory of gravity on noncommutative spacetime has been proven to be a challenging problem. We believe that the construction of noncommutative gravity should be based on a guiding symmetry principle, analogous to those of classical gravitational theories such as GR and Einstein-Cartan gravity, and of gauge field theories of particle physics. In paper **I** [25], we introduced the idea that noncommutative gravity should be built as a gauge theory of the twisted Poincaré symmetry. This idea and the emerging problems are reviewed in Chapter 8.

The more general case with $\theta^{\mu\nu}$ being an antisymmetric tensor field has also been considered. For example we have studied the possibility to construct generally covariant star-products between tensor-valued differential forms [26] or Lie-valued differential forms [27], which might enable consistent formulation of gauge and gravitational theories when $\theta^{\mu\nu}$ is an antisymmetric tensor field. These works are beyond the scope of this thesis.

1.3.2 Emergent gravity

As another alternative for the regular attempts to quantize gravity, we could interpret gravity as an emergent phenomenon. Gravity could be a residual effect arising from an unknown quantum theory that does not include gravity at the fundamental level. On the other hand, gravity could be a thermodynamic (statistical) effect arising from an unknown fundamental quantum theory.

The field equations of GR have been derived locally on Rindler causal horizons as a thermodynamic equation of state [28]. This and other such intriguing connections between gravity and thermodynamics offer some support for the idea of emergent gravity.

As a recent example of an emergent theory of gravity, we mention the so-called entropic gravity hypothesis [29], where gravity emerges as an entropic force due to thermodynamics of an unknown quantum theory of holographic screens. In this reinterpretation, laws of thermodynamics are not only upheld in gravitational phenomena, but rather they are the cause of gravity. Masses tend to gravitate towards each other because that increases the entropy of the system. Thermodynamics of holographic screens can be seen as a generalization of black hole thermodynamics. In particular, the entropy of a holographic screen that coincides with the event horizon of a black hole matches the Bekenstein-Hawking entropy [30, 31]. This is the maximum amount of entropy that can be fitted into a given region of space. Microscopic origin of the black hole entropy has been studied in string theory (see [32] for a major contribution) and in other candidates of quantum gravity as well.

We have studied the relation of entropic gravity to quantum mechanics [33, 34] in the light of the GRANIT experiment, where gravitationally bound quantum states of ultra-cold neutrons have been observed for the first time [35]. Methods for observing resonance transitions between gravitationally-bound neutron states have been developed [36, 37]. We found the claimed contradiction of entropic gravity with the existence of gravitationally bound quantum states of neutrons [38] to be based on questionable assumptions. It is plausible that both entropic gravity and quantum mechanics could emerge from a theory of holographic screens in a consistent manner. In order to become a viable theory of gravity, entropic gravity will have to be able to accommodate graviton as an emergent concept, much like that in AdS/CFT duality or as phonon in solid state physics. However, detailed account of these works is beyond the scope of this thesis.

1.4 Inflation and dark energy — accelerated expansion of the Universe

At the cosmological scale the Universe is incredibly homogeneous, isotropic and spatially flat. This has been observed both by the mappings of distribution of galaxies and in particular by the mappings of the cosmic microwave background radiation (CMB): most recently the WMAP [39] and Planck [40] projects. If the observable part of Universe had been expanding by a relatively stable rate since its birth, its distant regions would have never been in causal contact with each other. Hence a thermal equilibrium could have never been attained and we would expect to find the Universe to be much more inhomogeneous and anisotropic. This is called the horizon problem. The flatness is problematic because the curvature redshifts more slowly during the expansion of the Universe than matter and radiation. Thus the energy density of the Universe would have had to be extremely close to the critical density at the beginning. Currently, the best explanation to horizon and flatness problems is provided by the idea of cosmic inflation. During inflation the infant Universe expanded exponentially by some 80 orders of magnitude in a fraction of a second, and at the end of it the "Big Bang" expansion began. This means our whole observable Universe used to be a small causally connected region which was blown up to a huge size during inflation. The exponential expansion during inflation ironed out any inhomogeneities, anisotropies and spatial curvature, leaving only the primordial quantum fluctuations as seeds for the structure formation that followed, once the Universe had cooled down sufficiently.

According to observations of distant Type Ia supernovae the current expansion of the Universe is accelerating [41–43]. This view is supported by the observations on CMB radiation, as well as other observations on the redshifts of galaxies, e.g., [44, 45]. The cause of this acceleration is the so-called dark energy, which is uniformly distributed throughout the Universe. Dark energy has a negative pressure that drives the accelerating expansion. The simplest theoretical explanation for dark energy is the cosmological constant, which provides a constant negative pressure. It can indeed be made to fit the observational data very well, as the impressive experimental success of the standard six-parameter ΛCDM model demonstrates. However, the vacuum energy density predicted by QFT is some 120 orders of magnitude greater than the observed value of the cosmological constant. This has lead people to suspect that the vacuum energy density should be zero, and hence the source of dark energy should be looked elsewhere. In addition, the observed value of the cosmological constant happens to be comparable to the current value of the density of matter. This is a coincidence begging to be answered, because the vacuum energy and matter densities evolve at different rates when the Universe expands: vacuum energy density is constant and matter and radiation decrease as $\rho_{\rm mat} \propto a^{-3}$ and $\rho_{\rm rad} \propto a^{-4}$ with the growing scale factor $a(t)$ of space, respectively.

The next simplest alternative for dark energy is a scalar field that slowly evolves down a potential so that it attains negative pressure. This provides a dynamical dark energy that drives the expansion of the Universe at late-times quite similarly as the cosmological constant. There exists several scalar field models of dark energy [46]. One of the most studied models is called quintessence, where a regular scalar field is minimally coupled to spacetime and it slowly rolls down a potential that is chosen to obtain desired expansion of spacetime. In order to solve the coincidence problem with quintessence, the energy density of the scalar field has to track the density of matter and radiation in a specific way during the evolution of the Universe [47]. The density of quintessence tracks the density of radiation from below during the radiation dominated era. During matter domination quintessence attains negative pressure and starts to catch up with the density of matter, until it overtakes matter and becomes the dominant cause for the expansion. Thus scalar field models can provide an alternative phenomenological solution to the coincidence problem.

The expansion history of the Universe is one of the main motivations for considering modifications of GR. Instead of adding components like inflaton (the field causing inflation) and quintessence, we can try to modify the EH action (1.1) in order to achieve similar effects. One of the most popular modification is $f(R)$ gravity, where the scalar curvature in EH action is replaced by its function,

$$S_{f(R)} = \frac{1}{2\kappa}\int_{\mathcal{M}} \mathrm{d}^4x\sqrt{-g}f(R) + S_{\rm mat}[g_{\mu\nu},\psi]\,. \tag{1.6}$$

It is known to be able to realize practically any kind of expansion history and at the same time agree with local Solar system tests of gravity [48–51]. (For reviews and further references, see [52] and also [53].) We consider metric formalism exclusively. Assuming $f'(R)$ is invertible, continuous and one-to-one, the action of $f(R)$ gravity admits a classically equivalent representation as a scalar-tensor theory. First we rewrite the action in a dynamically equivalent form by introducing a pair of scalar fields ϕ, χ as

$$S_{f(R)} = \frac{1}{2\kappa} \int_{\mathcal{M}} \mathrm{d}^4 x \sqrt{-g} \, [\phi(R - \chi) + f(\chi)] + S_{\mathrm{mat}}[g_{\mu\nu}, \psi] \,. \tag{1.7}$$

The Lagrange multiplier ϕ enforces the relation $\chi = R$. The equation of motion for χ gives $\phi = f'(\chi)$, which can be inverted $\chi = f'^{-1}(\phi)$ and inserted back into the action. The result is a special case of the well known Brans-Dicke theory [54] with parameter $\omega = 0$, given in the Jordan frame as

$$\begin{aligned} S_{f(R)} &= \frac{1}{2\kappa} \int_{\mathcal{M}} \mathrm{d}^4 x \sqrt{-g} \, [\phi R - V(\phi)] + S_{\mathrm{mat}}[g_{\mu\nu}, \psi] \,, \\ V(\phi) &= \phi f'^{-1}(\phi) - f\left(f'^{-1}(\phi)\right) . \end{aligned} \tag{1.8}$$

The scalar field ϕ couples nonminimally to the curvature of spacetime and thus behaves very differently compared to any matter scalar fields, which are assumed to couple minimally to spacetime. As in any scalar-tensor theory, the action can be written in Einstein frame form

$$\begin{aligned} S'_{f(R)} = \int_{\mathcal{M}} \mathrm{d}^4 x \sqrt{-\tilde{g}} & \left[\frac{\tilde{R}}{2\kappa} - \frac{1}{2} \tilde{g}^{\mu\nu} \partial_\mu \tilde{\phi} \partial_\nu \tilde{\phi} - U(\tilde{\phi}) \right] \\ & + S_{\mathrm{mat}} \left[e^{-\sqrt{2\kappa/3}\,\tilde{\phi}} \tilde{g}_{\mu\nu}, \psi \right] \end{aligned} \tag{1.9}$$

by performing a conformal transformation of the metric and a redefinition of the scalar field

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} = \phi g_{\mu\nu} \,, \qquad \tilde{\phi} = \sqrt{\frac{3}{2\kappa}} \ln \left(\frac{\phi}{\phi_0} \right) . \tag{1.10}$$

The form of the potential for the scalar field is again defined by the function f as

$$U(\tilde{\phi}) = \frac{\phi f'^{-1}(\phi) - f\left(f'^{-1}(\phi)\right)}{2\kappa\phi^2} \,, \qquad \phi = e^{\sqrt{2\kappa/3}\,\tilde{\phi}} \,. \tag{1.11}$$

This demonstrates the fact that depending on the chosen conformal frame, a gravitational scalar field can appear as a minimally coupled scalar field, or as a scalar field that couples nonminimally to R. Hence it is no wonder that $f(R)$ gravity is able to exhibit similar cosmological features as GR amended with scalar fields that produce inflation and dark energy, such as inflaton and quintessence. However, the transformation to Einstein frame (1.9) modifies the matter part of the action by introducing an intricate coupling between the gravitational fields $\tilde{\phi}$, $\tilde{g}_{\mu\nu}$ and the matter fields. This implies that the usual energy-momentum tensor of matter (1.2) is not conserved in the Einstein frame, and hence the energy density does not depend on the scale factor $a(t)$ in the same way as it does in the $f(R)$ and Jordan frame forms.

In order to achieve desired expansion history, we have to choose a suitable form for the function $f(R)$. That can be seen as a kind of fine-tuning. Choosing a relatively simple form, such as for example $f(R) = R + \alpha R^a + \beta R^{-b}$ with some fixed positive a and b, should provide a model that is able to produce both the inflation and the current accelerating expansion without an excessive amount of parameters.

Adding an explicit coupling between an arbitrary function of R and the Lagrangian density of matter into $f(R)$ gravity causes an interesting extra force [55], which is relevant particularly in the galactic scales.

1.5 Dark matter — gravitational dynamics of galaxies

According to the theory of nucleosynthesis, measurements of abundances of certain light isotopes and the latest CMB radiation measurements [40] only 4.9% of the energy content of the Universe consists of known types of baryonic matter, assuming that GR is the correct theory of gravity. Over five times more (26.8%) consists of unknown massive nonrelativistic particles which do not interact electromagnetically, referred to as nonbaryonic cold dark matter.

The rotation curves of galaxies, namely the graph of velocities of orbiting stars and gas as a function of their distance from the galactic center of mass, provide a quite direct evidence for the existence of dark matter. In order for GR to reproduce the observed rotation curves, large amounts of dark matter must reside in galaxies, and in particular in their halos. Observations on gravitational lensing of galaxies and galaxy clusters provide another independent way to observe the effect of dark matter.

An alternative explanation to the observed dynamics of galaxies and galaxy systems was proposed by Milgrom [56]. His modified Newtonian dynamics (MOND) amends Newton's law of universal gravitation for low accelerations, but leaves it intact when acceleration is much higher than a scale of order $10^{-8}\,\mathrm{cm/s^2}$, which is fixed by observational data. Field theories that realize the MOND paradigm have been constructed: first a nonrelativistic field theory [57] and more recently relativistic field theories, in particular the tensor-vector-scalar (TeVeS) theory [58, 59]. MOND and its relativistic realizations have earned some remarkable experimental success especially at the scale of galaxies. However, at the scale of galaxy clusters these theories are not consistent with observations unless some form of dark matter exists. The missing mass could comprise of dark baryons. In order to be consistent with observations on the gravitational lensing of the famous bullet cluster 1E0657-56, the dark baryons would have to be in a collisionless form. On the other hand, the high collision speed of the pair of galaxy clusters that make up the bullet cluster appears to present a challenge to the ΛCDM model. See [62] for an extensive review. The modified $f(R)$ gravitational theory [55] may provide an alternative way to realize the MOND effect [60, 61].

Chapter 2

Hamiltonian formulation and quantization of gauge theories

In Hamiltonian formalism, the degrees of freedom of a dynamical system are described by pairs of conjugated variables, namely the generalized coordinates and their canonically conjugated momenta. These canonical variables form the configuration space of the system, referred to as the phase space. The phase space is a symplectic manifold. The system is described by a Hamiltonian function H on the phase space that generates the time evolution of the canonical variables and of any dynamical variable f thereof

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \{f, H\} + \frac{\partial f}{\partial t}\,, \tag{2.1}$$

where the Poisson bracket is induced by the symplectic two-form ω,

$$\{f, g\} = \omega^{\mu\nu}\partial_\mu f \partial_\nu g\,.$$

Unlike in the Lagrangian formalism, the equations of motion are first-order differential equations in time.

When a dynamical system involves constraints between the canonical variables, the system is constrained to live in a subspace of the phase space. In a gauge theory, the system is not only constrained, but also degenerate in the sense that several configurations in phase space are associated with the same physical state. This is referred to as gauge invariance. The full power of Hamiltonian formalism is realized in the analysis and quantization of theories that possess such invariances. Hamiltonian formulation and canonical quantization of systems with constraints was originally created by Dirac [63–65], followed by Bergmann and collaborators [66–68]. Since then it has been developed to a comprehensive framework for analysis and quantization of gauge theories. Hamiltonian formalism is regarded as the most reliable and complete formulation of gauge theories.

The main drawback of Hamiltonian formalism is the loss of manifest covariance due to the decomposition of spacetime into time and space. In this chapter, we will explain some essential aspects of Hamiltonian analysis and quantization of gauge systems.

We should emphasize that when considering Hamiltonian formulation of field theory, we consider the traditional symplectic Hamiltonian formalism exclusively (for reviews, see [69–72]). It is an instantaneous Hamiltonian formalism on an infinite-dimensional phase space, where canonical variables are fields on a spatial slice of spacetime at each instant of time. Each local degree of freedom is described by a pair of canonical variables, whose evolution in time is generated by the Hamiltonian. This Hamiltonian formalism is the foundation for the canonical quantization of field theories.

A more faithful Hamiltonian counterpart of classical first-order Lagrangian field theory would be covariant Hamiltonian field theory, where all coordinates x^μ of spacetime are treated equally. Hence one would define canonical momenta p_i^μ corresponding to the derivatives of the fields q_i with respect to all coordinates x^μ. Thus the phase space of covariant Hamiltonian field theory is a finite-dimensional polysymplectic or multisymplectic manifold.

2.1 Hamiltonian analysis of gauge theories

Hamiltonian analysis begins by casting the action functional of a given dynamical system into the canonical form. Then the variational principle is redefined in terms of the canonical variables. Let us consider a system of N fields $q_i(x)$, $i = 1, \ldots, N$, on a given $(d + 1)$-dimensional spacetime $\mathcal{M}$, whose action can be expressed in the Lagrangian form as

$$S = \int_{\mathcal{M}} \mathrm{d}^{d+1}x \mathcal{L}(q, \partial_\mu q)\,. \tag{2.2}$$

Here the Lagrange density function $\mathcal{L}$ of the system depends on the field variables $q_i(x)$ and on their partial derivatives $\partial_\mu q_i(x)$ with respect to the coordinates x^μ of spacetime, but not on the coordinates explicitly.[1] Physical motion is postulated to be such that the action is extremal under all variations $\delta q_i(x)$ that vanish on the boundary of spacetime. This variational principle implies the Euler-Lagrange equations of motion,

$$\frac{\partial \mathcal{L}}{\partial q_i} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu q_i)} \right) = 0\,, \qquad i = 1, \ldots, N\,, \tag{2.3}$$

where a sum over the repeated index of the partial derivatives ∂_μ with respect to the coordinates of spacetime x^μ, $\mu = 1, \ldots, d + 1$, is assumed. We omit the field argument (x) for brevity when there is no chance of misunderstanding.

[1] In general, the Lagrangian density $\mathcal{L}$ could of course depend on the coordinates explicitly. However, in this work we will not encounter such theories. Theories involving higher-order derivatives of the fields $q_i(x)$ will be considered later in Sec. 2.2.

We assume there exists a well defined decomposition of spacetime into time t and three-dimensional space Σ_t for every t. The idea is that spacetime admits a foliation into a one-parameter family of nonintersecting spacelike hypersurfaces Σ_t labelled by time. Spacetime is realized as the union of the spatial hypersurfaces. Foliation of spacetime into time and space will be discussed further in Chapter 3, where we consider dynamical spacetime. For now it suffices to understand that the action can be expressed as an integral of a Lagrangian over time

$$S = \int_{t_1}^{t_2} \mathrm{d}t L[q, \partial_t q] \,, \tag{2.4}$$

where at any given time t the Lagrangian is regarded as a functional of the fields $q_i(t, \boldsymbol{x})$ on Σ_t and of their partial derivatives with respect to time $\partial_t q_i(t, \boldsymbol{x})$,

$$L[q, \partial_t q] = \int_{\Sigma_t} \mathrm{d}^d x \mathcal{L}(q, \partial_t q, Dq, \ldots) \,. \tag{2.5}$$

Spatial derivatives of the fields Dq_i are now regarded as dependent variables in the Lagrangian. In (2.5), $\mathcal{L}$ may even depend on higher-order spatial derivatives, represented by the ellipses. The momentum $p_i(t, \boldsymbol{x})$ canonically conjugate to the field $q_i(t, \boldsymbol{x})$ on each point of space Σ_t is defined by[2]

$$p_i = \frac{\delta L}{\delta(\partial_t q_i)} \,, \qquad i = 1, \ldots, N \,. \tag{2.6}$$

The canonical variables satisfy the canonical Poisson brackets:

$$\begin{aligned} \{q_i(t, \boldsymbol{x}), p_j(t, \boldsymbol{y})\} &= \delta_{ij}\delta(\boldsymbol{x} - \boldsymbol{y}) \,, \\ \{q_i(t, \boldsymbol{x}), q_j(t, \boldsymbol{y})\} &= 0 \,, \\ \{p_i(t, \boldsymbol{x}), p_j(t, \boldsymbol{y})\} &= 0 \,. \end{aligned} \tag{2.7}$$

The Poisson bracket can be defined as

$$\{f, g\} = \int_{\Sigma_t} \mathrm{d}^d x \sum_{i=1}^{N} \left(\frac{\delta f}{\delta q_i(\boldsymbol{x})} \frac{\delta g}{\delta p_i(\boldsymbol{x})} - \frac{\delta f}{\delta p_i(\boldsymbol{x})} \frac{\delta g}{\delta q_i(\boldsymbol{x})} \right) , \tag{2.8}$$

where the arguments f and g are functions or functionals of the canonical variables $q_i(\boldsymbol{x})$ and $p_i(\boldsymbol{x})$. From now on the time dependence of dynamical variables is assumed implicitly, so that in any expression all dynamical variables are taken at the same time t. Poisson brackets between arguments at different times are not actually considered in this instantaneous Hamiltonian formalism. One can regard such Poisson brackets to be identically zero, since arguments at different times reside on different spatial hypersurfaces Σ_t and thus they must commute.

For a regular Lagrangian the Hessian matrix is of full rank, and hence one can solve the time derivatives $\partial_t q_i$ in terms of the canonical variables. In a gauge theory, the Hessian matrix is singular

$$\operatorname{rank} \left(\frac{\delta^2 L}{\delta(\partial_t q_i)\delta(\partial_t q_j)} \right) = N - K \,, \qquad 0 < K \leq N \,, \tag{2.9}$$

[2] In the functional derivative of the Lagrangian $L(t)$ with respect to $\partial_t q_i(t, \boldsymbol{x})$, the field itself $q_i(t, \boldsymbol{x})$ is regarded as independent of its time derivative.

where we assume the rank of the Hessian is constant everywhere. Therefore some of the velocities $\partial_t q_i$ cannot be solved in terms of the canonical variables q_i and p_i. Instead there exists a set of local primary constraints between the canonical variables

$$\phi_k(q,p) \approx 0\,, \qquad k = 1,\ldots,K\,. \tag{2.10}$$

These constraints are assumed to be independent. The weak equality $\approx$ is understood in the sense introduced by Dirac [69]: a weak equality can be imposed only after all Poisson brackets have been evaluated, while a usual strong equality can be imposed anywhere.

The canonical Hamiltonian H_c is defined as the Legendre transform of the Lagrangian L as[3]

$$H_\mathrm{c}[q,p] = \int_{\Sigma_t} \sum_{i=1}^{N} p_i \partial_t q_i - L[q, \partial_t q]\,. \tag{2.11}$$

The total Hamiltonian is then obtained by including the primary constraints (2.10) as

$$H = H_\mathrm{c} + \int_{\Sigma_t} \sum_{k=1}^{K} \lambda_k \phi_k\,, \tag{2.12}$$

where the coefficients λ_k are arbitrary functions, called Lagrange multipliers. We emphasize that the canonical Hamiltonian H_c in (2.12) is defined by (2.11) up to the constraints ϕ_k, since we can absorb any term proportional to a constraint ϕ_k into the Lagrange multiplier λ_k. The total Hamiltonian generates time evolution for every function $f(q,p)$ on the phase space as

$$\partial_t f(\boldsymbol{x}) = \{f(\boldsymbol{x}), H_\mathrm{c}\} + \int_{\Sigma_t} \mathrm{d}^d y \sum_{k=1}^{K} \lambda_k(\boldsymbol{y})\,\{f(\boldsymbol{x}), \phi_k(\boldsymbol{y})\} \approx \{f(\boldsymbol{x}), H\}\,. \tag{2.13}$$

We denote $f(\boldsymbol{x}) \equiv f(q(\boldsymbol{x}), p(\boldsymbol{x}))$ for brevity. The equations of motion for the canonical variables can be equivalently obtained by applying the variational principle to the action

$$S[q,p,\lambda] = \int_{t_1}^{t_2} \mathrm{d}t \left(\int_{\Sigma_t} \sum_{i=1}^{N} p_i \partial_t q_i - H \right). \tag{2.14}$$

Each primary constraint ϕ_k has to satisfy a consistency condition that ensures the constraint is preserved under time evolution of the system,

$$\partial_t \phi_k \approx \{\phi_k, H\} \approx 0\,, \qquad k = 1,\ldots,K\,. \tag{2.15}$$

These consistency conditions may require introduction of further constraints, or lead to determination of Lagrange multipliers in terms of the canonical variables.

[3] From now on we shall omit the fixed integration measure $\mathrm{d}^d x$ from the integrals over space Σ_t. Except when it could cause an ambiguity on which fields are integrated.

Constraints that are required by the preservation of other constraints are called secondary constraints. The secondary constraints in turn have to satisfy similar consistency conditions (2.15) in order for them to be preserved in time. That may require introduction of further constrains. This procedure is repeated as long as it takes for every constraint to satisfy the consistency condition that ensures the constraint is preserved in time.[4] Finally, we obtain the complete set of constraints

$$\phi_m(q,p) \approx 0\,, \qquad m = 1, \ldots, M \geq K\,. \tag{2.16}$$

The primary and secondary constraints are treated equally, since there is no physical difference between them. The complete set of constraints defines a surface in the phase space on which the system is constrained to evolve. This is referred to as the constraint surface of the system. The total Hamiltonian (2.12) can now be extended to include all the constraints multiplied by arbitrary Lagrange multipliers. The extended Hamiltonian is defined as

$$H_\mathrm{e} = H_\mathrm{c} + \int_{\Sigma_t} \sum_{m=1}^{M} \lambda_m \phi_m\,. \tag{2.17}$$

The canonical Hamiltonian H_c is now defined by (2.11) up to the full set of constraints (2.16). The dynamical equation (2.13) and the canonical form of the action (2.14) are extended accordingly by replacing H with H_e.

Constraints are classified to first-class and second-class constraints. A constraint that has a weakly vanishing Poisson bracket with every constraint in the system is called a first-class constraint. The first-class constraints are typically linear combinations of the constraints ϕ_m with some functions of the canonical variables as coefficients. On the other hand, if a constraint has a nonvanishing Poisson bracket with some constraint or constraints, it is called a second-class constraint. The number of second-class constraints is even. The consistency conditions for the full set of constraints, i.e., $\{\phi_m, H_\mathrm{e}\} \approx 0$, can be used to solve the Lagrange multipliers of the second-class constraints, while the multipliers of the first-class constraints are left arbitrary.

The second-class constraints can be set to zero strongly, if we replace the canonical Poisson bracket with the Dirac bracket. Given a set of second-class constraints $\varphi_b \approx 0$, $b = 1, \ldots, B$, the Dirac bracket is defined by

$$\begin{aligned}\{f(\boldsymbol{x}), g(\boldsymbol{y})\}_\mathrm{D} = \{f(\boldsymbol{x}), g(\boldsymbol{y})\} - \int_{\Sigma_t} \mathrm{d}^d z \mathrm{d}^d z' \sum_{b,c=1}^{B} \{f(\boldsymbol{x}), \varphi_b(\boldsymbol{z})\} \\ \times C^{-1}_{bc}(\boldsymbol{z}, \boldsymbol{z}')\,\{\varphi_c(\boldsymbol{z}'), g(\boldsymbol{y})\}\,,\end{aligned} \tag{2.18}$$

where $C^{-1}(\boldsymbol{x}, \boldsymbol{y})$ is the inverse to the matrix $C(\boldsymbol{x}, \boldsymbol{y})$ which has the components

$$C_{bc}(\boldsymbol{x}, \boldsymbol{y}) = \{\varphi_b(\boldsymbol{x}), \varphi_c(\boldsymbol{y})\}\,. \tag{2.19}$$

[4] We refer to the monographs [69–72] for a full treatment of Dirac's algorithm for uncovering all the constraints of a system.

When we use the Dirac bracket, we can set the second-class constraints to zero strongly, $\varphi_b = 0$, because $\{\varphi_b, f\}_{\mathrm{D}} = 0$ for any function f on phase space. The first-class constraints form a closed algebra under the Dirac bracket.

Then the extended Hamiltonian takes the form

$$H_{\mathrm{e}} = H_{\mathrm{c}} + \int_{\Sigma_t} \sum_{a=1}^{A} \lambda_a \Phi_a \,, \tag{2.20}$$

where Φ_a, $a = 1, \ldots, A$, are the first-class constraints in the system and λ_a are their arbitrary Lagrange multipliers. Time evolution of a dynamical variable $f(q, p)$ is given as

$$\begin{aligned} \partial_t f(\boldsymbol{x}) &= \{f(\boldsymbol{x}), H_{\mathrm{c}}\}_{\mathrm{D}} + \int_{\Sigma_t} \mathrm{d}^d y \sum_{a=1}^{A} \lambda_a(\boldsymbol{y}) \{f(\boldsymbol{x}), \Phi_a(\boldsymbol{y})\}_{\mathrm{D}} \\ &\approx \{f(\boldsymbol{x}), H_{\mathrm{e}}\}_{\mathrm{D}} \,. \end{aligned} \tag{2.21}$$

Now the equations of motion generated by H_{e} contain as many arbitrary functions as there are first-class constraints in the system. The state of the system is defined by the values of the canonical variables q_i and p_i over the spatial space. Since the time evolution of dynamical variables involves arbitrary functions, at any later time, there must exist a set of values for q_i and p_i which correspond to the same physical state. That set is called a gauge orbit. For a dynamical variable the gauge orbit at any given time is obtained by going through all possible values of the arbitrary functions λ_a. The first-class constraints function as generators of gauge transformations, which do not change the physical state of the system. The gauge generator is defined as

$$G[\epsilon] = \int_{\Sigma_t} \sum_{a=1}^{A} \epsilon_a \Phi_a \,, \tag{2.22}$$

where ϵ_a are parameter fields, which do not depend on the canonical variables. Gauge invariant quantities are those that have a vanishing Dirac bracket with the gauge generator,

$$\delta_\epsilon f(\boldsymbol{x}) = \{f(\boldsymbol{x}), G[\epsilon]\}_{\mathrm{D}} = 0 \,. \tag{2.23}$$

Instead of describing the state of the system at a given time with the whole gauge orbit, we may as well choose a single point on each orbit. This can be accomplished by introducing gauge fixing conditions,

$$\chi_a(q, p) \approx 0 \,, \qquad a = 1, \ldots, A \,. \tag{2.24}$$

The number of gauge fixing conditions χ_a is equal to the number of the first-class constraints Φ_a. The gauge fixing conditions have to remove the gauge freedom completely. This is accomplished when the consistency conditions for χ_a, i.e.,

$$\partial_t \chi_a(\boldsymbol{x}) = \{\chi_a(\boldsymbol{x}), H_{\mathrm{c}}\}_{\mathrm{D}} + \int_{\Sigma_t} \mathrm{d}^d y \sum_{b=1}^{A} \lambda_b(\boldsymbol{y}) \{\chi_a(\boldsymbol{x}), \Phi_b(\boldsymbol{y})\}_{\mathrm{D}} = 0 \,, \tag{2.25}$$

fix all the Lagrange multipliers λ_a completely. This means the square matrix with components $\{\chi_a, \Phi_b\}$ has to be of full rank A everywhere on space. In addition, the chosen gauge has to be accessible from anywhere on the constraint surface via a gauge transformation. When the gauge has been fixed, the system has only second-class constraints, namely the used-to-be-first-class constraints Φ_a and their associated gauge fixing conditions χ_a, and the original second-class constraints φ_b. All of the constraints can now be set to zero strongly by replacing the canonical Poisson bracket with the Dirac bracket (2.18), where the set of second-class constraints now consists of the $2A + B$ constraints $\Phi_a, \chi_a, \varphi_b$, where $a = 1, \ldots, A$, $b = 1, \ldots, B$. Sometimes one may wish to fix the gauge only partially, reducing the amount of gauge freedom in the system, rather than removing it completely. In such a case the number of first-class constraints in the system would be reduced by the number of gauge fixing conditions, but not necessarily to zero as above.

We should emphasize that it is not always possible to fix the gauge globally, because the geometry of the constraint surface and the gauge orbits may be such that no gauge fixing surface $\chi_a = 0$ intersects every gauge orbit exactly once. The gauge surface may intersect some gauge orbits several times or leave some orbits untouched. This Gribov ambiguity is a reason for avoiding canonical gauge fixation.

There exists a converse point of view compared to gauge fixing, where second-class constraints are traded for first-class constraints. Indeed a theory with second-class constraints can be seen as a gauge fixed version of a theory with only first-class constraints. In the simplest example we would have two second-class constraints $p_1 \approx 0$ and $q_1 \approx 0$, which can be seen as the first-class constraint that generates translations of q_1 and its gauge fixing condition, respectively. In general, achieving this interpretation often requires one to introduce new variables into the system, unless half of the second-class constraints happen to be first-class constraints in the absence of the remaining half of the constraints, like in the aforementioned example. Having a system with only first-class constraints provides the advantage of avoiding the need to introduce the Dirac bracket, which can sometimes be highly complicated, especially for the purpose of quantization. There also exist some powerful methods for working with gauge theories with only first-class constraints, which will be remarked below.

The number of physical degrees of freedom in any constrained theory can be counted according to Dirac's formula:

$$\text{number of physical degrees of freedom} = \frac{2N - 2A - B}{2}\,, \tag{2.26}$$

where $2N$ is the number of canonical variables, A is the number of first-class constraints and B is the number of second-class constraints. In other words, each first-class constraint deletes one physical degree of freedom, while two second-class constraints are required for the same purpose. This is consistent with the fact that first-class constraints and their associated gauge fixing conditions constitute a system of second-class constraint.

2.2 Higher time derivative theories

All the conventional theories of physics to date are described by an action functional (2.2) that depends at most on the first-order time derivatives of the dynamical variables, and hence their equations of motion are second-order differential equations in time (2.3).[5] When the Lagrangian of a system depends on the derivatives of each dynamical variable q_i up to n_i-th order,

$$S = \int_{\mathcal{M}} \mathrm{d}^{d+1}x \mathcal{L}\left(q_i, \partial_\mu q_i, \ldots, \partial_\mu^{n_i} q_i\right), \tag{2.27}$$

the equations of motion

$$\frac{\partial \mathcal{L}}{\partial q_i} + \sum_{k_i=1}^{n_i} (-\partial_\mu)^{k_i} \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu^{k_i} q_i)}\right) = 0, \qquad i = 1, \ldots, N, \tag{2.28}$$

contain time derivatives up to $2n$-th order, where $n = \max(n_i)$ is greatest n_i in the Lagrangian. This means that instead of the usual two pieces of initial value data (boundary conditions) for each dynamical variable, we now need $2n_i$ pieces of initial value data for each q_i. In other words, each higher derivative of a dynamical variable in the Lagrangian adds an extra degree of freedom into the system. Thus we need $2n_i$ independent canonical variables $Q_i^{k_i}$ and $P_i^{k_i}$, $k_i = 1, \ldots, n_i$, in order to represent each dynamical variable q_i. It is essential to the nature of higher derivative theories that each dynamical variable q_i (with $n_i > 1$) carries several degrees of freedom.

We again assume that spacetime can be decomposed so that the action may be written as

$$S = \int_{t_1}^{t_2} \mathrm{d}t L\left[q_i, \partial_t q_i, \ldots, \partial_t^{n_i} q_i\right]. \tag{2.29}$$

2.2.1 Regular higher derivative theories — Ostrogradski's Hamiltonian

Hamiltonian formulation for higher derivative theories with regular Lagrangians was first developed by Ostrogradski [73]. His choice of canonical variables can be defined recursively as

$$Q_i^1 = q_i, \qquad Q_i^{k_i+1} = \partial_t Q_i^{k_i}, \qquad k_i = 1, \ldots, n_i - 1, \tag{2.30a}$$

and the momenta in descending order as

$$P_i^{n_i} = \frac{\delta L}{\delta(\partial_t^{n_i} q_i)}, \qquad P_i^{k_i} = \frac{\delta L}{\delta(\partial_t^{k_i} q_i)} - \partial_t P_i^{k_i+1}, \qquad k_i = n_i - 1, \ldots, 1. \tag{2.30b}$$

[5] The EH action (1.1) contains a second-order time derivative, but it can be absorbed into a covariant total derivative, which contributes a boundary surface term into the action. This will be discussed in Chapter 3.

Since the system is regular, the definition of the highest order momenta $P_i^{n_i}$ for each variable can be solved for the highest time derivative as

$$\partial_t^{n_i} q_i = \partial_t Q_i^{n_i} = A_i \left(Q_j^1, \ldots, Q_j^{n_j}, P_j^{n_j} \right) , \tag{2.31}$$

where the solution A_i satisfies

$$P_i^{n_i} = \left. \frac{\delta L}{\delta(\partial_t^{n_i} q_i)} \right|_{\partial_t^{(k_i-1)} q_i = Q_i^{k_i},\ k_i=1,\ldots,n_i}^{\partial_t^{n_i} q_i = A_i(Q_j^1,\ldots,Q_j^{n_j}, P_j^{n_j})} \tag{2.32}$$

and A_i only needs to depend on as many canonical variables as there are configuration space coordinates in the Lagrangian, namely $N + \sum_{i=1}^N n_i$ variables, which are chosen to be $Q_i^1, \ldots, Q_i^{n_i}$ and $P_i^{n_i}$. Then the canonical Hamiltonian is obtained as

$$\begin{aligned} H &= \int_{\Sigma_t} \sum_{i=1}^N \sum_{k_i=1}^{n_i} P_i^{k_i} \partial_t Q_i^{k_i} - L \left[Q_i^1, \ldots, Q_i^{n_i}, A_i \right] \\ &= \int_{\Sigma_t} \sum_{i=1}^N \left(\sum_{k_i=1}^{n_i-1} P_i^{k_i} Q_i^{k_i+1} + P_i^{n_i} A_i \right) - L \left[Q_i^1, \ldots, Q_i^{n_i}, A_i \right] , \end{aligned} \tag{2.33}$$

where A_i are understood to depend on $Q_i^1, \ldots, Q_i^{n_i}$ and $P_i^{n_i}$. The Hamiltonian equations of motion for the canonical variables $Q_i^{k_i}$ and $P_i^{k_i}$, $k_i = 1, \ldots, n_i$ reproduce the definitions of the canonical variables (2.30), the definition of A_i (2.31) and the higher-derivative Euler-Lagrange equations (2.28). Thus the Ostrogradski Hamiltonian is dynamically equivalent to the higher derivative Lagrangian. The choice of variables (2.30) is by no means the only possible one, as will be explained in the next subsection.

The major problem with the Hamiltonian (2.33) is its linearity with respect to the momenta $P_i^{k_i}$, $k_i = 1, \ldots, n_i - 1$. This means the Hamiltonian has no stable minima. In a discrete system with a finite number degrees of freedom, this is not necessarily a fatal problem. However, an interacting field theory described by the Hamiltonian (2.33) is necessarily unstable. In a higher derivative theory, some of the degrees of freedom carried by a dynamical variable q_i have positive energy, while every other degree of freedom has a negative energy. When such local degrees of freedom with positive and negative energies are interacting with each other, any state will violently decay into a tempest of compensating positive and negative energy excitations. Thus interacting higher derivative field theories with regular Lagrangians are unstable, and hence not well defined quantum theories.[6]

2.2.2 Higher derivative theories with constraints

Some decades after Dirac developed Hamiltonian formalism for constrained systems it was generalized to higher derivative theories with singular Lagrangians

[6] See [53] for a vivid description of the Ostrogradskian instability.

[74, 75] (see also [76, 77]). Higher derivative theories whose Lagrangians are singular can sometimes avoid the Ostrogradskian instability. That is because constraints can sometimes control the dynamics of a system in a way that prevents the system from reaching the parts of the phase space that exhibit the instabilities. Theories which possesses continuous symmetries always have singular Lagrangians, in particular gauge theories, and hence they have a chance to avoid the problem with unstable ghosts. Thus in each higher derivative gauge theory, the existence and behavior of ghosts has to be checked carefully.

Hamiltonian formulation of actions that involve higher-order time derivatives requires us to introduce a pair of new independent variables for each higher-order time derivative of each variable. Consider a generic prescription for the additional variables

$$Q_i^1 = q_i\,, \qquad Q_i^{k_i+1} = E_i^{k_i}\left(q_j, \partial_t q_j, \dots, \partial_t^{\Theta_{ij}} q_j\right), \qquad \Theta_{ij} = \min\left(k_i, n_j - 1\right), \tag{2.34}$$

where $k_i = 1, \dots, n_i - 1$ for each $i = 1, \dots, N$. Like in Ostrogradski's prescription we choose $Q_i^1 = q_i$, but now we are free to choose the remaining additional variables to reflect the character of a given theory, which can be a great advantage. The functions $E_i^{k_i}$ have to satisfy the conditions

$$\det\left(\frac{\partial E_i^k}{\partial(\partial_t^k q_j)}\right) \neq 0\,, \qquad k = 1, \dots, n-1\,, \tag{2.35}$$

where the matrix contains the columns and rows with indices i, j for which $n_i > k, n_j > k$. This ensures that we can solve (2.34) for the time derivatives of the original variables as

$$\partial_t^{k_i} q_i = F_i^{k_i}\left(Q_j^1, \dots, Q_j^{\Theta_{ij}+1}\right), \qquad k_i = 1, \dots, n_i - 1\,, \tag{2.36}$$

where the functions $F_i^{k_i}$ satisfy $\det(\partial F_i^k / \partial Q_j^{k+1}) \neq 0$, $k = 1, \dots, n-1$.

Assuming the highest time derivative for each variable q_i has the same order, $n_i = n$,[7] we can differentiate (2.36) with respect to time in order to obtain

$$\partial_t Q_i^{k_i} = G_i^{k_i}\left(Q_j^1, \dots, Q_j^{\Theta_{ij}+1}\right), \qquad k_i = 1, \dots, n-1\,. \tag{2.37}$$

Furthermore we can write the highest time derivatives of the original variables as

$$\partial_t^n q_i = F_i^n\left(Q_j^1, \dots, Q_j^n, \partial_t Q_i^n\right). \tag{2.38}$$

Then we replace the Lagrangian of the higher derivative action (2.29) with an equivalent first-order Lagrangian

$$L^*\left[Q_i^{k_i}, \partial_t Q_i^{k_i}, \lambda_i^{k_i}\right] = L\left[Q_i^{k_i}, \partial_t Q_i^n\right] + \int_{\Sigma_t} \sum_{i=1}^{N} \sum_{k_i=1}^{n-1} \lambda_i^{k_i}\left(\partial_t Q_i^{k_i} - G_i^{k_i}\left(Q_j^1, \dots, Q_j^{\Theta_{ij}+1}\right)\right), \tag{2.39}$$

[7] This necessary requirement was pointed out in [78], as well as the way to meet this requirement in general by adding total time derivatives into the Lagrangian.

where in the right-hand side L is obtained by substituting the solutions (2.36) and (2.38) into the original Lagrangian (2.29), and we have introduced Lagrange multiplier fields $\lambda_i^{k_i}$ in order to enforce the relations (2.37). We could further introduce additional variables as $v_i = \partial_t Q_i^n$ [74–76], which can be useful if the functional dependence of the Lagrangian L on $\partial_t Q_i^n$ is complicated, but this is not necessary in many cases. The Lagrangian L^* depends only on the first-order time derivatives of the variables $Q_i^{k_i}$, $k_i = 1, \dots, n$. Hence we can apply the Hamiltonian formalism of constrained systems reviewed in Sec. 2.1.

The canonical momenta $P_i^{k_i}$ and $\pi_i^{k_i}$ conjugate to $Q_i^{k_i}$ and $\lambda_i^{k_i}$, $k_i = 1, \dots, n-1$, respectively, define the primary constraints

$$\Pi_i^{k_i} = P_i^{k_i} - \lambda_i^{k_i} \approx 0 \,, \qquad \pi_i^{k_i} \approx 0 \,. \tag{2.40}$$

These constraints $\Pi_i^{k_i}$ and $\pi_i^{k_i}$ form a system of second-class constraints, and thus they can be set to zero strongly by introducing the Dirac bracket (2.18). The resulting Dirac bracket modifies the Poisson bracket if one of the arguments depends on $\lambda_i^{k_i}$ and the other argument depends on $Q_i^{k_i}$ or $\pi_i^{k_i}$. Therefore, in this case, introducing the Dirac bracket and imposing the second-class constraints strongly is simply equivalent to setting $\Pi_i^{k_i} = 0$ and $\pi_i^{k_i} = 0$ and removing the auxiliary variables $\lambda_i^{k_i}$ and $\pi_i^{k_i}$ from the system.

The momenta conjugated to Q_i^n are defined as

$$P_i^n = \frac{\delta L^*}{\delta(\partial_t Q_i^n)} = \frac{\delta L}{\delta(\partial_t Q_i^n)} \,. \tag{2.41}$$

Some of the velocities $\partial_t Q_i^n$ may not be solvable in terms of the canonical variables $Q_i^{k_i}$ and P_i^n. Suppose we can solve $N-K$ of the velocities $\partial_t Q_i^n$. This means there will exist primary constraints among the variables $Q_i^{k_i}$ and P_i^n,

$$\phi_k(Q, P^n) \approx 0 \,, \qquad k = 1, \dots, K \,. \tag{2.42}$$

From here on the analysis proceeds exactly as in Sec. 2.1. The total Hamiltonian is defined as

$$\begin{aligned} H = \int_{\Sigma_t} \sum_{i=1}^{N} \left[\sum_{k_i=1}^{n-1} P_i^{k_i} G_i^{k_i} \left(Q_j^1, \dots, Q_j^{\Theta_{ij}+1} \right) + P_i^n \partial_t Q_i^n \right] \\ - L\left[Q_i^{k_i}, \partial_t Q_i^n \right] + \int_{\Sigma_t} \sum_{k=1}^{K} \lambda_k \phi_k(Q, P^n) \,, \end{aligned} \tag{2.43}$$

where Lagrange multipliers λ_k for the primary constraints were introduced. Completing the system of constraints proceeds as usual by requiring that each constraint must be preserved under time evolution generated by the total Hamiltonian. Then the constraints can be classified and the extended Hamiltonian defined.

There generally exist many different choices for the additional variables defined by $Q_i^{k_i}$ in (2.34), which each yield a different Hamiltonian formulation of a given higher derivative action. Such Hamiltonian formulations are related by canonical transformations [76] and hence they are classically equivalent. But those canonical transformation may be highly nonlinear. Thus there is no guarantee that the different Hamiltonian formulations remain equivalent after quantization.

The Hamiltonian (2.43) is linear in the momenta $P_i^{k_i}$, $k_i = 1, \ldots, n-1$, similarly as the Ostrogradski's Hamiltonian (2.33). Now the presence of constraints might prevent the Ostrogradskian instability by restricting the behavior of the variables $Q_i^{k_i}$ and $P_i^{k_i}$. For that to be possible there would have to exist at least as many constraints as there are unstable directions in the Hamiltonian, i.e., at least $N(n-1)$ constraints. Even then the stability of a theory depends on the specific structure of the theory.

The instability problem affects many higher derivative theories. For instance, string field theory [79] and some other nonlocal theories are affected (see [53] and references therein). The Ostrogradski theorem – together with a chosen symmetry principle – greatly limits the class of viable models for any purpose.

No theory with higher-order time derivatives has received remarkable experimental success to date. For most purposes there is indeed little reason to even consider higher time derivative theories. Gravity and theories related to quantum structure of spacetime are exceptions, because quantization of gravity has turned out be a serious problem and higher-order time derivative theories may offer an advantage in this respect, as was discussed in Sec. 1.3.

2.3 Quantization

2.3.1 Canonical operator quantization

In canonical operator quantization, we replace the canonical variables q_i and p_i with linear operators $\hat{q}_i$ and $\hat{p}_i$ acting on a Hilbert space of states. Each function on the phase space $f(q,p)$ is assigned to an operator equivalent $\hat{f} = \widehat{f(q,p)}$, which involves an appropriate operator ordering due to the noncommutativity of the operators. Depending on whether the system contains second-class constraints, either the Poisson bracket or the Dirac bracket is replaced by the commutator of operators as

$$[\hat{q}_i, \hat{p}_j] = i\hbar \widehat{\{q_i, p_j\}}_{\mathrm{D}} , \tag{2.44}$$

where in the right-hand side we have the operator equivalent of the evaluated Dirac bracket. If anticommuting fermionic degrees of freedom are included in the theory, we need to consider graded Dirac brackets and anticommutators as well. We first assume the system has only second-class constraints $\varphi_b \approx 0$. Then the general correspondence rule for canonical operator quantization reads

$$\left[\hat{f}, \hat{g}\right]_{\pm} = i\hbar \widehat{\{f, g\}}_{\mathrm{D}} , \tag{2.45}$$

where $\hat{f}$ and $\hat{g}$ are the operators corresponding to the functions $f(q,p)$ and $g(q,p)$, and the Dirac bracket is a graded one if necessary. The second-class constraints can be enforced as operator identities

$$\hat{\varphi}_b = 0\,, \tag{2.46}$$

because $\{\varphi_b, f\}_{\mathrm{D}} = 0$ for any f. The major difficulty with operator quantization is that one must find an operator representation of the Dirac bracket, which can be very complicated.

Then consider a theory with first-class constraints. One way to deal with first-class constraints is to work with a reduced phase space. Reduced phase space is obtained by taking the quotient of the constraint surface by the gauge orbits. This essentially corresponds to solving the constraints and deleting all unphysical degrees of freedom in favor of a set of independent canonical variables. This is often undesirable, since removing the gauge invariance can lead to a more complicated system. On the other hand, if the set of first-class constraints admits global gauge fixing conditions, then all constraints become second-class, and hence we can employ the operator quantization outlined above.

In Dirac quantization of a system with first-class constraints, one keeps all the gauge degrees of freedom and works with a larger space of states. Thus the quantum representation of the system contains unphysical information in Dirac quantization. Physical states are selected to be the states that are invariant under the gauge transformations generated by the first-class constraints,

$$\hat{G}[\epsilon]\,|\psi\rangle = 0\,, \qquad e^{i\hat{G}[\epsilon]}\,|\psi\rangle = |\psi\rangle\,, \tag{2.47}$$

where $\hat{G}[\epsilon]$ is the operator equivalent of the gauge generator (2.22). Such physically states coincide with the states of the reduced phase space quantization approach, but now there is no need to actually reduce the phase space nor the space of states. In many cases, it is desirable to retain the gauge invariance rather than reduce it. Indeed the most elegant methods for quantization of gauge theories are based on extending the phase space and introducing a new invariance. This is the approach taken in the celebrated Becchi-Rouet-Stora-Tyutin (BRST) formalism [80–82], which provides an elegant way to quantize a gauge system with first-class constraints. (For a review and references of further developments, see for example [72].)

2.3.2 Path integral quantization

In path integral quantization, we represent physically relevant transition amplitudes in terms of integrals over all possible paths or field configurations which are consistent with chosen boundary conditions (for a comprehensive presentation, see the monograph [83]). Path integral quantization has the great advantage of avoiding the need to find a complete operator representation of the Dirac bracket.

Let us consider a system with a set of second-class constraints ϕ_m, $m = 1, \ldots, M'$. Those constraints could be the original first-class constraints Φ_a and their associated gauge fixing conditions χ_a and the original second-class constraints φ_b, giving a total of $M' = M + A$ constraints. Path integral representation of a transition amplitude can be written as [84–87] (monographs [70–72, 83])

$$\begin{aligned} \langle \mathrm{f}|\mathrm{i}\rangle = \int \prod_{i=1}^{N} \mathcal{D}q_i \mathcal{D}p_i \prod_{m=1}^{M'} \delta(\phi_m) \det |\{\phi_m, \phi_n\}|^{1/2} \\ \times \exp\left[\frac{i}{\hbar}\int_{t_1}^{t_2} \mathrm{d}t \left(\int_{\Sigma_t} \sum_{i=1}^{N} p_i \partial_t q_i - H_\mathrm{c}[q,p]\right)\right], \end{aligned} \tag{2.48}$$

where the functional integral is over all configurations that satisfy chosen boundary conditions associated with the initial and final states of the system. The exponential in the integrand is the Hamiltonian form of the action times $i\hbar^{-1}$. When the set of second-class constraints ϕ_m consists of the original first-class constraints Φ_a and their gauge fixing conditions χ_a, and the gauge is chosen so that $\{\chi_a, \chi_b\} = 0$, the path integral can be rewritten as

$$\begin{aligned} \langle \mathrm{f}|\mathrm{i}\rangle = \int \prod_{i=1}^{N} \mathcal{D}q_i \mathcal{D}p_i \prod_{a=1}^{A} \mathcal{D}\lambda_a \, \delta(\chi_a) \det |\{\chi_a, \Phi_b\}| \\ \times \exp\left[\frac{i}{\hbar}\int_{t_1}^{t_2} \mathrm{d}t \left(\int_{\Sigma_t} \sum_{i=1}^{N} p_i \partial_t q_i - H_\mathrm{c}[q,p] - \int_{\Sigma_t} \sum_{a=1}^{A} \lambda_a \Phi_a\right)\right]. \end{aligned} \tag{2.49}$$

The integration measures are assumed to be appropriately normalized.

Assuming the Hamiltonian is quadratic in the canonical momenta, we can perform the Gaussian integrals over momenta, obtaining the Lagrangian form for the path integral over all field configurations $q_i(x)$ as

$$\langle \mathrm{f}|\mathrm{i}\rangle = \int \mathcal{D}\mu(q) \exp\left(\frac{i}{\hbar}\int_{\mathcal{M}} \mathrm{d}^{d+1}x \mathcal{L}(q, \partial_t q)\right). \tag{2.50}$$

The local integration measure $\mathcal{D}\mu(q)$ for the fields q_i incorporates any remaining constraints, namely the gauge conditions for Lagrangian formulation.

The generating functional of Green's functions is obtained by adding the auxiliary sources $J_i(x)$ for the fields $q_i(x)$ into the action functional S in the exponential of each path integral as $S + \int_{\mathcal{M}} \mathrm{d}^{d+1}x \sum_{i=1}^{N} J_i q_i$. The canonical form of the generating functional reads

$$\begin{aligned} Z[J] = \int \prod_{i=1}^{N} \mathcal{D}q_i \mathcal{D}p_i \prod_{m=1}^{M'} \delta(\phi_m) \det |\{\phi_m, \phi_n\}|^{1/2} \\ \times \exp\left[\frac{i}{\hbar}\int_{t_1}^{t_2} \mathrm{d}t \left(\int_{\Sigma_t} \sum_{i=1}^{N} p_i \partial_t q_i - H_\mathrm{c}[q,p] + \int_{\Sigma_t} \sum_{i=1}^{N} J_i q_i\right)\right]. \end{aligned} \tag{2.51}$$

The generating functionals for the cases (2.49) and (2.50) are obtained similarly.

Path integral quantization is especially important for the framework of QFT and for gauge field theories built on it.

2.3.3 Covariant perturbation theory

Even though the canonical quantization of a gauge theory outlined above is of utmost importance in order to understand the structure of the theory, it is not a convenient way to calculate S-matrix elements in Lorentz invariant gauge field theories. Such calculations are necessary for the comparison of the theory to the results of scattering experiments. For these calculations Lorentz covariant perturbation theory is superior. The BRST formalism [80–82, 88] in its Lagrangian form provides the calculation techniques and the required renormalizability proofs to achieve this in gauge field theories such as the electroweak theory and quantum chromodynamics in the standard model of particle physics. This approach was born out of he need to quantize GR [6, 89] and Yang-Mills theories of particle physics [90].

In BRST quantization, one amends the action functional with a gauge fixing action and a ghost action. The gauge fixing action introduces an auxiliary boson field B_a for each gauge fixing condition $\chi^a(q) = 0$, so that the field equations for B_a impose the gauge conditions. The ghost action introduces pairs of anticommuting Grassman fields called ghosts c^α and antighosts $\bar{c}_a$ for each gauge degree of freedom, which couple to the "matter fields" q_i via a gauge transformation $\delta_\alpha \chi^a$ of the gauge conditions χ^a, i.e., as $\bar{c}_a c^\alpha \delta_\alpha \chi^a$. When the original gauge invariant Lagrangian is Lorentz invariant, one can use the powerful antifield formalism [91, 92] for determining the ghost structure and obtaining a manifestly Lorentz covariant gauge fixed Lagrangian. The generating functional now formally looks like

$$
\begin{aligned}
Z[J] = \int \mathcal{D}q\mathcal{D}B\mathcal{D}c\mathcal{D}\bar{c} \exp\Big(\frac{i}{\hbar} \big(S[q] &+ S_{\text{gauge}}[q, B] \\
&+ S_{\text{ghost}}[q, c, \bar{c}] + S_J[q, J]\big)\Big) ,
\end{aligned}
\tag{2.52}
$$

where S is the original gauge invariant action, S_{gauge} fixes the gauge, S_{ghost} introduces the ghosts and antighosts and S_J adds the auxiliary sources J for the fields q. The gauge conditions are chosen to be Lorentz covariant and suited for perturbation theory calculations. The extended action satisfies a new invariance under the so-called BRST transformation, which is supersymmetric in the sense that it mixes the fermion and boson degrees of freedom. This extension of the theory brings about great elegance to the quantization of gauge theories. In particular, proofs of renormalization and anomaly cancellation are greatly simplified. Perturbative expansion of the S-matrix as a Dyson series can be guaranteed to be unitary and renormalizable at each loop order of the expansion, whenever the theory under consideration is free of pathological ghosts and renormalizable.

Chapter 3

Hamiltonian structure of spacetime in general relativity

Hamiltonian formulation of GR was among the very first theories that were studied after Dirac generalized Hamiltonian formalism for constrained systems [93, 94]. The canonical structure of GR was worked out by Arnowitt, Deser and Misner [95–97]. It is called the Arnowitt-Deser-Misner (ADM) formalism. For reviews and mathematical background, see [98].

GR is a generally covariant theory. Covariance under general coordinate transformations means that the coordinates of the theory are just parameters that can be chosen at will. That is, coordinates are not physical quantities. In fact any theory can be expressed in parameterized form by regarding the coordinates x^μ of the theory as dynamical variables that depend on an equal number of independent parameters y^μ as $x^\mu = X^\mu(y)$. Then the system is invariant under reparameterization $y^\mu = Y^\mu(z)$ in terms of another set of independent parameters z^μ. GR is such a parameterized theory by definition.

3.1 Geometry of hypersurfaces

Consider a three-dimensional hypersurface Σ embedded in four-dimensional spacetime $\mathcal{M}$. Metric tensor $g_{\mu\nu}$ of spacetime induces a metric on the hypersurface, which is defined by

$$h_{\mu\nu} = g_{\mu\nu} - \epsilon n_\mu n_\nu \,, \tag{3.1}$$

where n_μ is the unit normal to Σ and $\epsilon = n_\mu n^\mu$ is its norm. For a spacelike hypersurface the norm is $\epsilon = -1$, since n^μ is then timelike. Conversely, for a timelike hypersurface we have $\epsilon = 1$. We do not consider null hypersurfaces. The induced metric $h_{\mu\nu}$ is sometimes referred to as the first fundamental form of the hypersurface. With one spacetime index raised, $h^\mu{}_\nu = g^{\mu\rho} h_{\rho\nu} = h^{\mu\rho} h_{\rho\nu}$, it is the projection operator onto Σ:

$$h^{\mu}{}_{\nu} = \delta^{\mu}_{\nu} - \epsilon n^{\mu} n_{\nu} \,. \tag{3.2}$$

The subscript $\perp$ in front of a tensor is used to denote that it has been projected onto Σ, thus orthogonal to the normal n^{μ}. For example we denote

$$_{\perp}T^{\mu}{}_{\nu} = h^{\mu}{}_{\rho} h^{\sigma}{}_{\nu} T^{\rho}{}_{\sigma} \,. \tag{3.3}$$

We denote the metric compatible covariant derivatives on $(\mathcal{M}, g_{\mu\nu})$ and $(\Sigma, h_{\mu\nu})$ by ∇ and D, respectively. The covariant derivative DT of a (k,l)-tensor field T on Σ is given as the projection $_{\perp}\nabla T$ of the covariant derivative on spacetime, written in index notation as

$$D_{\mu} T^{\nu_1 \cdots \nu_k}{}_{\rho_1 \cdots \rho_l} = h^{\sigma}{}_{\mu} h^{\nu_1}{}_{\alpha_1} \cdots h^{\nu_k}{}_{\alpha_k} h^{\beta_1}{}_{\rho_1} \cdots h^{\beta_l}{}_{\rho_l} \nabla_{\sigma} T^{\alpha_1 \cdots \alpha_k}{}_{\beta_1 \cdots \beta_l} \,, \tag{3.4}$$

where in the right-hand side one considers the extension of T on spacetime.

Extrinsic curvature tensor of the hypersurface is defined as the component of $\nabla_{\mu} n_{\nu}$ that is fully tangent to Σ,[1]

$$K_{\mu\nu} = h^{\rho}{}_{(\mu} h^{\sigma}{}_{\nu)} \nabla_{\rho} n_{\sigma} = h^{\rho}{}_{\mu} \nabla_{\rho} n_{\nu} = \nabla_{\mu} n_{\nu} - \epsilon n_{\mu} a_{\nu} \,, \tag{3.5}$$

where we have defined the acceleration of an observer with velocity n_{μ} as

$$a_{\mu} = \nabla_{n} n_{\mu} = n^{\nu} \nabla_{\nu} n_{\mu} \,. \tag{3.6}$$

Incidentally, the extrinsic curvature (3.5) can be written as the Lie derivative of the induced metric $h_{\mu\nu}$ along the unit normal n to Σ,

$$K_{\mu\nu} = \frac{1}{2} \mathcal{L}_{n} h_{\mu\nu} \,. \tag{3.7}$$

The extrinsic curvature is sometimes referred to as the second fundamental form of Σ. The geometry of Σ is defined by the two fundamental forms of the hypersurface.

Decomposition of the Riemann curvature tensor of spacetime into components tangent and normal to the hypersurface Σ is given by the following projection relations:

i. Gauss relation

$$_{\perp}R_{\mu\nu\rho\sigma} \equiv h^{\alpha}{}_{\mu} h^{\beta}{}_{\nu} h^{\gamma}{}_{\rho} h^{\delta}{}_{\sigma} R_{\alpha\beta\gamma\delta} = {}^{(3)}R_{\mu\nu\rho\sigma} - \epsilon \left(K_{\mu\rho} K_{\nu\sigma} - K_{\mu\sigma} K_{\nu\rho} \right) . \tag{3.8}$$

ii. Codazzi relation

$$_{\perp}R_{\mu\nu\rho\boldsymbol{n}} \equiv h^{\alpha}{}_{\mu} h^{\beta}{}_{\nu} h^{\gamma}{}_{\rho} n^{\delta} R_{\alpha\beta\gamma\delta} = 2 D_{[\mu} K_{\nu]\rho} \,. \tag{3.9}$$

[1]We denote symmetrization and antisymmetrization of tensor indices by parentheses and square brackets, respectively. Normalization is chosen so that symmetrization has no effect on an already symmetric tensor.

iii. Ricci relation

$$
{}_{\perp}R_{\mu\boldsymbol{n}\nu\boldsymbol{n}} \equiv h^{\alpha}{}_{\mu}n^{\beta}h^{\gamma}{}_{\nu}n^{\delta}R_{\alpha\beta\gamma\delta} = K_{\mu\rho}K_{\nu}{}^{\rho} - \mathcal{L}_n K_{\mu\nu} + D_{(\mu}a_{\nu)} - \epsilon a_{\mu}a_{\nu}\,. \tag{3.10}
$$

These relations can be obtained by applying the Ricci identity to the covariant derivatives ∇_μ and D_μ. The remaining projections of Riemann tensor are either zero or related to the given ones due to the symmetries of the Riemann tensor. In the Gauss relation (3.8), ${}^{(3)}R_{\mu\nu\rho\sigma}$ is the Riemann tensor of the three-dimensional hypersurface Σ. In the used notation, the special tensor index $\boldsymbol{n}$ denotes contraction with the unit normal n^μ. Similar projection relations can be obtained for the Ricci tensor $R_{\mu\nu}$, scalar curvature R and the Weyl tensor $C_{\mu\nu\rho\sigma}$ by using the given relations (3.8)–(3.10).

Generalization to d-dimensional hypersurfaces embedded in a $(d+1)$-dimensional spacetime is straightforward.

3.2 Foliation of spacetime and its ADM parameterization

We consider a globally hyperbolic spacetime $\mathcal{M}$ that admits a foliation into a family of non-intersecting Cauchy surfaces Σ_t, which cover the spacetime. Each Cauchy surface Σ_t is a spacelike hypersurface, such that every never-ending causal curve intersects each Σ_t exactly once. The Cauchy surfaces are parameterized by level surfaces of a smooth scalar field t. Spacetime is the union of the Cauchy surfaces.

Then we introduce a timelike vector field t^μ that satisfies $t^\mu \nabla_\mu t = 1$. This vector field is decomposed into components normal and tangent to the spatial hypersurfaces Σ_t as $t^\mu = N n^\mu + N^\mu$, where $N = -n_\mu t^\mu$ is the lapse function and $N^\mu = h^\mu{}_\nu t^\nu$ is the shift vector on the spatial hypersurface Σ_t. The ADM variables consists of the lapse function, the shift vector and the induced metric (3.1) on Σ_t. Together they describe the foliation of spacetime.

Then we introduce a coordinate system on spacetime. We regard the smooth scalar field t as the time coordinate and introduce an arbitrary smooth coordinate system $(x^i, i = 1,2,3)$ on the spatial hypersurfaces Σ_t. The unit normal to Σ_t can now be written in terms of the lapse and shift variables as

$$
n_\mu = -N\nabla_\mu t = (-N, 0, 0, 0)\,, \qquad n^\mu = \left(\frac{1}{N}, -\frac{N^i}{N}\right). \tag{3.11}
$$

The invariant line element in spacetime is written

$$
\mathrm{d}s^2 = -N^2\mathrm{d}t^2 + h_{ij}\left(\mathrm{d}x^i + N^i\mathrm{d}t\right)\left(\mathrm{d}x^j + N^j\mathrm{d}t\right). \tag{3.12}
$$

The lapse function must be positive everywhere, $N > 0$, since $N\mathrm{d}t$ measures the lapse of proper time between the hypersurfaces Σ_t and Σ_{t+dt}. Integration over spacetime decomposes as $\int_{\mathcal{M}} \mathrm{d}^4x\sqrt{-g} = \int \mathrm{d}t \int_{\Sigma_t} \mathrm{d}^3x N\sqrt{h}$.

In the given ADM coordinate basis, the components of the metric of spacetime read

$$g_{00} = -N^2 + N_i N^i \,, \qquad g_{0i} = g_{i0} = N_i \,, \qquad h_{ij} = h_{ij} \,, \tag{3.13}$$

where $N_i = h_{ij} N^j$. Contravariant components of the metric of spacetime are

$$g^{00} = -\frac{1}{N^2} \,, \qquad g^{0i} = g^{i0} = \frac{N^i}{N^2} \,, \qquad g^{ij} = h^{ij} - \frac{N^i N^j}{N^2} \,, \tag{3.14}$$

where $h^{ij} h_{jk} = \delta^i_k$. Indices of tensors that are tangent to Σ_t can be lowered and raised by the induced metric h_{ij} and its inverse h^{ij}.

The extrinsic curvature tensor (3.5) is written

$$K_{ij} = \frac{1}{2} \mathcal{L}_n h_{ij} = \frac{1}{2N} \left(\partial_t h_{ij} - 2 D_{(i} N_{j)} \right) , \qquad K = h^{ij} K_{ij} \,, \tag{3.15}$$

where ∂_t denotes the partial derivative with respect to time t. The Lie derivative $\mathcal{L}_n A_{i_1 \cdots i_k}$ for any covariant tensor $A_{i_1 \cdots i_k}$ on Σ_t is given as

$$\mathcal{L}_n A_{i_1 \cdots i_k} = \frac{1}{N} \left(\partial_t A_{i_1 \cdots i_k} - \mathcal{L}_{\vec{N}} A_{i_1 \cdots i_k} \right) , \tag{3.16}$$

where $\mathcal{L}_{\vec{N}}$ denotes the Lie derivative along the shift vector N^i on the spatial hypersurface.

In the ADM coordinate basis, the time-components of tensors tangent to the spatial hypersurface Σ_t are defined by the spatial components of the tensor and the shift vector, depending on the way in which the spacetime is foliated. For example, the time-component of a covector A_μ which is tangent to Σ_t is obtained from $A_\mu n^\mu = 0$ as $A_0 = A_i N^i$. For contravariant tensors the time-components evidently vanish, e.g., $A^0 = 0$.

An observer whose velocity n_μ is normal to the spatial hypersurface has the acceleration (3.6) that is tangent to the hypersurface. It is given by the spatial derivative of the lapse function as

$$a_i = D_i \ln N = \frac{D_i N}{N} \,. \tag{3.17}$$

In the projection relation (3.10) for Riemann tensor, we give the terms involving a_i in terms of covariant derivatives of N as

$$D_{(i} a_{j)} + a_i a_j = \frac{D_i D_j N}{N} \,. \tag{3.18}$$

For GR, it suffices to obtain the decomposition of the scalar curvature with respect to the spatial hypersurfaces as

$$\begin{aligned} R &= {}^{(3)}\!R - 3 K_{ij} K^{ij} + K^2 + 2 h^{ij} \mathcal{L}_n K_{ij} - \frac{2}{N} D^i D_i N \\ &= {}^{(3)}\!R + K_{ij} K^{ij} - K^2 + 2 \nabla_\mu \left(n^\mu K - a^\mu \right) . \end{aligned} \tag{3.19}$$

In the second expression, the second-order time derivatives found in $h^{ij} \mathcal{L}_n K_{ij}$ have been absorbed into the covariant total derivative $\nabla_\mu (n^\mu K)$.

3.3 Hamiltonian formulation of general relativity

3.3.1 Einstein-Hilbert action with boundary surface terms

Variation of the EH action (1.1) with respect to the metric includes a nonvanishing surface integral over the boundary $\partial\mathcal{M}$ of spacetime. In order to obtain a variational principle that is consistent with the Einstein field equations, when only the variation of the metric (and not its derivatives) is fixed to zero on the boundary $\partial\mathcal{M}$, we shall add a surface term into the EH action so that the surface term in the variation of the original action gets cancelled. The amended EH action is defined by

$$S_{\mathrm{EH}} = \frac{1}{2\kappa} \int_{\mathcal{M}} \mathrm{d}^4x \sqrt{-g} R + \frac{1}{\kappa} \oint_{\partial\mathcal{M}} \mathrm{d}^3x \sqrt{|\gamma|} K \,. \tag{3.20}$$

Here $\gamma_{\mu\nu}$ is the induced metric on the boundary and K is the trace of the extrinsic curvature of the boundary. This completion to EH action was originally found in [99] and further considered in [100].

When the decomposition of scalar curvature (3.19) is substituted into the action (3.20), an additional surface term appears, giving the total surface contribution to the action as

$$S_{\mathrm{surface}} = \frac{1}{\kappa} \oint_{\partial\mathcal{M}} \mathrm{d}^3x \sqrt{|\gamma|} K + \frac{1}{\kappa} \oint_{\partial\mathcal{M}} \mathrm{d}^3x \sqrt{|\gamma|} r_\mu \left(n^\mu K - a^\mu \right) , \tag{3.21}$$

where r_μ is the outward-pointing unit normal to the boundary of spacetime. We should emphasize that in the first surface term K refers to the extrinsic curvature of the boundary $\partial\mathcal{M}$, while in the last term K refers to the extrinsic curvature of the spatial hypersurfaces Σ_t. In our globally hyperbolic spacetime, the boundary $\partial\mathcal{M}$ consists of the initial and final Cauchy surfaces, say Σ_1 and Σ_2, respectively, and of the timelike hypersurface $\mathcal{B}$ that connects those spatial hypersurfaces. The timelike part of the boundary is the union $\mathcal{B} = \bigcup_t \mathcal{B}_t$ of the two-dimensional boundaries $\mathcal{B}_t$ of the Cauchy surfaces Σ_t (at spatial infinity). On the initial and final Cauchy surfaces Σ_1 and Σ_2, the surface integrals cancel each other entirely. Thus only the integral over $\mathcal{B}$ survives in the surface terms,

$$S_{\mathrm{surface}} = \frac{1}{\kappa} \int_{\mathcal{B}} \mathrm{d}^3x \sqrt{-\gamma} \left(K_{\mathcal{B}} + r_\mu n^\mu K - r_\mu a^\mu \right) . \tag{3.22}$$

Here the trace of the extrinsic curvature of $\mathcal{B}$ is denoted by $K_{\mathcal{B}} = \nabla_\mu r^\mu$, so that it is not confused with K, which refers to the trace of the extrinsic curvature of the surfaces Σ_t on its intersection with the boundary $\mathcal{B}$. If the surfaces $\mathcal{B}$ and Σ_t are assumed to be orthogonal, the normals to $\mathcal{B}$ and Σ_t are orthogonal as well, i.e., $r_\mu n^\mu = 0$, and hence we further obtain [101]

$$S_{\mathrm{surface}} = \frac{1}{\kappa} \int_{\mathcal{B}} \mathrm{d}^3x \sqrt{-\gamma} h^{\mu\nu} \nabla_\mu r_\nu = \frac{1}{\kappa} \int \mathrm{d}t \oint_{\mathcal{B}_t} \mathrm{d}^2x N \sqrt{\sigma}\, {}^{(2)}\!K \,, \tag{3.23}$$

where σ_{ab} is the induced metric on $\mathcal{B}_t$ and ${}^{(2)}K$ is the extrinsic curvature of $\mathcal{B}_t$ embedded in Σ_t. This general expression for the surface term generalizes the ones encountered in asymptotically flat or anti-de Sitter spacetimes (compare for example to [102]). In the nonorthogonal case, one has to include extra two-dimensional surface terms involving the intersection angle $\eta = r_\mu n^\mu$ of the hypersurfaces $\mathcal{B}$ and Σ_t as [103]

$$S_{\text{surface}} = \frac{1}{\kappa}\int_{\mathcal{B}} \mathrm{d}^3x\sqrt{-\gamma}\,(K_{\mathcal{B}} + \eta K - r_\mu a^\mu) + \frac{1}{\kappa}\int_{\mathcal{B}_1}^{\mathcal{B}_2} \mathrm{d}^3x\sqrt{\sigma}\,\sinh^{-1}\eta\,, \quad (3.24)$$

where we denote the difference of the integrals over the two-dimensional final and initial surfaces $\mathcal{B}_2$ and $\mathcal{B}_1$ as $\int_{\mathcal{B}_1}^{\mathcal{B}_2} = \int_{\mathcal{B}_2} - \int_{\mathcal{B}_1}$. In the following treatment, we assume the orthogonal case (3.23) for simplicity.

The EH action (3.20) can now be written in the form (2.4) as[2]

$$\begin{aligned} S_{\text{EH}}[N, N^i, h_{ij}] = \frac{1}{2\kappa}\int_{t_1}^{t_2} \mathrm{d}t \Bigg[\int_{\Sigma_t} N\sqrt{h}\left(K_{ij}K^{ij} - K^2 + {}^{(3)}R\right) \\ + 2\oint_{\mathcal{B}_t} N\sqrt{\sigma}\,{}^{(2)}K\Bigg]. \quad (3.25)\end{aligned}$$

The corresponding Lagrangian can be read from (3.25) as a functional of N, N^i, h_{ij} and $\partial_t h_{ij}$.

If spacetime is spatially noncompact, we must choose a reference background and define the physical action as the difference to the reference action. Given a reference background g_0, ψ_0 for spacetime and matter configuration, the physical action is defined as

$$S_{\text{phys}}[g,\psi] = S[g,\psi] - S[g_0,\psi_0]\,, \quad (3.26)$$

where the action S consists of (3.25) plus the contribution of matter. An admissible reference background g_0, ψ_0 induces the same configuration on the fixed timelike boundary $\mathcal{B}$ as the fields g, ψ, at least asymptotically (at spatial infinity).

3.3.2 Hamiltonian formalism

Let us consider Hamiltonian formulation of the pure EH action (3.25). The canonical variables consist of the ADM variables N, N^i, h_{ij} and their canonically conjugated momenta p_N, p_i, p^{ij}, respectively, which are defined according to (2.6). The canonical Poisson brackets are postulated in the form (equal time t is understood)

$$\begin{aligned} \{N(\boldsymbol{x}), p_N(\boldsymbol{y})\} &= \delta(\boldsymbol{x}-\boldsymbol{y})\,, \quad (3.27)\\ \left\{N^i(\boldsymbol{x}), p_j(\boldsymbol{y})\right\} &= \delta^i_j\,\delta(\boldsymbol{x}-\boldsymbol{y})\,,\\ \left\{h_{ij}(\boldsymbol{x}), p^{kl}(\boldsymbol{y})\right\} &= \delta_i^{(k}\delta_j^{l)}\,\delta(\boldsymbol{x}-\boldsymbol{y})\,, \end{aligned}$$

[2] From now on we shall also omit the fixed integration measure d^2x from integrals over the boundary $\mathcal{B}_t$.

with all the other Poisson brackets among the variables vanishing.

Since the action is independent of the time derivatives of the lapse N and the shift N^i variables, their canonically conjugated momenta are the local primary constraints

$$p_N \approx 0\,, \qquad p_i \approx 0\,. \tag{3.28}$$

The momentum canonically conjugate to h_{ij} is defined by

$$p^{ij} = \frac{1}{2\kappa} G^{ijkl} K_{kl}\,, \qquad G^{ijkl} = \sqrt{h}\left[\frac{1}{2}\left(h^{ik}h^{jl} + h^{il}h^{jk}\right) - h^{ij}h^{kl}\right], \tag{3.29}$$

where G^{ijkl} is the DeWitt metric [104] (metric of the space of metrics). It has an inverse

$$G_{ijkl} = \frac{1}{2\sqrt{h}}\left(h_{ik}h_{jl} + h_{il}h_{jk} - h_{ij}h_{kl}\right), \tag{3.30}$$

which satisfies $G_{ijmn}G^{mnkl} = \delta_i^{(k}\delta_j^{l)}$. No more primary constraints are required since every $\partial_t h_{ij}$ can be solved in terms of the canonical variables.

The total Hamiltonian (2.12) is obtained as

$$\begin{aligned} H = &\int_{\Sigma_t} \left(N\mathcal{H}_0 + N^i\mathcal{H}_i + v_N p_N + v^i p_i\right) \\ &- \frac{1}{\kappa}\oint_{\mathcal{B}_t} N\sqrt{\sigma}\,{}^{(2)}K + 2\oint_{\mathcal{B}_t} N_i r_j p^{ij}\,, \end{aligned} \tag{3.31}$$

where we have defined the quantities

$$\mathcal{H}_0 = 2\kappa G_{ijkl}p^{ij}p^{kl} - \frac{1}{2\kappa}\sqrt{h}\,{}^{(3)}R \tag{3.32}$$

and

$$\mathcal{H}_i = -2h_{ij}D_k p^{jk} = -2h_{ij}\partial_k p^{jk} - \left(2\partial_j h_{ik} - \partial_i h_{jk}\right)p^{jk}\,. \tag{3.33}$$

The four primary constraints p_N and p_i are multiplied by arbitrary Lagrange multiplier fields v_N and v^i. The second boundary term in the Hamiltonian (3.31) comes from an integration by parts in the term

$$\int_{\Sigma_t} 2p^{ij}D_{(i}N_{j)} = -2\int_{\Sigma_t} N_i D_j p^{ij} + 2\oint_{\mathcal{B}_t} N_i r_j p^{ij}\,,$$

where r_i is the outward-pointing unit normal to the boundary $\mathcal{B}_t$.

The consistency conditions (2.15) for the primary constraints command us to impose the secondary constraints

$$\mathcal{H}_0 \approx 0\,, \qquad \mathcal{H}_i \approx 0\,. \tag{3.34}$$

We shall call them the *Hamiltonian constraint* and the *momentum constraint*, respectively. The Hamiltonian constraint (3.32) has the familiar form, a kinetic term minus a potential term, where extrinsic curvature squared plays the part of kinetic energy and intrinsic curvature acts as the potential. In empty space the extrinsic and intrinsic curvature contributions cancel each other exactly. No more constraints are required by the consistency conditions for the secondary constraints (3.34). All the constraints turn out to be first-class constraints.

We define a smeared version of the Hamiltonian constraint as the functional

$$\mathcal{H}_0[\xi] = \int_{\Sigma_t} \xi \mathcal{H}_0 \,, \tag{3.35}$$

where ξ is an arbitrary test function on Σ_t. A smeared momentum constraint is defined as

$$\Phi\big[\vec{X}\big] = \int_{\Sigma_t} X^i \mathcal{H}_i \,, \tag{3.36}$$

where $\vec{X}$ is an arbitrary test vector on Σ_t. These smeared first-class constraints shall act as generators of gauge transformations (2.22). The Hamiltonian and momentum constraints satisfy the following Poisson brackets among themselves

$$\{\mathcal{H}_0[\xi], \mathcal{H}_0[\eta]\} = \Phi\big[\xi \vec{D}\eta - \eta \vec{D}\xi\big] \,, \tag{3.37}$$

$$\Big\{\Phi\big[\vec{X}\big], \mathcal{H}_0[\xi]\Big\} = \mathcal{H}_0[\vec{X}(\xi)] \,, \tag{3.38}$$

$$\Big\{\Phi\big[\vec{X}\big], \Phi\big[\vec{Y}\big]\Big\} = \Phi\big[\big[\vec{X}, \vec{Y}\big]\big] \,, \tag{3.39}$$

where the vector $\vec{D}\eta$ has the components $(\vec{D}\eta)^i = h^{ij} D_j \eta$ and the action of a vector on a function is defined as usual, $\vec{X}(\xi) = X^i \partial_i \xi$, as is the commutator of vectors,

$$\big[\vec{X}, \vec{Y}\big]^i = X^j \partial_j Y^i - Y^j \partial_j X^i \,.$$

The Poisson bracket between Hamiltonian constraints (3.37) is obtained by a direct calculation. The momentum constraint (3.36) can be written as

$$\Phi\big[\vec{X}\big] = \int_{\Sigma_t} p^{ij} \mathcal{L}_{\vec{X}} h_{ij} - 2 \oint_{\mathcal{B}_t} X_i r_j p^{ij} \,, \tag{3.40}$$

where $\mathcal{L}_{\vec{X}} h_{ij} = 2D_{(i} X_{j)}$. Thus the momentum constraint generates infinitesimal spacetime-dependent spatial diffeomorphism for the canonical variables h_{ij}, p^{ij} on the hypersurface Σ_t. The metric h_{ij} behaves as a regular tensor field under the spatial diffeomorphisms, while the canonically conjugated momentum p^{ij} behaves as a tensor density of unit weight,[3]

$$\Big\{h_{ij}, \Phi\big[\vec{X}\big]\Big\} = \mathcal{L}_{\vec{X}} h_{ij} = X^k \partial_k h_{ij} + \partial_i X^k h_{kj} + \partial_j X^k h_{ik} \,,$$

$$\Big\{p^{ij}, \Phi\big[\vec{X}\big]\Big\} = \mathcal{L}_{\vec{X}} p^{ij} = X^k \partial_k p^{ij} + \partial_k X^k p^{ij} - \partial_k X^i p^{kj} - \partial_k X^j p^{ik} \,.$$

The Poisson brackets (3.38) and (3.39) follow from the fact that the Hamiltonian and momentum constraints are scalar and vector densities of unit weight

[3] On the boundary $\mathcal{B}_t$ of Σ_t, we may obtain some extra contributions from the required integrations by parts. Unless we consider that the diffeomorphisms become identity on the spatial boundary, i.e., the gauge parameter $X^i \to 0$ on $\mathcal{B}_t$ in the generator (3.36).

under spatial diffeomorphism, respectively. The momentum constraints form a Lie algebra under the Poisson bracket (3.39), which is reminiscent of the Lie algebra formed by Lie derivatives. The Poisson bracket between Hamiltonian and momentum constraints (3.38) too has a Lie algebraic form. However, in the right-hand side of the Poisson bracket between Hamiltonian constraints (3.37), the test vector multiplying the momentum constraint depends on the metric as $h^{ij}\left(\xi D_j\eta - \eta D_j\xi\right)$. These "field-dependent structure constants" break the Lie algebra structure.

The primary constraints p_N and p_i clearly have vanishing Poisson brackets with every constraint, since none of the constraints depend on N or N^i. Time evolution of the lapse and shift variables is specified by the arbitrary Lagrange multipliers of the primary constraints: $\partial_t N = v_N$, $\partial_t N^i = v^i$. Thus we can see that N and N^i serve as arbitrary multipliers of the Hamiltonian and momentum constraints in the Hamiltonian (3.31). Their values can be set at will as a gauge choice, provided that $N > 0$ everywhere so that the foliation of spacetime is well defined.

The secondary constraints $\mathcal{H}_0 = 0$ and $\mathcal{H}_i = 0$ together with canonical equations of motion

$$\partial_t h_{ij} = 4\kappa N G_{ijkl} p^{kl} + \mathcal{L}_{\vec{N}} h_{ij}\,, \tag{3.41}$$

$$\begin{aligned}\partial_t p^{ij} = &-\frac{4\kappa N}{\sqrt{h}}\left[p^i{}_k p^{jk} - \frac{1}{2}p^{ij}p - \frac{1}{4}h_{ij}\left(p_{ij}p^{ij} - \frac{1}{2}p^2\right)\right] \\ &-\frac{N\sqrt{h}}{2\kappa}\left({}^{(3)}R^{ij} - \frac{1}{2}h^{ij\,(3)}R\right) \\ &+\frac{\sqrt{h}}{2\kappa}\left(D^i D^j N - h^{ij}D^k D_k N\right) + \mathcal{L}_{\vec{N}} p^{ij}\,,\end{aligned} \tag{3.42}$$

express the Einstein field equations in canonical first-order form. We denote $p = h_{ij}p^{ij}$. Here the gauge freedom of the system presents itself by the four arbitrary multipliers N and N^i, and by the fact that the coordinate system on Σ_t has been left unspecified. The constraints $\mathcal{H}_0$ and $\mathcal{H}_i$ do not involve time derivatives of the canonical variables. They provide the so-called initial-value conditions of GR, whose preservation in time is ensured by the closure of the constraint algebra.

We explain the gauge fixing procedure in the general framework discussed in Chapter 2. The simplest way to fix the gauge freedom associated with the primary constraints $p_N = 0$ and $p_i = 0$ is to impose the lapse and shift variables to constant values everywhere. There do exist useful field-dependent choices for the conditions on N and N^i, but we do not consider them here. The gauge freedom associated with the secondary constraints $\mathcal{H}_0 = 0$ and $\mathcal{H}_i = 0$ can be fixed by introducing four conditions among the components of the metric h_{ij}. Thus we can choose the gauge conditions as

$$\begin{gathered}\sigma_\mu = 0\,, \qquad \sigma_0 = N - 1\,, \qquad \sigma_i = N^i\,, \\ \chi_\mu(h_{ij}) = 0\,, \qquad \mu = 0,\ldots,3\,.\end{gathered} \tag{3.43}$$

The four gauge conditions χ_μ have to be such that they fix four components of the metric h_{ij}, i.e., $\mathrm{rank}(\delta\chi_\mu/\delta h_{ij}) = 4$. These conditions are often referred to as coordinate conditions. This is because the conditions (3.43) essentially fix the coordinate system on spacetime and define how the spacetime is foliated. The secondary constraints $\mathcal{H}_0 = 0$ and $\mathcal{H}_i = 0$ can be regarded to fix four components of the momenta p^{ij}. This leaves us four independent canonical variables corresponding to the two physical degrees of freedom for the massless graviton. The same number of physical degrees of freedom can of course be obtained directly by applying Dirac's formula (2.26): $(20-2\times 8)/2 = 2$. The Dirac bracket can be defined (2.18) with the set of second-class constraints $\varphi_b = (p_N, p_i, \sigma_\mu, \mathcal{H}_\mu, \chi_\mu)$, $\mu = 0, \ldots, 3$. Since no other constraint or dynamical variable depends on the lapse and shift variables, these variables (N, p_N, N^i, p_i) simply drop out from the system. That leaves us eight nontrivial constraints $(\mathcal{H}_\mu, \chi_\mu)$ among the twelve dynamical variables h_{ij}, p^{ij}.

In the original ADM approach [97], the gauge freedom was completely removed in order to reduce the set of dynamical variables to the transverse-traceless components of the metric variables h_{ij}^{TT} and p_{TT}^{ij}, namely the two helicity states of the massless spin-2 graviton. Identification of the dependent and independent degrees of freedom has been developed considering the initial-value problem [105]. The canonical momentum p^{ij} can be otrhogonally and covariantly decomposed into a transverse-traceless part, a longitudinal (vector) part and a trace part. The longitudinal part of p^{ij} is fixed by the momentum constraint. The Hamiltonian constraint can be regarded to fix the conformal factor of the metric h_{ij}, leaving the conformally invariant metric independent. This provides a covariant description of the independent gravitational degrees of freedom. Coordinate conditions could of course be introduced to further reduce the system.

In a generally covariant system that is invariant under time reparameterization, Hamiltonian is typically a first-class constraint. The same is true for theories with more general diffeomorphism invariance. In a generally covariant system, time evolution is just the unfolding of a gauge transformation. In the present case of GR, we can see that the bulk part of the Hamiltonian (3.31) is indeed a sum of first-class constraints. However, the surface contribution on the boundary of spatial hypersurfaces does not vanish on the constraint surface. This indeed provides us the concept of total energy. First in order to obtain the physical Hamiltonian we need to subtract the reference background. The actual spacetime and the reference background should induce the same metric on the spatial boundary $\mathcal{B}_t$, at least asymptotically. Hence the volume element on the boundary is identical for them. Since the background is a stationary solution to the field equations, the constraints associated with the solution vanish. The canonical momentum p^{ij} vanishes for the background when the foliation is chosen appropriately. Thus the Hamiltonian for the background can be written $H_{\mathrm{b}} = -\frac{1}{\kappa}\oint_{\mathcal{B}_t} N\sqrt{\sigma}\,{}^{(2)}K_{\mathrm{b}}$, where ${}^{(2)}K_{\mathrm{b}}$ is the trace of the extrinsic curvature of the boundary for the background. Now the physical Hamiltonian is the difference $H_{\mathrm{phys}} = H - H_{\mathrm{b}}$. We can now define the total energy associated with the time translation along $t^\mu = Nn^\mu + N^\mu$ for any given solution as the value of the physical Hamiltonian on the constraint surface,

$$E = -\frac{1}{\kappa} \oint_{\mathcal{B}_t} N\sqrt{\sigma} \left({}^{(2)}K - {}^{(2)}K_{\mathrm{b}} \right) + 2 \oint_{\mathcal{B}_t} N_i r_j p^{ij} \,. \tag{3.44}$$

As a special case one obtains the ADM energy and momentum defined for asymptotically flat spacetimes. The ADM energy corresponds to unit time translation $N = 1, N^i = 0$,

$$E_{\mathrm{ADM}} = \frac{1}{2\kappa} \oint_S r^i \left(D^j \tilde{h}_{ij} - D_i \tilde{h}^j{}_j \right) , \tag{3.45}$$

where r^i is the unit normal to the spherical boundary S, the spatial metric with the background metric subtracted is denoted by $\tilde{h}_{ij} = h_{ij} - h_{0,ij}$, and the covariant derivative D_i is defined for the background. The momentum part of (3.44) represents how the total energy changes under boosts of t^μ (with constant N and N^i),

$$P_i N^i = 2 \oint_S N_i r_j p^{ij} \,. \tag{3.46}$$

3.3.3 Including matter

So far we have considered pure gravity without any field content. In GR, regular matter couples minimally to the metric of spacetime. Including matter fields without additional gauge symmetry is straightforward. Both the Hamiltonian constraint $\mathcal{H}_0$ and the momentum constraint $\mathcal{H}_i$ receive contributions from matter. As a simple example we can consider a scalar field ϕ with potential $V(\phi)$, whose additional contribution to $\mathcal{H}_0$ and $\mathcal{H}_i$ would be

$$\mathcal{H}_0^\phi = \frac{p_\phi^2}{2\sqrt{h}} + \frac{1}{2}\sqrt{h} h^{ij} D_i\phi D_j\phi + \sqrt{h} V(\phi) \,, \qquad \mathcal{H}_i^\phi = p_\phi D_i \phi \,, \tag{3.47}$$

where p_ϕ is the canonical momentum conjugate to ϕ. We can see that these contributions are proportional to projections of the energy-momentum tensor $T^\phi_{\mu\nu}$ of the scalar field as $\mathcal{H}_0^\phi = \sqrt{h}\, T^\phi_{\boldsymbol{nn}}$ and $\mathcal{H}_i^\phi = \sqrt{h}_\perp T^\phi_{i\boldsymbol{n}}$, where the projections are defined by $T^\phi_{\boldsymbol{nn}} = n^\mu n^\nu T^\phi_{\mu\nu}$ and ${}_\perp T^\phi_{i\boldsymbol{n}} = h^\mu{}_i n^\nu T^\phi_{\mu\nu}$. The right-hand side of the equation of motion (3.42) for p^{ij} receives a contribution from $\mathcal{H}_0^\phi$, which provides the primary coupling of the scalar field to the dynamics of spacetime.

In general, the Hamiltonian and momentum constraints of GR attain the form

$$\mathcal{H}_0 = -\sqrt{h} T_{\boldsymbol{nn}} \,, \qquad \mathcal{H}_i = -\sqrt{h}_\perp T_{i\boldsymbol{n}} \,, \tag{3.48}$$

where $\mathcal{H}_0$ and $\mathcal{H}_i$ are the constraints for pure gravity (3.32) and (3.33), and the energy-momentum tensor represents all matter. When matter fields with additional gauge symmetry are included, we naturally obtain entirely new constraints, in addition to the matter contributions to the constraints associated with the diffeomorphism symmetry of spacetime.

3.4 Quantization

As we already discussed in Sec. 2.3, canonical quantization can be performed in different ways. The first possible approach is to reduce the phase space in order to remove all unphysical degrees of freedom, so that each field configuration corresponds to a different state of the system. Alternatively, one can keep the gauge freedom and work with a larger space of state, where physical states are those that are left invariant under gauge transformations (2.47). The classic treatment of canonical quantization of GR along this approach was conducted by Wheeler and DeWitt [104, 106]. The first-class constraints become conditions on the state vector $|\psi\rangle$:

$$\hat{p}_N\,|\psi\rangle = 0\,, \qquad \hat{p}_i\,|\psi\rangle = 0\,, \qquad \hat{\mathcal{H}}_0\,|\psi\rangle = 0\,, \qquad \hat{\mathcal{H}}_i\,|\psi\rangle = 0\,. \tag{3.49}$$

We can choose the metric representation for the quantum states, where the canonical momenta become differential operators

$$p_N = \frac{\delta}{i\delta N}\,, \qquad p_i = \frac{\delta}{i\delta N^i}\,, \qquad p^{ij} = \frac{\delta}{i\delta h_{ij}}\,, \tag{3.50}$$

and the state vector becomes the wave function $\Psi[h_{ij}]$. The Hamiltonian constraint becomes the Wheeler-DeWitt equation

$$\left(G_{ijkl}\frac{\delta}{\delta h_{ij}}\frac{\delta}{\delta h_{kl}} + \sqrt{h}^{(3)}R\right)\Psi[h_{ij}] = 0\,. \tag{3.51}$$

which governs the dynamics. It corresponds to the Schrödinger equation.

For perturbative calculations of S-matrix elements and for studying renormalizability properties, the covariant quantization approach provides a more suitable approach. Unfortunately, GR is not a renormalizable theory, since divergences that are proportional to higher powers of the curvature invariants appear [3–7]. A possible solution to this problem is the inclusion of higher-order derivatives into the gravitational action. This will be considered in the following chapters.

Chapter 4

Hořava-Lifshitz gravity

4.1 Introduction to Hořava-Lifshitz gravity

In 2009, the so-called Hořava-Lifshitz (HL) theory of gravity was proposed [14] (see also [107, 108]). It is a candidate for a renormalizable QFT of gravity. The techniques used in the construction of HL gravity closely parallel methods developed in the theory of dynamical critical systems and quantum criticality. The theory of a Lifshitz scalar [109, 110] is a prototype of the class of condensed matter models that are most relevant to Hořava's proposal. HL gravity exhibits anisotropic scaling of space and time coordinates

$$\boldsymbol{x} \to b\boldsymbol{x}\,, \qquad t \to b^z t\,, \tag{4.1}$$

with a dynamic critical exponent z. The theory is designed so that it has a solution which describes an UV free-field fixed point whose scaling properties are given by (4.1) for a suitable z. At short distances HL gravity describes interacting nonrelativistic gravitons. The propagator for gravitons depends on the energy ω and the spatial momentum $\boldsymbol{k}$ as

$$\frac{1}{\omega^2 - c^2\boldsymbol{k}^2\left(1 + \sum_{n=1}^{z-1}(\xi_n\boldsymbol{k}^2)^n\right) - G(\boldsymbol{k}^2)^z}\,, \tag{4.2}$$

where G is a (running) coupling constant and the constants ξ_n have the dimensions $[\xi_n] = [\boldsymbol{k}^2]^{-1}$. Recall that in GR the gravitational coupling constant G_N has the dimension $[G_N] = -2$ in mass units. In the UV Lifshitz point the value of the critical exponent z is chosen so that the gravitational coupling constant is dimensionless. This improves the UV behavior and hence enables us to overcome the main obstacle for the perturbative renormalizability of GR in the framework of QFT.

Because space and time scale differently, unlike GR the theory does not possess full diffeomorphism invariance. Spacetime $\mathcal{M}$ is postulated to possess a preferred foliation $\mathcal{F}$ into spacelike hypersurfaces Σ_t of constant time t. Hence spacetime is invariant only under the foliation-preserving diffeomorphisms $\mathrm{Diff}_{\mathcal{F}}(\mathcal{M})$, whose infinitesimal generators are of the form

$$\delta \boldsymbol{x} = \boldsymbol{\zeta}(t, \boldsymbol{x}) \,, \qquad \delta t = f(t) \,. \tag{4.3}$$

For simplicity the topological structure of spacetime is assumed to be such that every leaf Σ_t of the foliation is topologically equivalent to a fixed manifold Σ. The preferred foliation of spacetime defines a global causal structure, which puts some of the fundamental problems of GR and quantum gravity into a new perspective. The preferred foliation implies an invariant notion of time that is susceptible only to time-dependent reparameterization (4.3). As a result, various aspects of the "problem of time" [111] associated with the attempt to quantize GR are eliminated. In a gravitational action, the preferred foliation of spacetime enables the inclusion of extra spatial covariant derivatives, which improve the UV behavior of the graviton propagator (4.2), while avoiding higher-order time derivatives which are known to produce problematic ghosts.

The downside of these considerable advantages is the loss of relativistic invariance at short distances. Assuming the breaking of Lorentz invariance happens at sufficiently short distances, this could be an acceptable feature. Indeed, since there are some good reasons to suspect that space is emergent, Lorentz symmetry is expected to be emergent as well. The intriguing question is whether Lorentz symmetry emerges together with space, e.g., at the Planck scale or at some larger distance scale. In HL gravity we assume that Lorentz symmetry can be broken well before space ceases to exist, so that we can use quantum field theory to describe the Lorentz noninvariant physics in the UV regime, but of course not at too large distances in order to conform with the experimental bounds on Lorentz violation. At low energies and large distances the critical exponent is expected to flow to the infrared (IR) fixed point $z = 1$, so that an isotropic state of spacetime is restored. Then the theory could have a chance to reproduce the predictions of GR with sufficient accuracy. The Lorentz invariance is absent in the fundamental description, but emerges at low energies as an approximate (accidental) symmetry. Likewise, the full diffeomorphism symmetry is restored at large distances, at least classically.

Since there exists a preferred decomposition of spacetime into time and space, it is convenient to use the ADM parameterization of the metric (see Sec. 3.2). For generality, we consider $(d+1)$-dimensional spacetime with the line element

$$\mathrm{d}s^2 = -N^2 c^2 \mathrm{d}t^2 + h_{ij} \left(N^i \mathrm{d}t + \mathrm{d}x^i\right) \left(N^j \mathrm{d}t + \mathrm{d}x^j\right) , \tag{4.4}$$

where $i, j = 1, \ldots, d$. In the UV fixed point of HL gravity − henceforth called a Lifshitz point − the space and time coordinates have the following scaling dimensions in terms of the dimension of momentum $[\boldsymbol{k}]$:

$$[\boldsymbol{x}] = [\boldsymbol{k}]^{-1} \,, \qquad [t] = [\boldsymbol{k}]^{-z} \,. \tag{4.5}$$

The scaling dimensions of the ADM variables can be read from (4.4), where $[\mathrm{d}s^2] = [\boldsymbol{k}]^{-2}$ and (4.5):

$$[h_{ij}] = [1] \,, \qquad [N] = [1] \,, \qquad [N^i] = [c] = \frac{[\boldsymbol{x}]}{[t]} = [\boldsymbol{k}]^{z-1} \,. \tag{4.6}$$

Here "[1]" denotes a dimensionless quantity. Note that we are not working in natural units, where we have a quantity c (speed of light) of dimension $[\boldsymbol{x}]/[t]$ that is set to unity, $c = 1$, so that $[\boldsymbol{x}] = [t]$. Instead we assume that there exists a quantity of dimension $[\boldsymbol{x}]^z/[t]$ that is set to unity, so that $[\boldsymbol{x}]^z = [t]$. The latter units are convenient for the construction of the action and for the power-counting argument. Later we will transform to natural units.

The $\mathrm{Diff}_{\mathcal{F}}(\mathcal{M})$ transformations (4.3) of the spatial metric h_{ij} and its canonically conjugated momentum p^{ij}, and the lapse and shift variables are

$$\begin{aligned}
\delta h_{ij} &= \zeta^k \partial_k h_{ij} + \partial_i \zeta^k h_{kj} + \partial_j \zeta^k h_{ik} + f \partial_t h_{ij} \,, \\
\delta p^{ij} &= \zeta^k \partial_k p^{ij} + \partial_k \zeta^k p^{ij} - \partial_k \zeta^i p^{kj} - \partial_k \zeta^j p^{ik} + f \partial_t p^{ij} \,, \\
\delta N &= \zeta^i \partial_i N + \partial_t f N + f \partial_t N \,, \\
\delta N^i &= \zeta^j \partial_j N^i - \partial_j \zeta^i N^j + \partial_t \zeta^i + \partial_t f N^i + f \partial_t N^i \,.
\end{aligned} \tag{4.7}$$

Without time reparameterization, $\delta x^i = \zeta^i(t, \boldsymbol{x})$ and $\delta t = f(t) = 0$, the transformations (4.7) are the spacetime-dependent spatial diffeomorphisms. Under time reparameterization, $\delta t = f(t)$ and $\delta x^i = \zeta^i = 0$, the spatial metric h_{ij} and its canonically conjugated momentum p^{ij} transform as scalars and both the lapse N and the shift N^i transform as scalar densities. The transformations under foliation-preserving diffeomorphism can be most conveniently obtained from the transformations of the metric (4.4) under spacetime diffeomorphisms, $\delta g_{\mu\nu} = -2\nabla_{(\mu}\epsilon_{\nu)}$, in the limit $c \to \infty$ [107].

The action functional of HL gravity is defined as

$$S_{\mathrm{HL}} = \int_{\mathcal{M}} \mathrm{d}t \mathrm{d}^d x N \sqrt{h} \left[\alpha_K \left(K_{ij} K^{ij} - \lambda K^2 \right) - \mathcal{V}(h_{ij}) \right] , \tag{4.8}$$

where α_K and λ are coupling constants and K_{ij} is the extrinsic curvature (3.15) of the hypersurface Σ_t. The volume element and the extrinsic curvature have the scaling dimensions

$$[\mathrm{d}t \mathrm{d}^d x N \sqrt{h}] = [\mathrm{d}t \mathrm{d}^d x] = [\boldsymbol{k}]^{-z-d} \tag{4.9}$$

and

$$[K_{ij}] = [\boldsymbol{k}]^z \,. \tag{4.10}$$

Since the action (4.8) must be dimensionless, we obtain the scaling dimension of the coupling α_K:

$$[\alpha_K] = [\boldsymbol{k}]^{d-z} \,. \tag{4.11}$$

Therefore the choice $z = d$ for the UV Lifshitz point makes α_K dimensionless. The potential $\mathcal{V}(h_{ij})$ of the Lagrangian density of (4.8) scales similarly as the kinetic term. As a result, the propagator behaves as (4.2) and the theory is rendered power-counting renormalizable. We mostly consider the usual case of three-dimensional space ($d = 3$) for which we choose $z = 3$. In the action (4.8), λ is another dimensionless coupling constant. In the IR Lifshitz point ($z = 1$), the coupling λ should flow to one, so that the full diffeomorphism symmetry is restored. In the UV Lifshitz point, the value of λ is not restricted by symmetry, since all the terms in the Lagrangian, $K_{ij}K^{ij}$, K^2 and the potential $\mathcal{V}(h_{ij})$ are invariant under $\mathrm{Diff}_{\mathcal{F}}(\mathcal{M})$ separately. Hence at high energies the value of λ is quite arbitrary, even insignificant, except for the special value $\lambda = d^{-1}$ that we will comment later.

4.1.1 Detailed balance condition

An additional symmetry was assumed in the original theory [14], namely the condition of detailed balance. It is implemented by defining the potential in the action (4.8) in terms of another action defined in the d-dimensional space. The purpose of the detailed balance condition was to enable simpler quantum behavior than in a generic theory and reduce the number of independent coupling constants. In condensed matter theory, the virtue of the detailed balance condition is in the simplification of the renormalization properties. Basically, renormalizability of the full theory respecting detailed balance depends on the renormalizability of the lower dimensional theory. The potential $\mathcal{V}(h_{ij})$ of the gravitational action (4.8) was defined by

$$\mathcal{V}(h_{ij}) = \frac{1}{4\alpha_K}\mathcal{G}_{ijkl}E^{ij}E^{kl}, \qquad E^{ij} = \frac{1}{\sqrt{h}}\frac{\delta W[h_{kl}]}{\delta h_{ij}}\,, \tag{4.12}$$

where the action $W[h_{kl}]$ is defined on the spatial hypersurface. We introduced the generalized DeWitt metric $\mathcal{G}^{ijkl}$ and its inverse $\mathcal{G}_{ijkl}$ as[1]

$$\begin{aligned}\mathcal{G}^{ijkl} &= \frac{1}{2}\left(h^{ik}h^{jl} + h^{il}h^{jk}\right) - \lambda h^{ij}h^{kl}\,, \\ \mathcal{G}_{ijkl} &= \frac{1}{2}\left(h_{ik}h_{jl} + h_{il}h_{jk}\right) - \frac{\lambda}{d\lambda - 1}h_{ij}h_{kl}\ ; \quad \lambda \neq \frac{1}{d}\,.\end{aligned} \tag{4.13}$$

In the case $z = d = 3$, the potential is given by the action

$$W = \frac{1}{w^2}\int_{\Sigma_t}\omega_3(\Gamma) + \mu\int_{\Sigma_t}\sqrt{h}\left({}^{(3)}R - 2\Lambda_W\right), \tag{4.14}$$

where the first term is the gravitational Chern-Simons term. Its contribution to E^{ij} is proportional to the Cotton tensor

$$C^{ij} = \varepsilon^{ikl}D_k\left({}^{(3)}R^j{}_l - \frac{1}{4}\delta^j{}_l{}^{(3)}R\right), \tag{4.15}$$

where ε^{ikl} is the Levi-Civita tensor. The potential defined by (4.14) is given by

$$E^{ij} = \frac{2}{w^2}C^{ij} - \mu\left({}^{(3)}R^{ij} - \frac{1}{2}h^{ij}\,{}^{(3)}R + h^{ij}\Lambda_W\right). \tag{4.16}$$

Comparing the IR limit of the HL action with the potential (4.12) to the EH action (1.1), one obtains the emergent speed of light

$$c = \frac{\mu}{2\alpha_K}\sqrt{\frac{\Lambda_W}{1-3\lambda}}\,, \qquad \frac{\Lambda_W}{1-3\lambda} > 0\,, \tag{4.17}$$

[1] Here the generalized DeWitt metric and its inverse are defined without the factors $\sqrt{h}$ and $1/\sqrt{h}$, respectively.

whose reality requires that Λ_W and $(1-3\lambda)$ must have the same sign. The effective cosmological constant is obtained as

$$\Lambda = \frac{3}{2}\Lambda_W \,. \tag{4.18}$$

In the IR limit, we expect λ to flow to one, and hence Λ_W is required to be negative (4.17). Thus the effective cosmological constant (4.18) is negative too. This disagrees with observational evidence [39–45] that suggests the present expansion of the Universe is accelerating and the cosmological constant is positive, yet very small, $\Lambda \sim 10^{-120}$ in Planck units. It was indeed found that GR is not recovered at large distances if the detailed balance condition is assumed [112, 113]. A generalized version of the theory without the detailed balance condition was introduced to correct this problem [114, 115].

4.1.2 Projectable Hořava-Lifshitz gravity

HL gravity comes in two flavours, with or without the so-called projectability condition that requires the lapse to depend only on the time coordinate, $N = N(t)$. The projectability condition is one of the features that makes the theory differ from GR. Recall that in GR, the projectability condition can always be enforced locally as a gauge choice. Moreover, for most physically relevant solutions of GR – such as Schwarzschild, Kerr and Friedmann-Lemaitre-Robertson-Walker spacetimes – the projectability condition can be achieved globally in some coordinate system. Although perturbations around those solutions need not respect the projectability of the lapse. Still imposing the projectability condition in the action has a major consequence to the structure of the theory. We consider first the projectable version of HL gravity.

The potential $\mathcal{V}(h_{ij})$ consists of all inequivalent spatial invariants with a scaling dimension equal or lower than that of the kinetic term, $[\mathcal{K}] = [\boldsymbol{k}]^{2z} = [\boldsymbol{k}]^{2d}$. The Riemann tensor (Riem) of Σ_t has the dimension $[\text{Riem}] = [\boldsymbol{k}]^2$ and the spatial covariant derivative scales $[D] = [\boldsymbol{x}]^{-1} = [\boldsymbol{k}]$. Thus the potential consists of terms of the form

$$(\text{Riem})^z\,,\ \ldots\,,\ (\text{Riem})^{z-3}(D\text{Riem})^2\,,\ \ldots\,,\ (\text{Riem})^{z-n-1}(D^{2n}\text{Riem})\,, \tag{4.19}$$

where $z = 0,\ldots,d$ and $n = 1,\ldots,z-1$. Each term in this list has dimension $[\boldsymbol{k}]^{2z}$. Only positive powers of the Riemann tensor are allowed in the action. We consider terms to be equivalent if they are related via (i) integration by parts and discarding boundary terms, (ii) commutator identities, (iii) Bianchi identities, and (iv) special relations appropriate to three-dimensional space: Weyl tensor vanishes; properties of Cotton tensor. We do not need explicit parity violation, and for simplicity, we choose to exclude it. Specializing to the three-dimensional space, there are only nine such inequivalent invariants with a dimension equal or lower than $[\boldsymbol{k}]^6$. Sorted according to ascending dimension, these terms can be chosen as

$$\begin{gathered}[1] : 1\,; \qquad [\boldsymbol{k}]^2 : {}^{(3)}\!R\,; \qquad [\boldsymbol{k}]^4 : {}^{(3)}\!R^2,\ {}^{(3)}\!R^{ij\,(3)}\!R_{ij}\,; \\ [\boldsymbol{k}]^6 : {}^{(3)}\!R^3\,,\ {}^{(3)}\!R^{(3)}\!R^{ij\,(3)}\!R_{ij}\,,\ {}^{(3)}\!R^i{}_j{}^{(3)}\!R^j{}_k{}^{(3)}\!R^k{}_i\,, \\ D^{i\,(3)}\!R^{jk}D_i{}^{(3)}\!R_{jk}\,,\ {}^{(3)}\!RD^{2\,(3)}\!R\,,\end{gathered} \tag{4.20}$$

where $D^2 = h^{ij}D_iD_j$ is the covariant Laplace operator. According to the power counting argument, in a $(3+1)$-dimensional spacetime all the terms with dimension $[\boldsymbol{k}]^6$ are marginal (renormalizable) and the rest of the terms with lower dimensions are relevant (super-renormalizable) [14, 107, 108, 116]. Note that the terms with highest derivatives of the curvature, such as

$$[\boldsymbol{k}]^4 : D^{2\,(3)}\!R\,; \qquad [\boldsymbol{k}]^6 : D^{4\,(3)}\!R\,, \tag{4.21}$$

have been discarded, since they only contribute boundary terms because of the projectability of N. Now the potential with dimension $[\mathcal{V}] = [\boldsymbol{k}]^6$ can be written

$$\begin{aligned}\mathcal{V}(h_{ij}) = \alpha_1\varrho^6 - \alpha_2\varrho^{4\,(3)}\!R + \beta_1\varrho^{2\,(3)}\!R^2 + \beta_2\varrho^{2\,(3)}\!R^{ij\,(3)}\!R_{ij} \\ + \gamma_1{}^{(3)}\!R^3 + \gamma_2{}^{(3)}\!R^{(3)}\!R^{ij\,(3)}\!R_{ij} + \gamma_3{}^{(3)}\!R^i{}_j{}^{(3)}\!R^j{}_k{}^{(3)}\!R^k{}_i \\ + \gamma_4 D^{i\,(3)}\!R^{jk}D_i{}^{(3)}\!R_{jk} + \gamma_5{}^{(3)}\!RD^{2\,(3)}\!R\,,\end{aligned} \tag{4.22}$$

where we have introduced a momentum scale ϱ in order to make all the coupling constants $\alpha_I, \beta_I, I = 1, 2$, and $\gamma_I, I = 1, \ldots, 5$, dimensionless.

We can without loss of generality rescale the time and space coordinates to set $\alpha_K = 1$ and $\alpha_2 = 1$, assuming these couplings are positive. The action (4.8) can be split into two parts, the relativistically invariant EH action and a Lorentz-violating action,

$$S_{\mathrm{HL}} = S_{\mathrm{EH}} + S_{\mathrm{LV}}\,. \tag{4.23}$$

The Lorentz-violating action S_{LV} consists of a single kinetic K^2-term and of the potential terms that contain higher-order spatial derivatives of the metric. It gives rise to an extra propagating gravitational scalar mode. This additional gravitational scalar mode appears because the full diffeomorphism symmetry of the kinetic term is explicitly broken down to the $\mathrm{Diff}_{\mathcal{F}}(\mathcal{M})$ symmetry by the parameter λ. The existence of such an extra gravitational degree of freedom is alarming, because it could be pathological and contradict observations. This point will be further discussed in Sec. 4.2.

In order to compare the action (4.23) to GR, we transform to natural physical units where $c = 1$ and $[t] = [\boldsymbol{x}]$. This can be accomplished by setting $t \to \varrho^{1-z}t$. After this transformation $\varrho^{1-z}t$ has the same dimension in natural units as the time coordinate t had in the original units (4.5). In physical units, the EH part of the actions reads

$$S_{\mathrm{EH}} = \varrho^2 \int_{\mathcal{M}} \mathrm{d}t\mathrm{d}^3x N\sqrt{h}\left(K_{ij}K^{ij} - K^2 + {}^{(3)}\!R - \alpha_1\varrho^2\right). \tag{4.24}$$

By comparing (4.24) to the ADM formulation of the EH action (3.25), we find that the momentum scale ϱ is essentially the Planck scale and the coupling constant α_1 is given by the cosmological constant Λ with respect to this scale:

$$\varrho^2 = \frac{1}{2\kappa} = \frac{1}{16\pi G_{\mathrm{N}}}\,, \qquad \varrho^2\alpha_1 = 2\Lambda\,. \tag{4.25}$$

From now on we will work in the physical units with $c = 1$.

Instead of the sum (4.23), the total action can be written in the form

$$S_{\mathrm{HL}} = \varrho^2 \int_{\mathcal{M}} \mathrm{d}t\mathrm{d}^3x N\sqrt{h}\left(K_{ij}K^{ij} - \lambda K^2 - \mathcal{V}(h_{ij})\right) \tag{4.26}$$

with the potential

$$\begin{aligned}\mathcal{V}(h_{ij}) = 2\Lambda - {}^{(3)}\!R + \varrho^{-2}\left(\beta_1{}^{(3)}\!R^2 + \beta_2{}^{(3)}\!R^{ij\,(3)}\!R_{ij}\right) + \varrho^{-4}\Big(\gamma_1{}^{(3)}\!R^3 \\ + \gamma_2{}^{(3)}\!R^{(3)}\!R^{ij\,(3)}\!R_{ij} + \gamma_3{}^{(3)}\!R^i{}_j{}^{(3)}\!R^j{}_k{}^{(3)}\!R^k{}_i \\ + +\gamma_4 D^{i\,(3)}\!R^{jk} D_i{}^{(3)}\!R_{jk} + \gamma_5{}^{(3)}\!R D^{2\,(3)}\!R\Big)\,.\end{aligned} \tag{4.27}$$

This is the form of the action we shall use in the Hamiltonian analysis in Sec. 4.2. We can see from the potential (4.27) that each of the higher-derivative terms become comparable to the spatial scalar curvature term $-{}^{(3)}\!R$ at a corresponding physical momentum scale:

$$\varrho_{\beta_I} = \varrho\,|\beta_I|^{-\frac{1}{2}}\,, \qquad \varrho_{\gamma_I} = \varrho\,|\gamma_I|^{-\frac{1}{4}}\,. \tag{4.28}$$

These momentum scales, at which each of the Lorentz breaking potential terms becomes significant, can be adjusted by the choice of the dimensionless couplings $\beta_I, I = 1, 2$, and $\gamma_I, I = 1, \ldots, 5$. In particular, the Lorentz breaking momentum scales can be set arbitrarily high in order to fulfil all current bounds on Lorentz symmetry breaking.

4.1.3 Nonprojectable Hořava-Lifshitz gravity and its extension

When the projectability condition is assumed the potential (4.27) is indeed the general potential that possesses the required symmetry $\mathrm{Diff}_{\mathcal{F}}(\mathcal{M})$. It is also the most general potential that implies a Hamiltonian of the conventional form of GR,

$$H_{\mathrm{c}} = \int_{\Sigma_t}\left(N\mathcal{H}_0 + N^i\mathcal{H}_i\right), \tag{4.29}$$

where both $\mathcal{H}_0$ and $\mathcal{H}_i$ are independent of the lapse N and the shift vector N^i. But in the case of HL gravity without the projectability condition, a Hamiltonian of this form does not define a physically consistent theory because of the spatial higher-order derivatives present in the Hamiltonian constraint $\mathcal{H}_0$; this argument is explained in Sec. 4.2.2. Moreover the theory contains an extra degree of freedom which is both unstable and strongly coupled at a very low cutoff scale [117].[2]

[2] The problem with strong coupling at too low energy scales apparently hampers the projectable version of the theory too.

We can consider the most general theory that conforms with the symmetry $\mathrm{Diff}_{\mathcal{F}}(\mathcal{M})$ [118]. Now the potential part of the action can contain the acceleration vector (3.17) in addition to curvature and covariant derivatives. This vector a_i transforms under $\mathrm{Diff}_{\mathcal{F}}(\mathcal{M})$ (4.3) as

$$\delta a_i = \zeta^j \partial_j a_i + \partial_i \zeta^j a_j + f \dot{a}_i \,, \tag{4.30}$$

i.e., a_i is a vector under the spatial spacetime-dependent diffeomorphisms and a scalar under time reparameterization. The vector (3.17) has the scaling dimension $[a_i] = [\boldsymbol{x}]^{-1} = [\boldsymbol{k}]$. In order to define the generic potential for the case $z = d = 3$, we include all permitted terms involving a_i with scaling dimension equal or lower to $[\boldsymbol{k}]^6$ into the potential. There are many new potential terms:

$$\begin{aligned}
&[\boldsymbol{k}]^2 : a_i a^i,\ D_i a^i\,; \qquad [\boldsymbol{k}]^4 : (a_i a^i)^2,\ a_i D^2 a^i,\ (D_i a^i)^2,\ a_i a_j {}^{(3)}\!R^{ij},\ \ldots\,; \\
&[\boldsymbol{k}]^6 : (a_i a^i)^3,\ a_i D^4 a^i,\ a_i a^i a_j D^2 a^j,\ a_i a^i a_j a_k {}^{(3)}\!R^{jk}, \\
&\qquad a_i a^j {}^{(3)}\!R^{ik}\, {}^{(3)}\!R_{jk},\ \ldots\,,
\end{aligned} \tag{4.31}$$

where the ellipses denote all the rest of the possible terms involving a_i. We do not include time derivatives of the vector a_i into the action. This is because a term that contains a single time derivative would violate time-reversal invariance, while terms that contain two or more time derivatives and at least one a_i have too large scaling dimension. Note that many of the terms involving a_i are related to other terms via integration by parts. For example, $D_i a^i$ and $D^2\, {}^{(3)}\!R$ are equivalent to $a_i a^i$ and $a_i a^i\, {}^{(3)}\!R + D_i a^i\, {}^{(3)}\!R$ up to boundary terms, respectively. A strong motivation for including all possible terms into the potential is that perturbative quantum corrections are bound to generate all kinds of permitted contributions, including the aforementioned terms that involve a_i, since there is no symmetry, nor any other mechanism, that prevents it. Thus the generic potential of nonprojectable HL gravity has the form

$$\begin{aligned}
\mathcal{V}(h_{ij}, a_i) = {}& \varrho^2 \alpha_1 - \alpha_2 {}^{(3)}\!R - \alpha_3 a_i a^i + \varrho^{-2} \Big(\beta_1 {}^{(3)}\!R^2 + \beta_2 {}^{(3)}\!R^{ij}\, {}^{(3)}\!R_{ij} \\
& + \beta_3 D^2\, {}^{(3)}\!R + \beta_4 a_i D^2 a^i + \beta_5 (D_i a^i)^2 + \beta_6 (a_i a^i)^2 \\
& + \beta_7 a_i a_j {}^{(3)}\!R^{ij} + \ldots \Big) + \varrho^{-4} \Big(\gamma_1 {}^{(3)}\!R^3 + \gamma_2 {}^{(3)}\!R\, {}^{(3)}\!R^{ij}\, {}^{(3)}\!R_{ij} \\
& + \gamma_3 {}^{(3)}\!R^i{}_j\, {}^{(3)}\!R^j{}_k\, {}^{(3)}\!R^k{}_i + \gamma_4 D^i\, {}^{(3)}\!R^{jk} D_i\, {}^{(3)}\!R_{jk} + \gamma_5 {}^{(3)}\!R D^2\, {}^{(3)}\!R \\
& + \gamma_6 D^4\, {}^{(3)}\!R + \gamma_7 a_i D^4 a^i + \gamma_8 a_i a^i a_j D^2 a^j + \gamma_9 (a_i a^i)^3 \\
& + \gamma_{10} a_i a^i a_j a_k {}^{(3)}\!R^{jk} + \gamma_{11} a_i a^j {}^{(3)}\!R^{ik}\, {}^{(3)}\!R_{jk} + \ldots \Big)\,.
\end{aligned} \tag{4.32}$$

As in (4.27) each higher-derivative term in the potential (4.32) is associated with a momentum scale (4.28) at which the term becomes comparable to the potential term $-{}^{(3)}\!R$ of GR. However, the coupling constants in the action cannot be chosen to have any values one wishes. Rather only certain parts of the parameter space are permitted in order to obtain a viable theory [117–119]. In particular, working on the flat background one finds that requiring the theory to be free of ghosts and instabilities the following conditions must be satisfied

$$(3\lambda - 1)(\lambda - 1) > 0\,, \qquad 0 < \alpha_3 < 2\,, \tag{4.33}$$

where the former condition is satisfied by $\lambda > 1$ (or alternatively by $\lambda < 1/3$). When the potential contains terms (4.31) that involve a_i, it is clear that $\mathcal{H}_0$ must depend on the lapse N in the Hamiltonian (4.29). Even though such a dependence is somewhat unconventional, there is no a priori reason why $\mathcal{H}_0$ could not depend on N and its spatial derivatives.

4.1.4 Renormalizability

Renormalizability of HL gravity has been investigated beyond the power-counting scheme in [120]. It was confirmed [14] that when the detailed balance condition holds, HL gravity can be understood as the result of stochastic quantization of topologically massive gravity. Then renormalizability of HL gravity is determined by the renormalizability of topologically massive gravity, which is thought to hold [121] although it has not been fully proven. Note that this result crucially depends on the condition of detailed balance. Therefore the conclusion does not necessarily hold for more general potentials like (4.27) or (4.32). At present the expectation of renormalizability of these theories is based mostly on the power-counting argument.

4.2 Hamiltonian formulation of Hořava-Lifshitz gravity

The action of HL gravity (4.26) is defined in terms of ADM variables, which make up the metric of spacetime (4.4). It differs from the ADM representation of EH action (3.25) due to the lack of full diffeomorphism symmetry. First we discuss Hamiltonian formulation of the projectable version of HL gravity. Then we consider HL gravity and its extension when N depends on both space and time.

4.2.1 Projectable Hořava-Lifshitz gravity

When the projectability condition is assumed, $N = N(t)$, HL gravity has a quite simple and consistent Hamiltonian structure. The algebra of constraints was shown to be closed for $z = 1, 2$ in [107], and this fact holds for any higher scaling exponent $z > 2$ as well. In order to obtain a power-counting renormalizable theory in $(d+1)$-dimensional spacetime, we choose the UV Lifshitz point as $z = d$. We consider the action in physical units, written in the form (2.4) as

$$S_{\mathrm{pHL}} = \varrho^2 \int_{t_1}^{t_2} \mathrm{d}t \int_{\Sigma_t} N\sqrt{h}\left(\mathcal{G}^{ijkl} K_{ij} K_{kl} - \mathcal{V}(h_{ij})\right). \tag{4.34}$$

The potential is of the form (4.27) but contains all the relevant and marginal terms with a scaling dimension equal or lower than $[\mathcal{V}(h_{ij})] = [\boldsymbol{k}]^{2d}$.

Since the action is independent of the time derivatives of N and N^i, their canonically conjugated momenta are the primary constraints

$$p_N \approx 0\,, \qquad p_i(\boldsymbol{x}) \approx 0\,, \tag{4.35}$$

respectively. Note that the projectability condition on N implies that the momentum p_N conjugate to N is projectable too. The momentum canonically conjugate to h_{ij} is defined as

$$p^{ij} = \varrho^2 \sqrt{h} \mathcal{G}^{ijkl} K_{kl}\,. \tag{4.36}$$

We assume $\lambda \neq d^{-1}$, so that the time derivative of the spatial metric $\partial_t h_{ij}$ can be solved in terms of the canonical variables similarly as in GR.[3] The total Hamiltonian is obtained in the form

$$H = \int_{\Sigma_t} \left(N\mathcal{H}_0 + N^i \mathcal{H}_i + v_N p_N + v^i p_i \right), \tag{4.37}$$

where we have defined

$$\mathcal{H}_0 = \frac{1}{\varrho^2\sqrt{h}} \left(p_{ij}p^{ij} - \frac{\lambda}{d\lambda - 1} p^2 \right) + \varrho^2 \sqrt{h}\, \mathcal{V}(h_{ij})\,. \tag{4.38}$$

The momentum constraint $\mathcal{H}_i$ is identical to the one of GR (3.33). $\mathcal{H}_0$ differs from GR for two reasons: the value of the parameter λ is not fixed to one and the potential $\mathcal{V}(h_{ij})$ contains spatial higher-order derivatives of the metric. The primary constraints (4.35) have been included with Lagrange multipliers v_N and v^i. Note that in the Hamiltonian, N and the whole term $v_N p_N$ could be taken out of the integral over Σ_t, since they are constants in space. We write them under the integral so that we do not have to restate the Hamiltonian when the projectability condition is revoked in the next subsection. Since we are interested in local dynamics, we assumed that boundary terms resulting from integration by parts can be dropped. In other words, we assume the spatial hypersurfaces are compact and have no boundary.

The required secondary constraints are

$$\Phi_0 = \int_{\Sigma_t} \mathcal{H}_0 \approx 0\,, \qquad \mathcal{H}_i(\boldsymbol{x}) \approx 0\,. \tag{4.39}$$

Note that the constraint Φ_0 is a single integrated constraint, unlike the local Hamiltonian constraint of GR. This is a direct consequence of the projectability condition. A global constraint like Φ_0 has no effect on local dynamics. Thus

[3] When $\lambda = d^{-1}$, we have to impose an extra primary constraint, $p = h_{ij}p^{ij} \approx 0$, because the DeWitt metric (4.13) is traceless, $\bar{\mathcal{G}}^{ijkl} = \frac{1}{2}\left(h^{ik}h^{jl} + h^{il}h^{jk}\right) - d^{-1}h^{ij}h^{kl}$. It has the traceless inverse, $\bar{\mathcal{G}}^{ijkl} = \frac{1}{2}\left(h_{ik}h_{jl} + h_{il}h_{jk}\right) - d^{-1}h_{ij}h_{kl}$, which satisfies $\bar{\mathcal{G}}_{ijmn}\bar{\mathcal{G}}^{mnkl} = \delta_i^{(k}\delta_j^{l)} - d^{-1}h_{ij}h^{kl}$. Thus we can solve the traceless components of K_{ij} in terms of canonical variables, but not the trace component K. In the total Hamiltonian, the trace contribution is absorbed into the primary constraint p that is multiplied by an arbitrary Lagrange multiplier.

as far as local dynamics is concerned only the local constraints $\mathcal{H}_i$ and p_i are relevant. We again define a global smeared version of the momentum constraint $\mathcal{H}_i$ as in (3.36), which is the generator of the spacetime-dependent spatial diffeomorphisms.

The Poisson brackets for the Hamiltonian constraint Φ_0 and the momentum constraint $\Phi[\vec{X}]$ now read

$$\{\Phi_0, \Phi_0\} = 0\,, \qquad \left\{\Phi_0, \Phi\big[\vec{X}\big]\right\} = 0\,. \tag{4.40}$$

Thus the algebra of constraints has a Lie algebra structure.

The Hamiltonian equations of motion generated by the total Hamiltonian (4.37) can be obtained via straightforward calculation. The metric and its canonical conjugated momentum satisfy the equations of motion:

$$\partial_t h_{ij} = \frac{2N}{\varrho^2\sqrt{h}}\left(p_{ij} - \frac{\lambda}{d\lambda - 1}h_{ij}p\right) + \mathcal{L}_{\vec{N}}h_{ij}\,, \tag{4.41}$$

$$\begin{aligned}\partial_t p^{ij} = &-\frac{2N}{\varrho^2\sqrt{h}}\left[p^i{}_k p^{jk} - \frac{\lambda}{d\lambda-1}p^{ij}p - \frac{1}{4}h^{ij}\left(p_{kl}p^{kl} - \frac{\lambda}{d\lambda-1}p^2\right)\right] \\ &- N\varrho^2\left(\int_{\Sigma_t}\mathrm{d}^d y\sqrt{h}\frac{\delta\mathcal{V}(h_{kl}(\boldsymbol{y}))}{\delta h_{ij}(\boldsymbol{x})} + \frac{1}{2}\sqrt{h}h^{ij}\mathcal{V}(h_{kl})\right) + \mathcal{L}_{\vec{N}}p^{ij}\,,\end{aligned} \tag{4.42}$$

where the functional derivative of the potential with respect to the metric has to be evaluated for the potential of each case $z = d$. The secondary constraints (4.39) provide the initial-value conditions for projectable HL gravity. The major difference compared to GR is that the Hamiltonian constraint, $\Phi_0 = 0$, tells that $\mathcal{H}_0$ has to vanish in average, but tells nothing about its local values.

Gauge fixing conditions can be introduced in a quite similar way as in GR. The difference is that the gauge conditions associated with p_N and Φ_0 must be global ones. That means the lapse N can still be fixed to a constant, but one of the gauge conditions on the metric h_{ij} must be an integrated constraint. We could for example impose the gauge conditions

$$N = 1\,, \qquad N^i = 0\,, \qquad \chi_0[h_{ij}] = \int_{\Sigma_t}\sqrt{h}f(h_{ij}) = 0\,, \qquad \chi_a(h_{ij}) = 0\,, \tag{4.43}$$

where $a = 1, \ldots, d$. The conditions χ_a have to fix d components of the metric, which requires $\mathrm{rank}(\partial\chi_a/\partial h_{ij}) = d$, and χ_0 cannot vanish due to the conditions χ_a.

We can count the number of physical degrees of freedom by using Dirac's formula (2.26). For comparison with GR we consider the case of three-dimensional space. For propagating modes we have 18 canonical variables $(N^i, h_{ij}, p_i, p^{ij})$ and six first-class constraints $(p_i, \mathcal{H}_i)$. Thus we obtain three propagating physical degrees of freedom. That is one more physical degrees of freedom than in GR. The extra degree of freedom is present due to the reduced diffeomorphism symmetry group $\mathrm{Diff}_{\mathcal{F}}(\mathcal{M})$. There exist two additional space-independent variables N, p_N and two additional global first-class constraints p_N and Φ_0, thus yielding two physical (nonpropagating) space-independent degrees of freedom, the same result as in GR.

So far we have considered pure gravity without any field content. When matter fields are added to the theory, the Hamiltonian and momentum constraints receive additional contributions. As a result the equation of motion for the momentum (4.42) is extended by $\{p^{ij}, H_{\text{fields}}\}$, where H_{fields} is the Hamiltonian for the included fields which are assumed to be coupled to the gravitational field conventionally. The momentum constraint and the global Hamiltonian constraint would be amended as

$$\int_{\Sigma_t} \mathcal{H}_0 = -\int_{\Sigma_t} \sqrt{h} T_{\boldsymbol{nn}}\,, \qquad \mathcal{H}_i = -\sqrt{h}_\perp T_{\boldsymbol{in}}\,, \tag{4.44}$$

where the projections of the energy-momentum tensor $T_{\mu\nu}$ represent all the included fields.

4.2.2 Dynamical inconsistency of nonprojectable Hořava-Lifshitz gravity

The Hamiltonian structure of original HL gravity without the projectability condition was found to be physically inconsistent for generic couplings [122]. Even before that a similar problem had been found in the theory with detailed balance [123].

We shall review this result for the potential of the form (4.27) with generic (nonvanishing) couplings. For definiteness we consider three-dimensional space. Generalization of the argument to other (higher) dimensions in noted. Hamiltonian analysis of this theory differs from the projectable theory constructed in Sec. 4.2.1 by the following way. Because the lapse N is a genuine scalar field on the spatial hypersurfaces, the primary constraint $p_N \approx 0$ is space-dependent too. As a result we obtain that (4.38) becomes the local Hamiltonian constraint

$$\mathcal{H}_0 \approx 0\,. \tag{4.45}$$

The smeared version of the Hamiltonian constraint $\mathcal{H}_0[\xi]$ is defined as in (3.35). The total Hamiltonian takes the same form (4.37) as in the projectable theory, but now all terms in the Hamiltonian density are local constraints, similarly as in GR.

The Poisson brackets (3.38) and (3.39) between the momentum constraint $\Phi\big[\vec{X}\big]$ and the Hamiltonian constraint $\mathcal{H}_0[\xi]$ have the same form as in GR. However, the Poisson bracket between Hamiltonian constraints (3.37) is changed drastically. The Poisson bracket between Hamiltonian constraints can be written

$$\{\mathcal{H}_0[\xi], \mathcal{H}_0[\eta]\} = \int_{\Sigma_t} \big\{\mathcal{H}_0[\xi], p^{ij}(\boldsymbol{x})\big\} \{h_{ij}(\boldsymbol{x}), \mathcal{H}_0[\eta]\} - (\xi \leftrightarrow \eta)\,. \tag{4.46}$$

Variation of the Hamiltonian constraint $\mathcal{H}_0[\xi]$ with respect to h_{ij} contains spatial derivatives of the variation δh_{ij} up to fourth order. There are two derivatives in the variation of the curvature and two more derivatives in the potential (4.27). Variation with respect to the momentum p^{ij} does not involve derivatives of δp^{ij}. Thus the Poisson bracket of Hamiltonian constraints has the form

$$\begin{aligned}\{\mathcal{H}_0[\xi], \mathcal{H}_0[\eta]\} = \int_{\Sigma_t} \eta \left(C_4^{ijkl} D_{ijkl}\xi + C_3^{ijk} D_{ijk}\xi + C_2^{ij} D_{ij}\xi + C_1^i D_i\xi \right) \\ - (\xi \leftrightarrow \eta)\,,\end{aligned} \tag{4.47}$$

where $C_n^{i_1\cdots i_n}$ ($n = 1, 2, 3, 4$) are symmetric tensor densities consisting of the canonical variables h_{ij} and p^{ij} and their spatial derivatives, and we denote symmetrized higher-order covariant derivatives by $D_{ij} = D_{(i}D_{j)}$ etc. For generic couplings the tensors densities C_n have quite complicated forms, which we do not present here. We may integrate (4.47) by parts in order to obtain

$$\{\mathcal{H}_0[\xi], \mathcal{H}_0[\eta]\} = \int_{\Sigma_t} \xi \left(E_3^{ijk} D_{ijk}\eta + E_2^{ij} D_{ij}\eta + E_1^i D_i\eta + E_0\eta \right), \tag{4.48}$$

where we have defined

$$\begin{aligned} E_3^{ijk} &= 4D_l C_4^{ijkl} - 2C_3^{ijk}\,, \\ E_2^{ij} &= 6D_{kl} C_4^{ijkl} - 3D_k C_3^{ijk} = \frac{3}{2} D_k E_3^{ijk}\,, \\ E_1^i &= 4D_{jkl} C_4^{ijkl} - 3D_{jk} C_3^{ijk} + 2D_j C_2^{ij} - 2C_1^i\,, \\ E_0 &= D_{ijkl} C_4^{ijkl} - D_{ijk} C_3^{ijk} + D_{ij} C_2^{ij} - D_i C_1^i\,. \end{aligned} \tag{4.49}$$

Note that the fourth-order spatial derivative of η cancels out, contrary to the result of [122]. But that does not change the final conclusion. More generally, for every even n, C_n contributes to E_m for $m < n$, but never to E_n. For every uneven n, the contribution of C_n to E_n is $-2C_n$. Thus the highest spatial derivative of η is always of odd order. This is clearly a general feature that holds in any theory where the Poisson bracket of Hamiltonian constraints is of the form (4.47) with nonvanishing coefficients C_n usually up to $n = d + 1$ for the renormalizable case $z = d$. This result was obtained in paper **IV** [124] in the context of more general HL theories considered in Chapter 5.

For the Hamiltonian constraint (4.45) to be preserved under time evolution, the Poisson bracket of the constraint $\mathcal{H}_0[\xi]$ with itself should vanish weakly. Plugging in $\xi = \delta(\boldsymbol{x} - \boldsymbol{y})$ and $\eta = N$ into (4.48) we obtain the consistency condition for the Hamiltonian constraint in the form

$$E_3^{ijk} D_{ijk}N + E_2^{ij} D_{ij}N + E_1^i D_i N + E_0 N \approx 0\,. \tag{4.50}$$

An important feature of the tensor densities E_n is that the number of spatial derivatives in the coefficient of a given coupling constant increases by one when n decreases by one. This is evident by looking at (4.49). As a result E_n have different asymptotic behavior in the asymptotically flat case. If E_0 behaves asymptotically as $O(r^{-a})$, then E_n behaves as $O(r^{-a+n})$. It should also be emphasized that none of the coefficients E_n in (4.50) vanish in HL gravity with generic couplings.

There are two possible ways to interpret the condition (4.50). We could regard it as a new secondary constraint that imposes further constraints on the variables h_{ij} and p^{ij}. Thus we would impose such extra constraints on h_{ij} and p^{ij} that are sufficient to satisfy (4.50). Considering N to be arbitrary, those constraints would have to imply

$$E_3^{ijk} \approx 0\,, \qquad E_2^{ij} \approx 0\,, \qquad E_1^{i} \approx 0\,, \qquad E_0 \approx 0\,. \tag{4.51}$$

For the generic potential (4.27) this over-constrains the variables h_{ij} and p^{ij}, leaving no room for gravitational dynamics due to lack of physical degrees of freedom. There does exist some special cases, namely ones with lower order spatial derivatives in the potential, where the constraints (4.51) can be imposed without sacrificing physical viability. This approach takes advantage of the gauge freedom in the variables h_{ij} and p^{ij}, and the fact that the rank of the operator acting on N in (4.50) varies on the constraint surface. Such a case will be discussed in Sec. 4.2.3. Since in general (4.51) is not viable, we are bound to consider the other interpretation. Namely that (4.50) is a condition on the lapse N, which has the role of a Lagrange multiplier in the action. This means that the Hamiltonian constraints are second-class for generic couplings. In the asymptotically flat case the lapse must tend to a constant at infinity. Indeed, since GR is supposed to be recovered at large distances, we may impose the familiar asymptotic behavior of GR. In asymptotically flat coordinates, we have [102]

$$\begin{aligned} N &= 1 + O\left(\frac{1}{r}\right), & N^i &= O\left(\frac{1}{r}\right), \\ h_{ij} &= \delta_{ij} + O\left(\frac{1}{r}\right), & p^{ij} &= O\left(\frac{1}{r^2}\right). \end{aligned} \tag{4.52}$$

It turns out that $N = 0$ is the only solution of the homogeneous equation (4.50), where $E_0 \neq 0$, that goes to a constant at infinity [122]. Thus the lapse is zero everywhere . This means that there is no time evolution. The Hamiltonian vanishes in the gauge $N^i = 0$ and hence any quantity constructed from the variables h_{ij} and p^{ij} is a constant of motion. Since vanishing N (everywhere) cannot be obtained from a physically sensible solution by a regular coordinate transformation, this is a genuine inconsistency of the theory rather than a poor choice of coordinates. Recall that the lapse N relates the coordinate time to the proper time measured by an observer with velocity n^μ normal to the spatial hypersurfaces. The fixing of the lapse $N = 0$ does not contradict with the existence of the time reparameterization symmetry, contrary to what one might expect. The time reparameterization symmetry is a trivial symmetry on-shell [122], meaning that these transformations completely vanish when the equations of motion are satisfied, because $\mathcal{H}_0(\boldsymbol{x})$ are second-class constraint.

In summary, we either have too few gravitational degrees of freedom or there is no time evolution at all. Thus we conclude that the theory is dynamically inconsistent.

4.2.3 Low-energy effective action

Let us consider an effective action whose potential (4.27) contains only the term $-{}^{(3)}R$. This is supposed to represent the IR limit of nonprojectable HL gravity. A consistent set of constraints can be obtained by imposing an additional constraint [122, 125, 126], namely $p \approx 0$. Furthermore one may interpret this constraint as a gauge fixing condition.

In this case the Hamiltonian constraint (4.38) differs from GR only by a term that is proportional to the trace of the momentum p^{ij} squared,

$$\mathcal{H}_0 = \mathcal{H}_0^{\mathrm{GR}} + \frac{\lambda - 1}{(d-1)(d\lambda - 1)} \frac{p^2}{\varrho^2 \sqrt{h}}\,, \tag{4.53}$$

where $\mathcal{H}_0^{\mathrm{GR}}$ is the Hamiltonian constraint of GR in d-dimensional space. Poisson bracket of the Hamiltonian constraints of GR are known to close for the momentum constraint (3.37). Now the secondary constraint (4.50) takes the following form after some multiples of the momentum constraint $\mathcal{H}_i$ and its covariant derivative are dropped

$$2D_i N D^i p + N D^2 p \approx 0\,. \tag{4.54}$$

Note that asymptotically the last term dominates with its $O(r^{-4})$ behavior compared to the first $O(r^{-5})$ term. This constraint can be rewritten as

$$D_i \left(N^2 D^i p\right) \approx 0\,. \tag{4.55}$$

If we interpret (4.55) as a condition for the lapse we obtain the result $N = 0$, which is an inconsistency, as was discussed in Sec. 4.2.2. However, in this deep IR effective case the interpretation of (4.55) as a constraint on the variables h_{ij} and p^{ij} actually works, because the higher-order spatial derivatives are absent in the Hamiltonian constraint. Since $N^2 > 0$ the constraint (4.55) is satisfied everywhere if

$$D_i p = \sqrt{h}\partial_i \left(\frac{p}{\sqrt{h}}\right) \approx 0\,. \tag{4.56}$$

That is, the scalar $g^{-1/2}p$ is a function of time only. The boundary condition $p^{ij}\big|_\infty = 0$ at spatial infinity requires this function to be zero. Thus we impose the constraint

$$p \approx 0\,. \tag{4.57}$$

Preservation of (4.57) under time evolution requires that its Poisson bracket with the Hamiltonian constraint should vanish weakly. One obtains that the Poisson bracket between p and the Hamiltonian constraint implies a condition on the lapse [126]

$$(D^2 - R)N = 0\,. \tag{4.58}$$

This is an elliptic partial differential equation (PDE) which always has a solution. At any given time t the lapse N is determined by the geometry of the spatial hypersurface Σ_t through the PDE (4.58) and the boundary conditions at spatial infinity. Thus there is no gauge freedom in N, unlike in GR where the lapse has to be specified as a gauge choice. Fixing the lapse does not contradict with the existence of the time reparameterization symmetry, because the time reparameterization symmetry is trivial when the equations of motion are satisfied [122].

The number of physical degrees of freedom is found to be two, which correspond to the two polarization modes of the graviton. Thus the extra mode present in HL gravity does not appear when the higher-derivative terms are dropped from the potential.

Instead of considering $\mathcal{H}_0$ and p as a pair of second-class constraints, we may interpret that the Hamiltonian constraint $\mathcal{H}_0$ is a first-class constraint and p is a gauge fixing condition associated with the gauge symmetry corresponding to $\mathcal{H}_0$. In order to turn $\mathcal{H}_0$ into a first-class constraint we must add a term involving the gauge condition to it. Indeed it is obvious that by adding a term $g^{-1/2}p^2$ into (4.53) we obtain the Hamiltonian constraint of GR, effectively setting λ to one. Recall that in GR, the Cauchy data is restricted by the Hamiltonian and momentum constraints so that we can freely specify the trace and the transverse-traceless part of the momentum p^{ij} and also spatial metric h_{ij} up to a conformal factor [105]. Then the longitudinal (vector) part of the momentum p^{ij} and the conformal factor are fixed by the the momentum and Hamiltonian constraint, respectively. Fixing the trace p of the momentum fixes the slicing of spacetime into spacelike hypersurfaces [104, 105]. The scalar quantity $g^{-1/2}p$ measures the rate of contraction of local three-volume elements with respect to local proper time.

Summing up, modifying GR with the parameter λ alone does not lead to a new theory of gravity, but rather produces a partially gauge fixed version of GR. Although such a truncated model could be viewed as an effective low-energy limit of HL gravity, it does not properly reflect the structure of the full theory, because the essential higher-order spatial derivatives have been left out.

This type of analysis has been extended to the models with quadratic curvature terms added into the potential. The case with ${}^{(3)}R$ and ${}^{(3)}R^2$ terms in the potential was studied in [127]. The linearized model with quadratic curvature terms in the potential was studied in [128]. It appears that the structure of the algebra of constraints and also the number of propagating degrees of freedom depend on which quadratic curvature terms are included.

4.2.4 Consistent extension of nonprojectable Hořava-Lifshitz gravity

Because of the reduced diffeomorphism symmetry group $\mathrm{Diff}_{\mathcal{F}}(\mathcal{M})$ there is an additional half scalar degree of freedom in HL gravity. Its time evolution is governed by an equation of motion that contains only first order time derivative, hence requiring only one piece of Cauchy data. It has been shown to be strongly coupled at all scales by considering perturbations about a reasonable vacuum [129], regardless whether the detailed balance is assumed or not. This suggests that perturbative GR cannot be reproduced in HL gravity [129], and that the theory can be ruled out by existing observations on the gravitational radiation of binary pulsars, which agree with linearized GR. The low-energy regime of the theory was further analyzed in [117, 130], where problems with instability and strong coupling of the extra degree of freedom were found. Since then these problems have been confirmed in various papers.

An extension of nonprojectable HL gravity was proposed [118] that is argued to be free of the pathologies of the original theory (instability, over-constrained evolution, or strong coupling at low energies), since the extra scalar mode has a healthy quadratic action. As we explained in Sec. 4.1.3, this is achieved by adding terms that involve the spatial vector $a_i = N^{-1} D_i N$ into the action. Hamiltonian formalism of this extension has been studied in [131–133]. We shall mostly follow [132] that provides a quite complete analysis, but also take into account the other works.

Hamiltonian and constraints

The action of the extended HL gravity has a similar form as the previous HL theories, but with a potential of the form (4.32) instead of (4.27),

$$S_{\mathrm{HL}} = \varrho^2 \int_{t_1}^{t_2} \mathrm{d}t \int_{\Sigma_t} N\sqrt{h} \left(\mathcal{G}^{ijkl} K_{ij} K_{kl} - \mathcal{V}(h_{ij}, a_i) \right) . \tag{4.59}$$

We assume the spatial hypersurfaces Σ_t are compact and have no boundary, so that integrals of total divergences may be dropped. Asymptotically flat spacetimes with boundary will be discussed later. Hamiltonian formulation of the action begins similarly as in Secs. 4.2.1 and 4.2.2. The momenta that are canonically conjugate to N and N^i, respectively, are the local primary constraints, $p_N \approx 0$ and $p_i \approx 0$. The momentum p^{ij} canonically conjugate to h_{ij} has the same form (4.36) as in the previous versions of HL gravity. The total Hamiltonian is obtained in the same form (4.37) as well, where we now define

$$\mathcal{H}_0 = \frac{1}{\varrho^2 \sqrt{h}} \left(p_{ij} p^{ij} - \frac{\lambda}{d\lambda - 1} p^2 \right) + \varrho^2 \sqrt{h}\, \mathcal{V}(h_{ij}, a_i) \, . \tag{4.60}$$

So far the only difference compared to the previous versions of HL gravity is that the potential $\mathcal{V}$ in (4.60) depends also on the vector a_i. This turns out to be a decisive difference.

The secondary constraints are obtained as

$$\mathcal{C} = \mathcal{H}_0 - \frac{1}{N} D_i V^i \approx 0 \, , \qquad \mathcal{H}_i \approx 0 \, , \tag{4.61}$$

where we defined the vector density

$$V^i(\boldsymbol{x}) = \varrho^2 \frac{\delta}{\delta a_i(\boldsymbol{x})} \int_{\Sigma_t} N\sqrt{h}\, \mathcal{V}(h_{ij}, a_i) \, . \tag{4.62}$$

The momentum constraint $\mathcal{H}_i$ retains the same form as in GR (3.33). The constraint $\mathcal{C}$, however, depends on the lapse N and its spatial derivatives. We shall refer to $\mathcal{C}$ as the *scalar constraint*, since its form and role are not quite the same compared to the Hamiltonian constraint in GR. Since the scalar constraint involves N, its Poisson bracket with the primary constraint p_N does not vanish

$$\{\mathcal{C}, p_N\} \neq 0\,. \tag{4.63}$$

Therefore they appear to be second-class constraints that may be used for eliminating N and p_N from the set of canonical variables. But we shall soon see that there exists two integrated combinations of these constraints that are first-class constraints.

For the general potential (4.32) the scalar constraint $\mathcal{C}$ is a quite complicated PDE for the lapse N. We assume that there exists a solution to the scalar constraint. This imposes a condition on the variables h_{ij} and p^{ij} as well, since such a solution may not exist for any h_{ij} and p^{ij}. The nature of the scalar constraint will be further discussed below in the case of the IR limit. The scalar constraint determines N up to a constant rescaling, a time-dependent prefactor. This freedom left in N is associated with the time reparameterization symmetry. We only need to make N satisfy the scalar constraint $\mathcal{C} = 0$ at one point in time, say on the initial spatial hypersurface at $t = t_1$. Then N can be made to satisfy the scalar constraint at any later time by fixing a Lagrange multiplier.

We define a smeared momentum constraint that generates spatial diffeomorphisms not only for h_{ij} and p^{ij} but also for N, p_N, N^i and p_i. It is defined by including multiples of the primary constraints p_N and p_i into $\mathcal{H}_i$:

$$\Phi\big[\vec{X}\big] = \int_{\Sigma_t} X^i \tilde{\mathcal{H}}_i\,, \qquad \tilde{\mathcal{H}}_i = -2h_{ij} D_k p^{jk} + D_i N p_N + \mathcal{L}_{\vec{N}} p_i\,. \tag{4.64}$$

The particular combination

$$\Pi_N = \int_{\Sigma_t} N p_N \approx 0 \tag{4.65}$$

of the primary constraints plays an important role in this theory. First observe that it generates constant rescaling of N and p_N,

$$\{N, \Pi_N\} = N\,, \qquad \{p_N, \Pi_N\} = -p_N\,, \tag{4.66}$$

and has vanishing Poisson bracket with every other variable. The vector a_i is invariant under the rescaling

$$\{a_i, \Pi_N\} = 0\,. \tag{4.67}$$

Therefore only explicit dependence on N and p_N matters when Poisson brackets with Π_N are evaluated. The scalar constraint $\mathcal{C}$ is invariant under the rescaling generated by Π_N,

$$\{\mathcal{C}, \Pi_N\} = 0\,, \tag{4.68}$$

although $\mathcal{C}$ depends on N. In fact, Π_N has a weakly vanishing Poisson bracket with every constraint established so far. We shall later see that no further secondary constraints appear in this theory, meaning that Π_N is a first-class constraint. The constraint Π_N is preserved under time evolution, since the definition of the scalar constraint (4.61) implies

$$\int_{\Sigma_t} N\mathcal{H}_0 = \int_{\Sigma_t} \left(N\mathcal{C} + D_i V^i\right) = \int_{\Sigma_t} N\mathcal{C}\,. \tag{4.69}$$

The last equality does not hold when the spatial hypersurface has a boundary, because then the total derivative yields a boundary term.

Scalar constraint in the IR limit

In the IR limit, the potential takes the form

$$\mathcal{V}(h_{ij}, a_i) = \varrho^2\alpha_1 - \alpha_2 R - \alpha_3 a_i a^i \tag{4.70}$$

and hence the vector density (4.62) is

$$V^i = -2\varrho^2\alpha_3\sqrt{h}D^i N\,. \tag{4.71}$$

Then the scalar constraint $\mathcal{C}$ reads

$$\frac{1}{\varrho^4 g}\mathcal{G}_{ijkl}p^{ij}p^{kl} + \varrho^2\alpha_1 - \alpha_2 R - \alpha_3\frac{D_i N D^i N}{N^2} + 2\alpha_3\frac{D^2 N}{N} \approx 0\,, \tag{4.72}$$

where we multiplied $\mathcal{C}$ with the factor $\varrho^{-2}g^{-1/2} > 0$ in order to get a scalar equation, rather than a density equation. As in the general case above we consider the scalar constraint to be an equation for N, but at the same time imposing a condition on h_{ij} and p^{ij} so that a solution to the equation exists. We may simplify this constraint by writing it in terms of the square root of the lapse $\nu = \sqrt{N}$. Then we obtain

$$L\nu \approx \varrho^2\alpha_1\nu\,, \tag{4.73}$$

where L is the Schrödinger-like linear differential operator

$$L = -4\alpha_3 D^2 + \alpha_2 R - \frac{1}{\varrho^4 h}\mathcal{G}_{ijkl}p^{ij}p^{kl} \tag{4.74}$$

and $\varrho^2\alpha_1$ is the eigenvalue associated with the function ν. In the original analyses [132, 133], the case with zero eigenvalue was considered exclusively, $\varrho^2\alpha_1 = 2\Lambda = 0$, i.e., the case of vanishing cosmological constant. In the asymptotically flat case we too assume $\alpha_1 = 0$. A solution ν to (4.73) exists if the spectrum of the operator (4.74) contains the eigenvalue $\varrho^2\alpha_1$. This means the variables h_{ij} and p^{ij}, which define the potential part of the operator (4.74) (the last three terms), have to be such that $\varrho^2\alpha_1$ is in the spectrum of the operator. Moreover, since ν is positive, $\varrho^2\alpha_1$ must be the lowest eigenvalue of the operator (4.74), and hence ν is the ground state "wave function". Then the solution ν to (4.73) determines the lapse $N = \nu^2$ up to a constant rescaling.

Stability of the secondary constraints

The total Hamiltonian can be written as

$$H = \int_{\Sigma_t} \left(N\mathcal{H}_0 + N^i \tilde{\mathcal{H}}_i + w_N N p_N + w^i p_i \right) , \tag{4.75}$$

where the extended momentum constraint (4.64) has been introduced and we wrote the Lagrange multipliers as

$$v_N = w_N N + N^i D_i N , \qquad v^i = w^i + \mathcal{L}_{\vec{N}} N^i , \tag{4.76}$$

where w_N and w^i are arbitrary.

The extended momentum constraint $\tilde{\mathcal{H}}_i$ is preserved under time evolution, because every constraint transforms as a density of unit weight under spatial diffeomorphisms. Furthermore the time evolution of $\mathcal{H}_i$ differs from that of $\tilde{\mathcal{H}}_i$ only by terms that are constraints.

In order to ensure the preservation of the scalar constraint $\mathcal{C}$ under time evolution, we impose

$$\partial_t \mathcal{C}(\boldsymbol{x}) \approx \{\mathcal{C}(\boldsymbol{x}), H\} \approx \int_{\Sigma_t} N \left(\{\mathcal{C}(\boldsymbol{x}), \mathcal{H}_0\} + w_N \{\mathcal{C}(\boldsymbol{x}), p_N\} \right) = 0 . \tag{4.77}$$

Neither of the Poisson brackets at the right-hand side vanishes in general, not even weakly. Therefore we can ensure the stability of the scalar constraint by solving the Lagrange multiplier w_N from the linear inhomogeneous equation (4.77), which is a PDE with spatial derivatives of w_N up to $2d$-th order. It is the N-dependence of the scalar constraint that enables us to establish the stability of this constraint in a way that is physically viable. The Lagrange multiplier w_N is not determined uniquely by (4.77). Rather w_N is determined up to solutions of the homogeneous part of the equation (4.77). Every solution of the homogeneous equation is a gauge freedom in the time evolution and corresponds to a first-class constraint. Since the scalar constraint is invariant under the constant rescaling generated by the constraint (4.65), the homogeneous equation has a constant solution $w_N = c_N$ (an arbitrary function of time). In principle, such a homogeneous PDE might admit nonconstant solutions. However, for a generic choice of parameters in the action, we expect that such extra solutions do not exist, because no extra first-class constraints appear to exist. Thus the general solution to (4.77) has the form

$$w_N = \bar{w}_N[N, h_{ij}, p^{ij}] + c_N , \tag{4.78}$$

where $\bar{w}_N[N, h_{ij}, p^{ij}]$ is the particular solution to (4.77). Substituting the solution (4.78) into the Hamiltonian (4.75) ensures the preservation of the scalar constraint in time.

In the IR limit, (4.77) is a second order elliptic PDE for w_N. It has been confirmed to possess a particular solution [132].

Hamiltonian as a sum of first-class constraints

Since we are dealing with a theory that possesses time reparameterization symmetry, we expect to find a Hamiltonian that is a sum of first-class constraints. Those constrains should generate the two types of foliation-preserving diffeomorphisms: global time reparameterizations, and spatial diffeomorphisms. We have already seen that the primary constraint p_i, the momentum constraint $\mathcal{H}_i$, and the linear combination Π_N of the primary constraints p_N are first-class constraints. In addition, there exists a linear combination of the scalar constraints $\mathcal{C}$ and the primary constraints p_N that has a (weakly) vanishing Poisson bracket with every constraint. Namely, we have the first-class constraint

$$\Pi_0 = \int_{\Sigma_t} N\left(\mathcal{H}_0 + \bar{w}_N[N, h_{ij}, p^{ij}]p_N\right) \approx 0\,. \tag{4.79}$$

The total Hamiltonian is obtained as the sum of first-class constraints

$$H = \Pi_0 + c_N \Pi_N + \Phi\big[\vec{N}\big] + \int_{\Sigma_t} w^i p_i\,. \tag{4.80}$$

Elimination of second-class constraints and equations of motion

We can now set the second-class constraints to zero strongly by introducing the Dirac bracket (2.18). The present theory has two local second-class constraints p_N and $\mathcal{C}$. But two global combinations of these constraints are first-class constraints, namely (4.65) and (4.79). A constraint surface that involves first-class constraints has a degenerate induced symplectic form and hence the Dirac bracket is undefined. Therefore setting the constraints p_N and $\mathcal{C}$ to zero everywhere is not permitted. That would also make the Hamiltonian vanish completely. We shall first impose a gauge fixing condition that removes the constant rescaling freedom of the lapse by setting N to one in average:

$$\int_{\Sigma_t} N\sqrt{h} = \int_{\Sigma_t} \sqrt{h} \equiv V_\Sigma\,. \tag{4.81}$$

Then we impose the second-class constraints strongly as

$$p_N = 0\,, \qquad \mathcal{C} = \varrho^2 \mathcal{C}_0 \sqrt{h}\,, \tag{4.82}$$

where $\mathcal{C}_0$ is a constant on Σ_t that ensures the first-class constraint (4.79) and the Hamiltonian along with it are not set to zero strongly. N is now completely determined by the constraints (4.81) and (4.82). Thus we can eliminate the variables N and p_N from the phase space. After that we may express $\mathcal{C}_0$ in terms of the variables h_{ij} and p^{ij}. In this case the Dirac bracket between h_{ij} and p^{ij} actually reduces to the canonical Poisson bracket, because the second-class constraints are of the form $N = N[h_{ij}, p^{ij}]$ and $p_N = 0$, and hence they do not impose constraints on h_{ij} and p^{ij}.

In the partially reduced phase space, the Hamiltonian (4.80) takes the form

$$H = \varrho^2 V_\Sigma\, \mathcal{C}_0 + \Phi\big[\vec{N}\big] + \int_{\Sigma_t} w^i p_i\,, \tag{4.83}$$

where $\mathcal{C}_0 = \mathcal{C}_0[h_{ij}, p^{ij}]$ is understood. The equations of motion are

$$\begin{aligned} \partial_t h_{ij} &= \frac{\delta H}{\delta p^{ij}} = \varrho^2 V_\Sigma \frac{\delta \mathcal{C}_0}{\delta p^{ij}} + \mathcal{L}_{\vec{N}} h_{ij} \,, \\ \partial_t p^{ij} &= -\frac{\delta H}{\delta h_{ij}} = -\varrho^2 V_\Sigma \frac{\delta \mathcal{C}_0}{\delta h_{ij}} + \mathcal{L}_{\vec{N}} p^{ij} \,. \end{aligned} \tag{4.84}$$

In the IR limit we may express $-\mathcal{C}_0$ as the smallest eigenvalue of a linear operator

$$(L - \varrho^2 \alpha_1)\nu = -\mathcal{C}_0 \nu \,, \tag{4.85}$$

where the operator L is given in (4.74) and $\nu = \sqrt{N}$. The first-order changes of the eigenvalue $-\mathcal{C}_0$ with respect to h_{ij} and p^{ij}, which are required in the equations of motion, can be obtained via first-order perturbation theory

$$V_\Sigma \frac{\delta \mathcal{C}_0}{\delta h_{ij}} = -\frac{\delta}{\delta h_{ij}} \int_{\Sigma_t} \sqrt{h}\nu (L - \varrho^2 \alpha_1)\nu \,, \tag{4.86}$$

and similarly for p^{ij}. Written in terms of N, the variations read

$$\begin{aligned} V_\Sigma \frac{\delta \mathcal{C}_0}{\delta h_{ij}} &= \frac{2N}{\varrho^4 \sqrt{h}} \left[p^i{}_k p^{kj} - \frac{\lambda}{d\lambda - 1} p^{ij} p + \frac{1}{2} h^{ij} \left(p_{kl} p^{kl} - \frac{\lambda}{d\lambda - 1} p^2 \right) \right] \\ &\quad + N\sqrt{h} h^{ij} \left(-\varrho^2 \alpha_1 + \alpha_2 R + \alpha_3 a_k a^k \right) + \alpha_3 N \sqrt{h} a^i a^j \\ &\quad + \alpha_2 \sqrt{h} \left(D^i D^j - h^{ij} D^2 - R^{ij} \right) N, \\ V_\Sigma \frac{\delta \mathcal{C}_0}{\delta p^{ij}} &= \frac{2N}{\varrho^4 \sqrt{h}} \left(p_{ij} - \frac{\lambda}{d\lambda - 1} h_{ij} p \right). \end{aligned} \tag{4.87}$$

The equations of motion (4.84) are naturally equivalent to those obtained by varying the action (4.59) and finding its extremal.

Asymptotically flat spacetime

Let us then consider an asymptotically flat four-dimensional spacetime. The familiar asymptotic falloff and boundary conditions (4.52) on the ADM variables are imposed on the spatial boundary $\partial\Sigma_t$. The extrinsic and intrinsic curvature tensors behave in the asymptotic region as $K_{ij} = O(r^{-2})$ and $R_{ij} = O(r^{-3})$. The acceleration vector (3.17) behaves as $a_i = O(r^{-2})$.

The primary and secondary constraints are the same as above. The asymptotic value of the lapse $N \to 1$, however, breaks the invariance under the constant rescaling of N generated by (4.65). Therefore the homogeneous part of (4.77) no longer admits the constant solution, but rather only the trivial solution, $w_N = 0$. Thus the general solution of the inhomogeneous equation (4.77) is the particular solution $w_N = \bar{w}_N[N, h_{ij}, p^{ij}]$. This means that the constraints p_N and $\mathcal{C}$ are now purely second-class constraints without any first-class combinations.

In asymptotically flat spacetime boundary terms need to be included. For the Hamiltonian (4.75) to define a consistent variational principle we must add some boundary terms into the Hamiltonian in order to cancel a total derivative in its variation. We obtain all the required boundary terms by considering a variation of the Hamiltonian with respect to all the variables, so that the asymptotic behavior of the variables is preserved by the variation. The variations of variables other than the metric h_{ij} only produce vanishing boundary terms whose integrands falloff quicker than $O(r^{-2})$ in the limit $r \to \infty$. The variation of h_{ij} yields the boundary term

$$-\varrho^2\alpha_2 \oint_{\mathcal{B}_t} \sqrt{\sigma}\, r^i \delta^{jk}(\partial_j \delta h_{ik} - \partial_i \delta h_{jk})\,, \tag{4.88}$$

where the integral is over the spherical boundary $\partial\Sigma_t$ of radius $r \to \infty$, γ is the determinant of the induced metric on the boundary and r^i is the outward-pointing unit normal to the boundary. This boundary term originates from the variation of the spatial scalar curvature term $-\alpha_2{}^{(3)}R$ in the potential (4.32). All the rest of the terms in the potential can only produce vanishing boundary terms whose integrands falloff quicker than $O(r^{-2})$ in the limit $r \to \infty$, because they contain higher spatial derivatives. Therefore we ensure the consistency of the variational principle by redefining the Hamiltonian as

$$H' = H + \varrho^2\alpha_2 \oint_{\mathcal{B}_t} \sqrt{\sigma}\, r^i \delta^{jk}(\partial_j h_{ik} - \partial_i h_{jk})\,. \tag{4.89}$$

The added boundary term differs from the one in GR only by the coupling constant α_2. In the Hamiltonian (4.75), the first term can no longer be written as in (4.69), but instead we obtain

$$\int_{\Sigma_t} N\mathcal{H}_0 = \int_{\Sigma_t} N\mathcal{C} + \oint_{\mathcal{B}_t} s_i V^i\,. \tag{4.90}$$

The asymptotic behavior of $V^i = O(r^{-2})$ is dominated by the IR part (4.71) of V^i with all the rest of the terms giving vanishing contributions because of the higher number of spatial derivatives. Hence we obtain

$$\oint_{\mathcal{B}_t} s_i V^i = -2\varrho^2\alpha_3 \oint_{\mathcal{B}_t} \sqrt{\sigma}\, r^i \partial_i N\,. \tag{4.91}$$

Thus the Hamiltonian can be written as a sum of local constraints plus a boundary term

$$H = \int_{\Sigma_t} \left(N\mathcal{C} + \bar{w}_N N p_N + N^i \tilde{\mathcal{H}}_i + w^i p_i\right) + E\,, \tag{4.92}$$

where the boundary term is the total energy

$$E = \varrho^2 \oint_{\mathcal{B}_t} \sqrt{\sigma}\, r^i \left[\alpha_2 \delta^{jk}(\partial_j h_{ik} - \partial_i h_{jk}) - 2\alpha_3 \partial_i N\right]. \tag{4.93}$$

In GR, the ADM energy depends only on the first derivatives of the metric on the boundary. In nonprojectable HL gravity, also the derivative of the lapse along the inward-pointing unit normal to the boundary contributes to the total energy.

In the limit $r \to \infty$, the metric can be treated as a perturbation of flat spacetime. In Newtonian gauge,

$$N = 1 + \psi \,, \qquad h_{ij} = (1 - 2\phi)\delta_{ij} \,, \tag{4.94}$$

the energy takes the form

$$E = \varrho^2 \oint_{\mathcal{B}_t} \sqrt{\sigma}\, r^i \partial_i \left(4\alpha_2\phi - 2\alpha_3\psi\right) . \tag{4.95}$$

In the static spherically symmetric solutions, $\phi = \psi$ and the asymptotic form of the solution to $O(r^{-1})$ is given by [118]

$$\psi = \phi = -\frac{r_0}{2r} \,, \tag{4.96}$$

where r_0 is a parameter with dimensions of length. The energy of this solution is

$$E = 8\pi\varrho^2 \left(\alpha_2 - \frac{\alpha_3}{2}\right) r_0 \,. \tag{4.97}$$

In the weak field limit, identifying the parameter r_0 as the Schwarzschild radius, $r_0 = 2G_\mathrm{N}M$, and the total energy as the mass, $E = M$, we obtain the effective Newton constant G_N in terms of the gravitational coupling constant ϱ^2 of the action (4.59) and the coupling constants α_2 and α_3 of its potential (4.32)

$$G_\mathrm{N} = \frac{1}{16\pi\varrho^2(\alpha_2 - \frac{1}{2}\alpha_3)} \,, \tag{4.98}$$

as was already obtained in [118] via Lagrangian methods.

Elimination of second-class constraints and construction of reduced phase space proceeds quite similarly as in the case of compact space. We impose the second-class constraints strongly, $p_N = 0$ and $\mathcal{C} = 0$, and eliminate the variables p_N and N in favor of the metric variables. Together with the boundary condition $N|_\infty = 1$ at spatial infinity the constraint $\mathcal{C} = 0$ determines the lapse completely as a functional of the variables h_{ij} and p^{ij}. The reduced phase space consists of the variables h_{ij} and p^{ij} modulo the gauge orbits of the foliation-preserving diffeomorphisms.

It should be possible to generalize the treatment in the presence of boundaries to spacetimes other than the asymptotically flat ones. Similarly as in GR, a generic spacetime with boundary can be considered [101, 103] (see Sec. 3.3). In HL gravity, however, that has not been accomplished yet.

Chapter 5

Modified Hořava-Lifshitz gravity

In papers **II** [15] and **III** [16], we proposed the modified $F(R)$ HL gravity that aims to combine the interesting cosmological aspects of $f(R)$ gravity and the possible renormalizability of HL gravity. In particular, we demonstrated that the solution of spatially-flat Friedmann-Robertson-Walker (FRW) equation has two branches: one that coincides with the usual $f(R)$ gravity for a certain choice of parameters, and one that is totally new and typical only for modified HL gravity. It was shown that unlike to standard HL gravity, our $F(R)$ HL gravity enables the possibility to unify the early-time inflation with the late-time acceleration in accord with the scenario of Ref. [51]. Hamiltonian formulation of the theory was constructed and analyzed. Hamiltonian analysis of the non-projectable version of this theory, where the lapse N depends also on the spatial coordinates, was conducted in paper **IV** [124].

$F(R)$ modifications of HL gravity were proposed and studied for the first time in [134]. Our proposal includes those models as special cases. Some further work on $F(R)$ HL gravities can also be found in [135].

In this chapter, we review the papers **II** [15], **III** [16] and **IV** [124] briefly. The used notation and conventions are harmonized to match the previous chapters.

5.1 Actions for modified Hořava-Lifshitz gravity

A generalized action that is invariant under foliation-preserving diffeomorphisms (4.3) consists of invariants constructed from the extrinsic and intrinsic curvature tensors of the spatial hypersurface Σ_t, namely K_{ij} and ${}^{(3)}R_{ijkl}$, and their covariant derivatives. Contractions of those tensors are taken with the induced metric h_{ij} on Σ_t. In addition, we shall include the total derivative term

$$2\nabla_\mu \left(n^\mu K - a^\mu\right) = 2\nabla_\mu \left(n^\mu \nabla_\nu n^\nu - n^\nu \nabla_\nu n^\mu\right), \tag{5.1}$$

which is a part of the decomposition of the scalar curvature of spacetime (3.19). It was excluded from the definition of original HL gravity because it only contributes a surface term. Here we consider an action that can depend on the included invariants nonlinearly, and hence the term (5.1) no longer appears as a total derivative in the Lagrangian. The gravitational action has the form

$$\begin{aligned} S_{\mathrm{gHL}} = \int_{\mathcal{M}} \mathrm{d}^4x N\sqrt{h} F\big(& h_{ij}, K_{ij}, D_i K_{jk}, \dots, D_{i_1}\cdots D_{i_n} K_{jk}, \dots, \\ & {}^{(3)}R_{ijkl}, D_i{}^{(3)}R_{jklm}, \dots, D_{i_1}\cdots D_{i_n}{}^{(3)}R_{jklm}, \dots, \\ & \nabla_\mu \left(n^\mu \nabla_\nu n^\nu - n^\nu \nabla_\nu n^\mu\right)\big)\,, \end{aligned} \tag{5.2}$$

where all possible contractions of the involved tensors are understood. We consider four-dimensional spacetime exclusively in this chapter. The Lagrangian of the action contains a second-order time derivative of the metric in the term (5.1), similarly as the usual $f(R)$ gravity. Time derivatives higher than second-order are excluded in order to avoid the problem with unstable ghost fields, which are often encountered in higher derivative theories (see Sec.2.2). The lapse is considered to be projectable, $N = N(t)$. One could further generalize the theory by revoking the projectability condition and considering an extended action which also depends on the vector $a_i = N^{-1}D_i N$, like we did for the usual HL gravity in Secs. 4.1.3 and 4.2.4. That would further increase the number of invariants permitted in the action.

In FRW spacetime with a flat spatial part and a nontrivial lapse, the metric is given by

$$\mathrm{d}s^2 = -N(t)^2 \mathrm{d}t^2 + a(t)^2 \sum_{i=1}^{3} \left(\mathrm{d}x^i\right)^2 . \tag{5.3}$$

Then the action (5.2) takes the form

$$S_{\mathrm{gHL}} = \int_{\mathcal{M}} \mathrm{d}^4x N\sqrt{h} F\left(\frac{H}{N}, \frac{3}{a^3 N}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{a^3 H}{N}\right)\right), \tag{5.4}$$

where we denote the Hubble parameter by $H = \dot{a}/a$, with $\dot{a} = \frac{\mathrm{d}a}{\mathrm{d}t}$. The function F in the Lagrangian of (5.4) depends only on two variables, $\frac{H}{N}$ and $\frac{3}{a^3 N}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{a^3 H}{N}\right)$.

As a specific example of the general theory (5.2), we consider modified $F(R)$ HL gravity. The main motivation for the modified $F(R)$ HL gravity is cosmology – similar to the reasons to consider the usual generally covariant $f(R)$ gravity. The action is defined by

$$\begin{aligned} S_{F(\tilde{R})} &= \varrho^2 \int_{\mathcal{M}} \mathrm{d}^4x N\sqrt{h} F(\tilde{R})\,, \\ \tilde{R} &= K_{ij}K^{ij} - \lambda K^2 + 2\mu \nabla_\mu \left(n^\mu \nabla_\nu n^\nu - n^\nu \nabla_\nu n^\mu\right) - \mathcal{V}(h_{ij})\,. \end{aligned} \tag{5.5}$$

We assume the function F is normalized as $F'(0) = 1$. The parameters λ and μ are dimensionless coupling constants. The potential part $\mathcal{V}(h_{ij})$ is a function of the metric h_{ij} on the three-dimensional space and the covariant derivatives

D_i defined by this metric. In paper **II** [15], the potential part of $\tilde{R}$ was defined according to the detailed balance condition (4.12). Here as in papers **III** [16] and **IV** [124], we consider that the potential part $\mathcal{V}(h_{ij})$ has the more general form given by (4.27). In the IR Lifshitz point, both the critical exponent z and the coupling constants λ and μ should flow to one, and the potential to $\mathcal{V}(h_{ij}) = -{}^{(3)}R$ effectively, so that the relativistic invariance could emerge at long distances as a symmetry of the effective action.

5.2 Hamiltonian formalism

Hamiltonian formulation of the general action (5.2) would be very complicated and indeed not that enlightening because of the extreme generality of the action. In the case of FRW spacetime, the action (5.4) is greatly simplified. We shall first outline Hamiltonian formulation of the action (5.4) following paper **III** [16]. Then the modified $F(R)$ HL gravity will be considered, summing up results from papers **II** [15], **III** [16] and **IV** [124].

5.2.1 Hamiltonian analysis of the general action in the FRW spacetime

We consider Hamiltonian formulation of the action (5.4) in FRW spacetime. This is quite similar to Misner's Hamiltonian formulation of FRW universe in canonical GR [136].

Introducing four additional variables α, A, β, B enables us to replace the action (5.4) with

$$S_{\mathrm{gHL}} = \int_{\mathcal{M}} \mathrm{d}^4x N\sqrt{h} \left[\alpha \left(A - \frac{H}{N} \right) + \beta \left(B - \frac{3}{a^3 N} \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{a^3 H}{N} \right) \right) + F\left(A, B\right) \right] . \tag{5.6}$$

The variations of the action with respect to α and β yield

$$A = \frac{H}{N}\,, \qquad B = \frac{3}{a^3 N} \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{a^3 H}{N} \right), \tag{5.7}$$

respectively. Integration by parts permits the removal of the second-order time derivative of a and the time derivative of N, assuming the boundary terms vanish due to appropriate boundary conditions. As a result β becomes a dynamical variable. Thus the action can be rewritten as

$$S_{\mathrm{gHL}} = \int \mathrm{d}t \int_{\Sigma_t} N\sqrt{h} \left[\alpha \left(A - \frac{H}{N} \right) + \beta B + \frac{3\dot{\beta} H}{N^2} + F\left(A, B\right) \right] . \tag{5.8}$$

This action is equivalent to (5.4) in the sense that they produce equivalent equations of motion. The advantage of the action (5.8) over (5.4) is the simpler dependence on the variables a and N, which will be crucially important in the Hamiltonian analysis.

In the Hamiltonian formalism, the canonical variables consist of the generalized coordinates h_{ij}, N, α, A, β, and B, and their canonically conjugated momenta π^{ij}, π_N, π_α, π_A, π_β and π_B, respectively. The action (5.8) does not depend on the time derivative of N, α, A or B. Thus we have the primary constraints

$$\Phi_1 = \pi_N \approx 0\,, \qquad \Phi_2 = \pi_\alpha \approx 0\,, \qquad \Phi_3 = \pi_A \approx 0\,, \qquad \Phi_4 = \pi_B \approx 0\,. \tag{5.9}$$

The momenta canonically conjugate to β and h_{ij} are

$$\pi_\beta = \frac{3a^3 H}{N}\,, \qquad \pi^{ij} = \frac{a}{6}\left(-\alpha + \frac{3\dot{\beta}}{N}\right)\delta^{ij}\,, \tag{5.10}$$

respectively. No more primary constraints are required, since the velocities $\dot{\beta}$ and $\dot{h}_{ij}$ can be solved in terms of the canonical variables. The canonical Hamiltonian (2.11) is obtained as

$$\begin{aligned} H_{\mathrm{c}} &= \int_{\Sigma_t} N\mathcal{H}\,, \qquad (5.11)\\ \mathcal{H} &= \frac{\pi_\beta}{3}\left(\frac{2}{a}\sum_{i=1}^{3}\pi^{ii} + \alpha\right) - a^3\left(\alpha A + \beta B + F(A,B)\right). \end{aligned}$$

The total Hamiltonian is defined by including the primary constraints with Lagrange multipliers.

The necessary secondary constraints are defined as

$$\begin{aligned} \Phi_0 &= \int_{\Sigma_t}\mathcal{H} \approx 0\,, & \Phi_5 &= -\frac{\pi_\beta}{3} + a^3 A \approx 0\,, \qquad (5.12)\\ \Phi_6 &= \alpha + \frac{\partial F(A,B)}{\partial A} \approx 0\,, & \Phi_7 &= \beta + \frac{\partial F(A,B)}{\partial B} \approx 0\,. \end{aligned}$$

The consistency conditions for the secondary constraints do not require introduction of further constraints. The complete algebra of the constraints under Poisson bracket can be found in paper **III** [16].

The total Hamiltonian can be written as a sum of the two first-class constraints H_0 and $\Phi_1 = \pi_N$, multiplied by two arbitrary time-dependent multipliers N and λ_1:

$$H_T = NH_0 + \lambda_1\Phi_1\,. \tag{5.13}$$

We have defined the first-class Hamiltonian constraint as

$$H_0 = \Phi_0 + \int_{\Sigma_t}\sum_{n=2}^{4}u_n\Phi_n\,, \tag{5.14}$$

where u_n are Lagrange multipliers that have been solved from the consistency conditions of the secondary constraints. Note that (5.14) is a combination of secondary and primary constraints. Usually a secondary first-class constraint would require us to define an extended Hamiltonian where the constraint would be added with an additional arbitrary multiplier. In this case, however, that would only lead to a redefinition of the multiplier N. The first-class constraints are associated with the remaining gauge symmetry of the system, invariance under time reparameterization.

Let us consider the case where all the second-order partial derivatives of the function $F(A,B)$ are nonzero. First we introduce the Dirac bracket (2.18) for the second-class constraints Φ_b, $b=2,\dots,7$. Then we can use those constraints to set $\pi_A=\pi_B=\pi_\alpha=0$ and to solve the auxiliary variables A, B and α in terms of the dynamical variables as

$$A=\frac{\pi_\beta}{3a^3}\,,\qquad \beta=-\frac{\partial F(\frac{\pi_\beta}{3a^3},B)}{\partial B}\ \Rightarrow\ B=\tilde{B}\left(\beta,\frac{\pi_\beta}{3a^3}\right), \tag{5.15}$$

and

$$\alpha=-\frac{\partial F\left(A,\tilde{B}\left(\beta,\frac{\pi_\beta}{3a^3}\right)\right)}{\partial A}\Big|_{A=\frac{\pi_\beta}{3a^3}}\,. \tag{5.16}$$

The total Hamiltonian is now given in terms of the dynamical variables as

$$H_T=\int_{\Sigma_t}N\mathcal{H}+\lambda_1\pi_N\,, \tag{5.17}$$

$$\mathcal{H}=\frac{2\pi_\beta}{3a}\sum_{i=1}^{3}\pi^{ii}-a^3\left[\beta\,\tilde{B}\left(\beta,\frac{\pi_\beta}{3a^3}\right)+F\left(\frac{\pi_\beta}{3a^3},\tilde{B}\left(\beta,\frac{\pi_\beta}{3a^3}\right)\right)\right],$$

where N acts as an arbitrary multiplier. The equation of motion for h_{ij} can be solved for the scale factor as

$$a(t)^3=a(t_0)^3+\int_{t_0}^{t}\mathrm{d}tN\pi_\beta\,. \tag{5.18}$$

For the canonically conjugated momentum, we obtain the equation of motion

$$\dot{\pi}^{ij}=\delta^{ij}N\left(\frac{\pi_\beta}{3a^3}\sum_{k=1}^{3}\pi^{kk}+\frac{3a}{2}\left[\beta\tilde{B}+F(A,\tilde{B})\right]-\frac{\pi_\beta}{2a^2}\frac{\partial F(A,\tilde{B})}{\partial A}\right), \tag{5.19}$$

where we set $A=\frac{\pi_\beta}{3a^3}$ and the arguments of $\tilde{B}$ have been omitted for brevity, $\tilde{B}\equiv\tilde{B}\left(\beta,\frac{\pi_\beta}{3a^3}\right)$. For the variable β, we obtain the equation of motion

$$\dot{\beta}=\frac{N}{3}\left(\frac{2}{a}\sum_{i=1}^{3}\pi^{ii}-\frac{\partial F(A,\tilde{B})}{\partial A}\Big|_{A=\frac{\pi_\beta}{3a^3}}\right). \tag{5.20}$$

For its conjugated momentum π_β, we obtain the equation of motion

$$\dot{\pi}_\beta=Na^3\tilde{B}\left(\beta,\frac{\pi_\beta}{3a^3}\right). \tag{5.21}$$

Specifying the form of the function F enables the analysis of FRW spacetime in any theory of the general form (5.2). We can conclude that when the second partial derivatives of the function $F(A,B)$ do not vanish, the proposed general action defines a consistent constrained theory. Other cases are considered in paper **III** [16] as well.

5.2.2 Modified $F(R)$ Hořava-Lifshitz gravity

We introduce two auxiliary fields A and B in order to rewrite the action (5.5) of modified $F(R)$ HL gravity as

$$S_{F(\tilde{R})} = \varrho^2 \int_{\mathcal{M}} \mathrm{d}^4x N\sqrt{h} \left[B \left(\tilde{R} - A \right) + F(A) \right] . \tag{5.22}$$

The variation of the action with respect to B gives $A = \tilde{R}$, which can be inserted back into the action in order to produce the original action (5.5). The variation with respect to A yields $B = F'(A)$, where F' denotes the derivative of F with respect to its argument. The action (5.22) is equivalent to the original action (5.5) in terms of the equations of motion.

The "modified scalar curvature" $\tilde{R}$ in the action (5.5) can be written

$$\tilde{R} = \mathcal{G}^{ijkl} K_{ij} K_{kl} + 2\mu \nabla_\mu \left(n^\mu K \right) - \frac{2\mu}{N} D^2 N - \mathcal{V}(h_{ij}) , \tag{5.23}$$

where $D^2 = h^{ij} D_i D_j$ is the covariant Laplace operator. Introducing (5.23) into the action and performing integration by parts yields the action

$$\begin{aligned} S_{F(\tilde{R})} = \varrho^2 \int_{t_1}^{t_2} \mathrm{d}t \int_{\Sigma_t} \sqrt{h} \Big[& NB \left(\mathcal{G}^{ijkl} K_{ij} K_{kl} - \mathcal{V}(h_{ij}) - A \right) + NF(A) \\ & -2\mu K \left(\partial_t B - N^i D_i B \right) - 2\mu N D^2 B \Big] . \end{aligned} \tag{5.24}$$

We are mainly interested in the propagating degrees of freedom in the bulk. Hence we assume the spatial hypersurfaces Σ_t are compact and have no boundary, so that integrals of total divergences may be dropped. We assume that the lapse is projectable, $N = N(t)$.

We denote the momenta canonically conjugate to the variables h_{ij}, N, N^i, A and B by p^{ij}, p_N, p_i, p_A and p_B, respectively. Since the Lagrangian is independent of the time derivatives of N, N^i and A, we define the primary constraints

$$p_N \approx 0 , \qquad p_i(\boldsymbol{x}) \approx 0 , \qquad p_A(\boldsymbol{x}) \approx 0 . \tag{5.25}$$

The momenta canonically conjugate to the metric h_{ij} and the scalar field B are defined as

$$p^{ij} = \varrho^2 \sqrt{h} \left[B \mathcal{G}^{ijkl} K_{kl} - \frac{\mu}{N} h^{ij} \left(\partial_t B - N^i D_i B \right) \right] , \tag{5.26}$$

$$p_B = -2\mu \varrho^2 \sqrt{h} K . \tag{5.27}$$

We assume $\mu \neq 0$ so that the momentum p_B does not vanish. We recall that the generalized DeWitt metric $\mathcal{G}^{ijkl}$ (4.13) has the inverse $\mathcal{G}_{ijkl}$ when $\lambda \neq 1/3$. However, as long as $\mu \neq 0$, the invertibility of $\mathcal{G}^{ijkl}$ is not that significant in our theory, because K is given by (5.27) as $K = -p_B / 2\mu\varrho^2\sqrt{h}$. Therefore we can write

$$\mathcal{G}^{ijkl} K_{kl} = K^{ij} + \frac{\lambda}{2\mu\varrho^2\sqrt{h}} h^{ij} p_B . \tag{5.28}$$

Thus we can solve $\partial_t K_{ij}$ and $\partial_t B$ in terms of the canonical variables for any value of λ. Hence no more primary constraints are needed.

The total Hamiltonian (2.12) has the familiar form

$$H = \int_{\Sigma_t} \left(N\mathcal{H}_0 + N^i\mathcal{H}_i + v_N p_N + v^i p_i + v_A p_A \right) , \tag{5.29}$$

where we have defined

$$\begin{aligned} \mathcal{H}_0 = \frac{1}{\varrho^2\sqrt{g}} & \left[\frac{1}{B}\left(p_{ij}p^{ij} - \frac{1}{3}p^2 \right) - \frac{1}{3\mu} p p_B - \frac{1-3\lambda}{12\mu^2} B p_B^2 \right] \\ & + \varrho^2\sqrt{g}\left[B\left(\mathcal{V}(h_{ij}) + A\right) - F(A) + 2\mu D^2 B \right] , \end{aligned} \tag{5.30}$$

and

$$\mathcal{H}_i = -2h_{ij}D_k p^{jk} + D_i B p_B \, , \tag{5.31}$$

and the primary constraints are included with Lagrange multipliers.

The consistency conditions that ensure the preservation of the primary constraints imply the following secondary constraints:

$$\Phi_0 = \int_{\Sigma_t} \mathcal{H}_0 \approx 0 \, , \qquad \mathcal{H}_i(\boldsymbol{x}) \approx 0 \, , \qquad \Phi_A(\boldsymbol{x}) = B(\boldsymbol{x}) - F'(A(\boldsymbol{x})) \approx 0 \, . \tag{5.32}$$

The smeared momentum constraint (3.36) is the generator of spatial diffeomorphisms for the dynamical variables h_{ii}, p^{ij}, B, p_B. It can be extended to a generator of spatial diffeomorphisms for all variables by including multiples of the primary constraints (5.25):

$$\Phi\left[\vec{X}\right] = \int_{\Sigma_t} X^i \tilde{\mathcal{H}}_i \, , \tag{5.33}$$

where we have defined

$$\tilde{\mathcal{H}}_i = -2h_{ij}D_k p^{jk} + D_i B p_B + D_i A p_A + D_i N p_N + \mathcal{L}_{\vec{N}} p_i \, . \tag{5.34}$$

The momentum constraints satisfy the standard Lie algebra (3.39). The Hamiltonian constraint Φ_0 and the momentum constraint satisfy the same Poisson brackets as in projectable HL gravity (4.40).

Since $F''(A) = 0$ would essentially reproduce the original HL gravity, we assume $F''(A) \neq 0$. Then p_A and Φ_A are second-class constraints,

$$\{p_A(\boldsymbol{x}), \Phi_A(\boldsymbol{y})\} = F''(A(\boldsymbol{x}))\delta(\boldsymbol{x} - \boldsymbol{y}) \, . \tag{5.35}$$

By replacing the canonical Poisson bracket with the Dirac bracket (2.18) and setting $p_A = 0$ and $\Phi_A = 0$, we can eliminate the pair of auxiliary variables A, p_A. We can solve A in terms of B from the algebraic constraint $\Phi_A = 0$. For the remaining canonical variables the Dirac bracket coincides with the Poisson bracket.

The total Hamiltonian is a sum of the first-class constraints p_N, p_i, Φ_0 and $\mathcal{H}_i$. We can conclude that our modified $F(R)$ HL gravity contains an extra scalar degree of freedom compared to the usual HL gravity. In the constructed Hamiltonian formalism, the extra scalar is represented by the pair of canonical variables B, p_B. The extra scalar degree of freedom is similar to the one present in the standard $f(R)$ gravity. For an appropriately chosen function F and parameters in the Lagrangian, the extra scalar possesses a healthy potential and it is not an unstable ghost. In order the avoid a divergence of the kinetic part of $\mathcal{H}_0$, we should consider $B > 0$, and then choosing $\lambda \geq 1/3$ ensures the p_B^2 term is nonnegative, while λ could flow to one in the IR limit.

Gauge fixing conditions could be introduced in a similar way as explained in Sec. 4.2.1 in the context of usual projectable HL gravity. Canonical quantization can then be performed as discussed in Sec. 2.3.

5.2.3 Nonprojectable $F(R)$ Hořava-Lifshitz gravity

Let us then consider what happens to modified $F(R)$ HL gravity when we revoke the projectability condition, letting the lapse depend on both space and time. Now the primary constraint $p_N \approx 0$ associated with the lapse becomes a local constraint. As a result $\mathcal{H}_0$ becomes the local Hamiltonian constraint, $\mathcal{H}_0 \approx 0$. The total Hamiltonian is given by (5.29)–(5.31) as a sum of local constraints.

As in the case of original nonprojectable HL gravity analyzed in Sec. 4.2.2, the problem arises in the Poisson bracket between Hamiltonian constraints. In the renormalizable case ($z = d = 3$), the Poisson bracket is obtained in the form (4.48). The tensor densities $E_n^{i_1 \cdots i_n}$ in (4.49) are constructed from the canonical variables h_{ij}, p^{ij}, B, p_B and their covariant derivatives. The consistency condition (4.50) again has two possible interpretations. If it is regarded as a condition on N, the only solution for generic couplings is $N = 0$, which is physically impossible since it collapses the time dimension of spacetime. If we regard (4.50) as a constraint on the canonical variables h_{ij}, p^{ij}, B, p_B, we end up with too many constraints.

We illustrate this problem by considering the effective potential in the IR limit. We consider the potential (omitting an irrelevant constant term)

$$\mathcal{V}(h_{ij}) = \alpha\,{}^{(3)}\!R\,. \tag{5.36}$$

We obtain the densities E_0 and E_1^i as

$$\begin{aligned} \frac{1}{2}E_1^i &= -\alpha\mathcal{H}^i - \frac{2(\alpha+1)}{3}D^i\pi + 4(-\alpha+\mu)\left(\pi^{ij} - \frac{1}{3}h^{ij}\pi\right)\frac{D_jB}{B} \\ &\quad - \frac{1-2\alpha-3\lambda}{3\mu}BD^i\pi_B + \left(\frac{3\lambda+\mu-2\alpha-1}{3\mu} + \alpha\right)D^iB\pi_B\,, \\ E_0 &= \frac{1}{2}D_iE_1^i\,, \end{aligned} \tag{5.37}$$

where the momentum constraint term $-\alpha\mathcal{H}^i$ can be dropped. Now the required constraint reads

$$E_1^iD_iN + E_0N \approx 0 \qquad \text{or} \qquad D_i\left(N^2E_1^i\right) \approx 0\,. \tag{5.38}$$

In order to satisfy this constraint without fixing N, we need to introduce new constraints that imply $E_1^i \approx 0$. In the usual HL gravity, it was sufficient to impose $p \approx 0$, but in our theory that would be insufficient. It would take three constraints – such as $p \approx 0$, $p_B \approx 0$ and $B \approx$ constant – to satisfy the required constraint. The consistency conditions for those constraints imply further constraints. As a result the system is over-constrained. When the full potential (4.27) is considered, the number of required constraints can be further increased.

Thus we can conclude that the nonprojectable version of $F(R)$ HL gravity is physically inconsistent in a quite similar way as the usual HL gravity. In fact, the inconsistency is even more serious than in the usual HL gravity. The low-energy effective action of $F(R)$ HL gravity does not admit the interpretation of being a partially gauge fixed form of a generally covariant theory, unlike the usual nonprojectable HL gravity which can be interpreted as partially gauge fixed GR, as was discussed in Sec. 4.2.3. This suggests that the mentioned interpretation might be specific to the original HL gravity, i.e, not generalizable to $\mathrm{Diff}_{\mathcal{F}}(\mathcal{M})$ invariant theories in general. It might be possible to obtain a consistent formulation of nonprojectable $F(R)$ HL gravity by extending the potential part $\mathcal{V}(h_{ij})$ of $\tilde{R}$ with terms that involve the vector $a_i = N^{-1} D_i N$ as in (4.32). That, however, has not been studied properly yet.

5.3 Renormalizability

Modified $F(R)$ HL gravity retains the renormalizability characteristics of the original HL gravity. That was shown in paper **III** [16]. We consider the action in another conformal frame, quite similarly to what we did for $f(R)$ gravity (1.9). First the action is written in the Jordan frame as in (5.22) with $B = F'(A)$. We choose the gauge $N = 1, N^i = 0$ for simplicity. Then we perform a conformal transformation to Einstein frame. We define a new scalar field by

$$\varphi = \frac{1}{3} \ln F'(A) \,. \tag{5.39}$$

We can solve it algebraically for $A(\varphi)$ so that $F'(A(\varphi)) = e^{3\varphi}$. The conformally transformed metric $\bar{h}_{ij}$ is defined by

$$h_{ij} = e^{-\varphi} \bar{h}_{ij} \,. \tag{5.40}$$

Such a field redefinition should keep the S-matrix invariant and preserve the renormalization characteristics of the theory. The action takes the form

$$\begin{aligned} S_{F(\tilde{R})} = \varrho^2 \int_{\mathcal{M}} \mathrm{d}^4 x \sqrt{\bar{h}} \bigg[& \frac{1}{4} \bar{h}^{ik} \bar{h}^{jl} \partial_t \bar{h}_{ij} \partial_t \bar{h}_{kl} - \frac{\lambda}{4} \left(\bar{h}^{ij} \partial_t \bar{h}_{ij} \right)^2 \\ & + \left(-\frac{1}{2} + \frac{3\lambda}{2} - \frac{3\mu}{2} \right) \bar{h}^{ij} \partial_t \bar{h}_{ij} \partial_t \varphi \\ & + \left(\frac{3}{4} - \frac{9\lambda}{4} + \frac{9\mu}{2} \right) (\partial_t \varphi)^2 + \bar{\mathcal{V}} \left(\bar{h}_{ij}, \varphi \right) - V(\varphi) \bigg] \,, \end{aligned} \tag{5.41}$$

where the interactions of the fields $\bar{h}_{ij}$ and φ are defined by

$$\bar{\mathcal{V}}\left(\bar{h}_{ij}, \varphi\right) = \mathcal{V}\left(e^{-\varphi}\bar{h}_{ij}\right), \qquad V(\varphi) = A(\varphi)F'(A(\varphi)) - F(A(\varphi)). \tag{5.42}$$

The transformed potential $\bar{\mathcal{V}}\left(\bar{h}_{ij}, \varphi\right)$ contains spatial derivatives of the fields $\bar{h}_{ij}$ and φ up to $2z$-th order. Thus the propagators of $\bar{h}_{ij}$ and φ behave as $|\boldsymbol{k}|^{-2z}$ in the UV region, similarly as in (4.2). In four-dimensional spacetime, we choose $z = 3$, which gives six spatial derivatives in the Lagrangian and propagators with $|\boldsymbol{k}|^{-6}$ UV behavior.

Decoupling of the fields $\bar{h}_{ij}$ and φ in the Einstein frame can be accomplished by certain choices of the parameters λ, μ and the coupling constants in the potential $\mathcal{V}$. Choosing $\mu = \lambda - 1/3$ diagonalizes the kinetic part of the Lagrangian. Then $\lambda > 1/3$ ensures the kinetic term for φ is positive and the theory is unitary. The interactions can be simplified by fixing couplings in the potential.

5.4 Cosmological aspects

We briefly discuss the cosmological aspects of modified $F(R)$ HL gravity, which were studied in papers **III** [16] and **IV** [124]. We consider the spatially flat FRW universe (5.3). The modified scalar curvature in the action (5.5) has the form

$$\tilde{R} = \frac{(3 - 9\lambda + 18\mu)H^2}{N^2} + \frac{6\mu}{N}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{H}{N}\right). \tag{5.43}$$

We assume that matter can be included similarly as in GR. The energy density ρ and the pressure p of the matter fluid are assumed to satisfy the standard conservation law

$$\dot{\rho} + 3H(\rho + p) = 0. \tag{5.44}$$

The field equations can be obtained by taking variations of the action with respect to h_{ij} and N. The one corresponding to the first Friedmann equation is obtained by varying N, setting $N = 1$ and integrating

$$F\big(\tilde{R}\big) - 6\left\{(1 - 3\lambda + 3\mu)\, H^2 + \mu\dot{H}\right\} F'\big(\tilde{R}\big) + 6\mu H\frac{\mathrm{d}F'\big(\tilde{R}\big)}{\mathrm{d}t} - \rho - \frac{C}{a^3} = 0, \tag{5.45}$$

where C is a constant of integration. If $C > 0$, the term Ca^{-3} may be regarded as cold dark matter. In the special case $\lambda = \mu = 1$ and $C = 0$, the equation for standard $f(R)$ gravity is recovered. The counterpart of the second Friedmann equation is obtained by varying h_{ij} and setting $N = 1$:

$$\begin{aligned} &F\big(\tilde{R}\big) - 2\,(1 - 3\lambda + 3\mu)\left(\dot{H} + 3H^2\right) F'\big(\tilde{R}\big) \\ &\qquad - 2\,(1 - 3\lambda)\, H\frac{\mathrm{d}F'\big(\tilde{R}\big)}{\mathrm{d}t} + 2\mu\frac{\mathrm{d}^2F'\big(\tilde{R}\big)}{\mathrm{d}t^2} + p = 0. \end{aligned} \tag{5.46}$$

Several kinds of functional forms of F realize exponentially expanding de Sitter cosmologies without any extra components like inflaton or dark energy. In the absence of matter, the above Friedmann equations reduce to

$$F\big(\tilde{R}\big) + 6\gamma^2(3\lambda - 3\mu - 1)F'\big(\tilde{R}\big) = 0 \tag{5.47}$$

for the de Sitter universe: $N(t) = 1$, $a(t)^2 = \exp(\gamma t)$. Compared to the standard $f(R)$ gravity, which is known to be able to describe expanding de Sitter cosmologies inherently, the existence of the two parameters λ and μ enables an even richer set of solutions. For example, reconstruction of nontrivial power law solutions for the scale factor provide a bigger set of exact solutions. Realistic models unifying inflation and current accelerated expansion (dark energy) are possible, which was demonstrated by an explicit model in paper **III** [16]. As in some other modified gravitational theories, the scale factor, the Hubble parameter or the effective energy density and pressure may become divergent after a finite period of time has passed. Hence care must be taken when specific models are constructed.

Chapter 6

Hamiltonian analysis of covariant renormalizable gravity

As an alternative candidate for a QFT of gravity, we consider the covariant renormalizable gravity (CRG) [17]. Unlike HL gravity, CRG is defined to be covariant under spacetime diffeomorphism and possesses local Lorentz invariance at the fundamental level. CRG achieves a similar UV behavior of the graviton propagator as HL gravity due to the presence of higher-order derivatives in its Lagrangian. Since the theory is generally covariant, higher-order time derivatives must be included as well. As a means to avoid the notorious problems with ghosts in higher derivative theories, Lorentz invariance of the theory is broken spontaneously at high energies. The spontaneous symmetry breaking is accomplished by introducing an exotic scalar field and imposing a constraint on the scalar field. Choosing a solution of the constraint breaks Lorentz symmetry spontaneously. As a result the higher-order time derivatives are supposed to vanish when small perturbations on Minkowski spacetime are considered in certain reference frames. Hence the theory might avoid the problem with ghosts. The scalar field needs to couple to spacetime in an unusual and highly nonminimal way in order to accomplish this feat. The spatial higher-order derivatives left in the Lagrangian are able ensure power-counting renormalizability, similarly to what we saw in the HL gravity.

However, we should note that the renormalizability of CRG, as well as of the HL theory, is mostly based on the power-counting arguments. There are many potential pathologies that could ultimately ruin the renormalizability of the theory, such as ghosts or strong coupling.

An improved version of CRG was proposed in [18], because the original CRG model did not improve the UV behavior sufficiently. A perturbative analysis on the Minkowski background showed that the extra degrees of freedom present in the new theory do not propagate. Only the massless graviton propagates, at least on the tree level. This is a notable feature for a generally covariant higher derivative gravitational theory, albeit with spontaneously broken symmetry.. Whether the introduction of the exotic constrained scalar field, followed by the spontaneous symmetry breaking, really are able to accomplish such gain, deserves further investigation. In this chapter, we review the papers **V** [19] and **VI** [20], where Hamiltonian formulations of the original and new versions of CRG were studied, respectively.

6.1 Original covariant renormalizable gravity

Original CRG [17] couples the energy-momentum tensor $T_{\mu\nu}$ of an exotic fluid to the Ricci tensor and scalar curvature as $T^{\mu\nu}R_{\mu\nu} + \beta TR$, and includes extra derivatives as the combination $T^{\mu\nu}\nabla_\mu\nabla_\nu + \gamma T\nabla^\mu\nabla_\mu$, in order obtain desired UV behavior for the graviton propagator. We consider a power-counting renormalizable theory in four-dimensional spacetime, which is defined by the action

$$S_{\mathrm{CRG}} = \int_{\mathcal{M}} \mathrm{d}^4x\sqrt{-g}\left[\frac{R}{2\kappa} - \alpha\left(T^{\mu\nu}R_{\mu\nu} + \beta TR\right)\left(T^{\mu\nu}\nabla_\mu\nabla_\nu + \gamma T\nabla^\mu\nabla_\mu\right)\right.$$
$$\left. \times\left(T^{\mu\nu}R_{\mu\nu} + \beta TR\right)\right]. \quad (6.1)$$

One considers small perturbations on the Minkowski background in linear approximation, $g_{\mu\nu} = \eta_{\mu\nu} + \tilde{g}_{\mu\nu}$, $|\tilde{g}_{\mu\nu}| \ll 1$, with the gauge conditions $\tilde{g}_{00} = \tilde{g}_{0i} = 0$. The fluid is assumed to be perfect with the equation of state parameter $w = p/\rho$ (ratio of pressure p to energy density ρ). When the parameters β are chosen as $\beta = (1-w)/(2(3w-1))$ and $\gamma = 1/(3w-1)$, one obtains that the higher-order time derivatives in the Lagrangian vanish

$$\begin{aligned} T^{\mu\nu}R_{\mu\nu} + \beta TR &= \frac{\rho(w+1)}{2}\left(\partial^i\partial^j\tilde{g}_{ij} - \partial^k\partial_k(\delta^{ij}\tilde{g}_{ij})\right), \\ \left(T^{\mu\nu}\nabla_\mu\nabla_\nu + \gamma T\nabla^\mu\nabla_\mu\right)f &= \rho(w+1)\partial^i\partial_i f\,. \end{aligned} \quad (6.2)$$

This is argued to ensure that the theory is power-counting renormalizable, since the Lagrangian contains six spatial derivatives of the metric, which modify the UV behavior of the graviton propagator to $|\boldsymbol{k}|^{-6}$ in momentum space.

The perfect fluid could be realized by a scalar field ϕ with the energy-momentum tensor

$$T^\phi_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left(\frac{1}{2}\partial_\rho\phi\partial^\rho\phi + V(\phi)\right), \quad (6.3)$$

provided that ϕ is imposed to satisfy the constraint

$$\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + U(\phi) = 0\,. \quad (6.4)$$

When we assume $U(\phi) > 0$, the constraint (6.4) implies that the vector $\partial_\mu\phi$ is timelike. Then at least locally, one can choose the direction of time to be parallel to $\partial_\mu\phi$, so that the constraint (6.4) yields

$$\frac{1}{2}\left(\frac{\mathrm{d}\phi}{\mathrm{d}t}\right)^2 = U(\phi)\,. \tag{6.5}$$

Choosing a solution to the constraint on ϕ breaks Lorentz invariance. On the flat background metric one obtains the energy density $\rho_\phi = U(\phi)+V(\phi)$, the pressure $p_\phi = U(\phi) - V(\phi)$, and the equation of state parameter $w = p_\phi/\rho_\phi$ associated with $T^\phi_{\mu\nu}$. For simplicity, it is assumed that the potentials are constants: $U(\phi) = U_0$, $V(\phi) = V_0$. In (6.2), we obtain the factor $\rho_\phi(w+1) = 2U_0$. Using the constraint (6.4) and fixing the parameters β and γ in terms of the potentials U_0 and V_0, one obtains the action as

$$S_{\mathrm{CRG}} = \int_{\mathcal{M}} \mathrm{d}^4x\sqrt{-g}\left[\frac{R}{2\kappa} - \alpha\partial^\mu\phi\partial^\nu\phi G_{\mu\nu}\left(\partial^\mu\phi\partial^\nu\phi\nabla_\mu\nabla_\nu\right.\right.$$
$$\left.\left. -\ \partial^\mu\phi\partial_\mu\phi\nabla^\nu\nabla_\nu\right)\partial^\rho\phi\partial^\sigma\phi G_{\rho\sigma}\right], \tag{6.6}$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ is the Einstein tensor.[1]

6.1.1 First-order ADM representation of the action

We obtain the ADM representation of the action (6.6) by decomposing it with respect to a foliation of spacetime $\mathcal{M}$ into a family of Cauchy surfaces Σ_t. In the action, the required expressions are

$$\begin{aligned}\partial^\mu\phi\partial^\nu\phi\, G_{\mu\nu} = D^i\phi D^j\phi &\Bigg[{}^{(3)}R_{ij} + KK_{ij} - 2K_{ik}K^k{}_j - \frac{1}{N}D_iD_jN \\ &+ \frac{1}{N}\mathcal{L}_{Nn}K_{ij} - \frac{1}{2}h_{ij}\left(R + K_{kl}K^{kl} + K^2 + 2\nabla_nK - \frac{2}{N}D^kD_kN\right)\Bigg] \\ &+ 2\left(\nabla_n\phi\right)D^i\phi\left(D_iK - D^jK_{ji}\right) + \frac{1}{2}\left(\nabla_n\phi\right)^2\left(K^2 - K_{ij}K^{ij} + {}^{(3)}R\right)\end{aligned} \tag{6.7}$$

and

$$\begin{aligned}\left(\partial^\mu\phi\partial^\nu\phi\nabla_\mu\nabla_\nu - \partial^\mu\phi\partial_\mu\phi\nabla^\nu\nabla_\nu\right)f = D^i\phi D^j\phi &\left[\left(D_iD_j - h_{ij}D^kD_k\right.\right. \\ &\left.\left. -\ h_{ij}(D^k\ln N)D_k\right)f - \left(K_{ij} - h_{ij}K\right)\nabla_nf + h_{ij}\nabla_n\nabla_nf\right] \\ &- 2\left(\nabla_n\phi\right)D^i\phi\left(D_i\nabla_n + K_{ij}D^j\right)f + \left(\nabla_n\phi\right)^2\left(D^iD_i - K\nabla_n\right)f\,.\end{aligned} \tag{6.8}$$

Here we denote the Lie derivative along the vector $Nn^\mu = (1, -N^i)$ by $\mathcal{L}_{Nn}$ and the covariant derivative of along the unit normal n^μ by ∇_n:

$$\mathcal{L}_{Nn}K_{ij} = \partial_tK_{ij} - \mathcal{L}_{\vec{N}}K_{ij}\,, \qquad \nabla_nf = \frac{1}{N}\left(\partial_tf - N^iD_if\right), \tag{6.9}$$

[1] Alternatively, one can use the constraint (6.4) to write $\partial_\mu\phi\partial^\mu\phi = -2U_0$ in the Lagrangian, so that $\partial^\mu\phi\partial^\nu\phi G_{\mu\nu} = \partial^\mu\phi\partial^\nu\phi R_{\mu\nu} + U_0R$ and $-\partial^\mu\phi\partial_\mu\phi\nabla^\nu\nabla_\nu = 2U_0\nabla^\mu\nabla_\mu$.

where f is a scalar function on spacetime. Substituting these results and the decomposition of the scalar curvature (3.19) into (6.6) gives us the ADM form of the action. Unfortunately, the result is a complicated higher derivative action, whose Lagrangian contains higher-order time derivatives of the ADM variables and the scalar field ϕ up to fourth order. As a result extra degrees of freedom associated with the higher time derivatives are expected to be present in this theory. The general action turned out to be an unrewarding target to analyze due to its complexity.

We can simplify the action considerably by identifying a foliation of spacetime defined by the constrained scalar field. There exists a foliation of space-time into spatial hypersurfaces Σ_t whose unit normal is given by

$$n^\mu = -\frac{\partial^\mu \phi}{\sqrt{-\partial_\nu \phi \partial^\nu \phi}} = -\frac{\partial^\mu \phi}{\sqrt{2U_0}} \,. \tag{6.10}$$

When this foliation is chosen the constraints on the scalar field (6.4) becomes identical with the normalization requirement of the unit normal, $n_\mu n^\mu = -1$. The gradient of the scalar field can now be written

$$\partial^\mu \phi = -\sqrt{2U_0} n^\mu \,, \tag{6.11}$$

where the unit normal is given in terms of the ADM variables (3.11). Since (6.11) implies

$$\partial_\mu \phi = -\sqrt{2U_0} n_\mu = \left(\sqrt{2U_0} N, 0, 0, 0\right) , \tag{6.12}$$

the leaves Σ_t of the foliation are surfaces of constant ϕ in ADM coordinates. We can integrate to get

$$\phi(t) = \phi(t_0) + \sqrt{2U_0} \int_{t_0}^{t} dt' N(t') \,, \tag{6.13}$$

which implies N is constant on Σ_t too, i.e., projectable in the language of HL gravity. The symmetry is reduced to foliation-preserving diffeomorphisms (4.3). The results (6.7) and (6.8) for the spacetime decomposition of the action can now be written

$$\partial^\mu \phi \partial^\nu \phi G_{\mu\nu} = U_0 \left(K^2 - K_{ij} K^{ij} + {}^{(3)}\!R\right) , \tag{6.14}$$

$$\left(\partial^\mu \phi \partial^\nu \phi \nabla_\mu \nabla_\nu - \partial^\mu \phi \partial_\mu \phi \nabla^\nu \nabla_\nu\right) f = 2U_0 \left(D^i D_i - K \nabla_n\right) f \,. \tag{6.15}$$

The action still contains second-order time derivatives of the metric. Hence we introduce extra variables ζ_1, λ_1, ζ_2 and λ_2 and rewrite the action (6.6) into an equivalent first-order form,

$$\begin{aligned} S_{\mathrm{CRG}} = \int_{\mathcal{M}} \mathrm{d}^4 x \sqrt{-g} \Big\{ \frac{R}{2\kappa} - \alpha \big[\zeta_1 \zeta_2 + \lambda_1 \big(\zeta_1 - (\partial^\mu \phi \partial^\nu \phi \nabla_\mu \nabla_\nu \\ - \partial^\mu \phi \partial_\mu \phi \nabla^\nu \nabla_\nu) \, \zeta_2 \big) + \lambda_2 \left(\zeta_2 - \partial^\mu \phi \partial^\nu \phi G_{\mu\nu} \right) \big] \Big\} \,. \end{aligned} \tag{6.16}$$

The ADM representation of the action is then obtained in the form (2.4) as

$$
\begin{aligned}
S_{\mathrm{CRG}} = \int_{t_1}^{t_2} \mathrm{d}t \int_{\Sigma_t} N\sqrt{h} \Bigg\{ & \frac{\mathcal{G}^{ijkl} K_{ij} K_{kl} + {}^{(3)}R}{2\kappa} - \alpha \big[\zeta_1 \zeta_2 \\
& + \lambda_1 \left(\zeta_1 + 2U_0 \left(K \nabla_n \zeta_2 - D^i D_i \zeta_2 \right) \right) \\
& + \lambda_2 \left(\zeta_2 + U_0 \left(\mathcal{G}^{ijkl} K_{ij} K_{kl} - {}^{(3)}R \right) \right) \big] \Bigg\} , \qquad (6.17)
\end{aligned}
$$

where the DeWitt metric is used, $\mathcal{G}^{ijkl} = \frac{1}{2}(h^{ik}h^{jl} + h^{il}h^{jk}) - h^{ij}h^{kl}$. The scalar field ζ_2 is the required extra variable that absorbs the second-order time derivatives. This first-order action is the basis of the following Hamiltonian formulation.

6.1.2 Hamiltonian formalism

Let us consider Hamiltonian formulation of the first-order ADM form (6.17) of the action of the original CRG. First we shall define the canonical momenta. Since the action is independent of the time derivatives of N, N^i, ζ_1, λ_1 and λ_2, their canonically conjugated momenta, p_N, p_i, p_{ζ_1}, p_{λ_1} and p_{λ_2}, respectively, are the primary constraints:

$$
p_N \approx 0 , \quad p_i(\boldsymbol{x}) \approx 0 , \quad p_{\zeta_1}(\boldsymbol{x}) \approx 0 , \quad p_{\lambda_1}(\boldsymbol{x}) \approx 0 , \quad p_{\lambda_2}(\boldsymbol{x}) \approx 0 . \qquad (6.18)
$$

The momenta canonically conjugate to h_{ij} and ζ_2 are defined by

$$
p^{ij} = \sqrt{h} \left(\frac{1}{2\kappa} \mathcal{G}^{ijkl} K_{kl} - \alpha U_0 \left(\lambda_1 h^{ij} \nabla_n \zeta_2 + \lambda_2 \mathcal{G}^{ijkl} K_{kl} \right) \right) , \qquad (6.19)
$$

$$
p_{\zeta_2} = -2\alpha U_0 \sqrt{h}\, \lambda_1 K . \qquad (6.20)
$$

Since $\alpha \neq 0$ and $U_0 > 0$, it is possible to solve the time derivatives of the variables h_{ij} and ζ_2 in terms of the canonical variables. Thus no more primary constraints are required for performing the Legendre transformation that gives us the canonical Hamiltonian. The total Hamiltonian is obtained as

$$
H = \int_{\Sigma_t} \left(N\mathcal{H}_0 + N^i \mathcal{H}_i + v_N p_N + v^i p_i + v_{\zeta_1} p_{\zeta_1} + v_{\lambda_1} p_{\lambda_1} + v_{\lambda_2} p_{\lambda_2} \right) , \qquad (6.21)
$$

where the v fields are Lagrange multipliers and we have defined

$$
\begin{aligned}
\mathcal{H}_0 = \frac{1}{\sqrt{h}} \frac{2\kappa}{1 - 2\kappa\alpha U_0 \lambda_2} \Bigg[& p_{ij} p^{ij} - \frac{1}{3} p^2 - \frac{1}{3} \left(\frac{1 - 2\kappa\alpha U_0 \lambda_2}{2\kappa\alpha U_0 \lambda_1} \right) p p_{\zeta_2} \qquad (6.22) \\
& + \frac{1}{6} \left(\frac{1 - 2\kappa\alpha U_0 \lambda_2}{2\kappa\alpha U_0 \lambda_1} \right)^2 p_{\zeta_2}^2 \Bigg] + \sqrt{g} \Bigg\{ - \frac{1 + 2\kappa\alpha U_0 \lambda_2}{2\kappa} {}^{(3)}R \\
& + \alpha \left[\zeta_1 \zeta_2 + \lambda_1 \left(\zeta_1 - 2U_0 D^i D_i \zeta_2 \right) + \lambda_2 \zeta_2 \right] \Bigg\}
\end{aligned}
$$

and

$$
\mathcal{H}_i = -2h_{ij} D_k p^{jk} + D_i \zeta_2 p_{\zeta_2} . \qquad (6.23)
$$

We immediately recognize that $\mathcal{H}_i$ has the familiar form of a momentum constraint, whose smeared form (3.36) generates diffeomorphisms on the spatial hypersurface Σ_t for the fields h_{ij} and ζ_2, and their conjugated momenta.

The secondary constraints, which ensure the primary constraints (6.18) are satisfied at all times, are defined as

$$\Phi_0 = \int_{\Sigma_t} \mathcal{H}_0 \approx 0\,, \qquad \Phi_i(\boldsymbol{x}) = \mathcal{H}_i \approx 0\,, \qquad \Phi_4(\boldsymbol{x}) = \zeta_2 + \lambda_1 \approx 0\,, \tag{6.24}$$
$$\Phi_5(\boldsymbol{x}) = \frac{1}{g}\left(pp_{\zeta_2} - \frac{1-2\kappa\alpha U_0\lambda_2}{2\kappa\alpha U_0\lambda_1}p_{\zeta_2}^2\right) + 3\alpha^2 U_0\lambda_1^2\left(\zeta_1 - 2U_0 D^i D_i\zeta_2\right) \approx 0\,,$$
$$\Phi_6(\boldsymbol{x}) = \frac{1}{g}\left(p_{ij}p^{ij} - \frac{1}{3}p^2 - \frac{1}{6}\left(\frac{1-2\kappa\alpha U_0\lambda_2}{2\kappa\alpha U_0\lambda_1}\right)^2 p_{\zeta_2}^2\right)$$
$$+ \frac{(1-2\kappa\alpha U_0\lambda_2)^2}{4\kappa^2 U_0}\left(\zeta_2 - U_0 R\right) \approx 0\,.$$

The global Hamiltonian constraint Φ_0 and the smeared momentum constraint $\Phi\big[\vec{X}\big]$ satisfy similar Poisson brackets as in projectable HL theories, namely (4.40). Obtaining the Poisson bracket between Φ_0 and $\Phi\big[\vec{X}\big]$ is easiest if we extend the momentum constraint to a generator diffeomorphisms for all variables. This is again accomplished by including multiples of the primary constraints (6.18) into the generator $\Phi\big[\vec{X}\big]$. Then the primary constraints, $\mathcal{H}_0$ and $\mathcal{H}_i$ transform as scalar and vector densities, while the other secondary constraints Φ_s, $s = 4, 5, 6$, were defined to be scalars under spatial diffeomorphisms. The remaining Poisson brackets between the secondary constraints (6.24) turn out to be quite complicated expressions.

There exists four propagating physical degrees of freedom, since we have 26 canonical variables ($N^i, h_{ij}, \lambda_1, \zeta_1, \lambda_2, \zeta_2$, and their conjugated momenta), six first-class constraints ($p_i, \mathcal{H}_i$), and six second-class constraints ($p_{\zeta_1}, p_{\lambda_1}, p_{\lambda_2}, \Phi_4$, Φ_5, Φ_6). That is two more local physical degrees of freedom than in GR. As another comparison, our analysis shows that CRG (considered with respect to a foliation adapted to ϕ) has one more physical degree of freedom than projectable HL gravity. Interestingly, the number of physical modes is exactly the same as in the modified $F(R)$ HL gravity considered in Chapter 5. One extra physical degree of freedom ζ_2 has its origin in the higher order time derivatives present in the CRG action. The other extra propagating mode is caused by the projectability condition similarly as in HL gravity. No ghosts appear. But such extra degrees of freedom may be problematic since they may generate extra (long range) forces that are not in agreement with observations. One might be able to bring the number of physical degrees of freedom closer to that of GR by introducing some extra gauge symmetry, along with some new fields, which generates some new constraints.

Once the Dirac bracket (2.18) has been introduced, the second-class constraints Φ_4, Φ_5, Φ_6 can be used to solve the auxiliary variables $\zeta_1, \lambda_1, \lambda_2$ in terms of the dynamical variables ($h_{ij}, p^{ij}, \zeta_2, p_{\zeta_2}$), and the primary constraints $p_{\zeta_1}, p_{\lambda_1}, p_{\lambda_2}$ are used the eliminate the auxiliary momenta.

Gauge fixing conditions can be introduced in a similar way as explained in Sec. 4.2.1 in the context of projectable HL gravity (Sec. 4.2.1). Canonical quantization can be performed as discussed in Sec. 2.3.

Here we considered only the power-counting renormalizable theory corresponding to HL gravity with critical exponent $z = 3$. The super-renormalizable model corresponding to $z = 4$ was also studied in paper **V** [19]. It has a quite similar Hamiltonian structure as the theory considered above. In particular, there exists one auxiliary field less in the action and correspondingly there are two second-class constraints less in the Hamiltonian formulation of the theory. Most importantly the number and nature of physical degrees of freedom matches the theory reviewed above. Theories corresponding to even higher z can be treated in a similar fashion.

However, the original CRG models appear to be insufficient for renormalizability [18].

6.2 New covariant renormalizable gravity

The new version of covariant renormalizable gravity [18] is considered in this section. For definiteness we shall consider the specific model corresponding to critical exponent $z = 3$, which should be power-counting renormalizable in four-dimensional spacetime. The action reads

$$\begin{aligned} S_{\mathrm{CRG}} = \int_{\mathcal{M}} \mathrm{d}^4x\sqrt{-g}\,\Big[&\frac{R}{2\kappa} - \alpha P_\alpha{}^\mu P_\beta{}^\nu \left(R_{\mu\nu} - \frac{1}{2U_0}\partial_\rho\phi\nabla^\rho\nabla_\mu\nabla_\nu\phi \right) \\ &\times \left(\partial^\mu\phi\partial^\nu\phi\nabla_\mu\nabla_\nu - \partial_\mu\phi\partial^\mu\phi\nabla^\nu\nabla_\nu\right) P^{\alpha\mu} P^{\beta\nu} \left(R_{\mu\nu} - \frac{1}{2U_0}\partial_\rho\phi\nabla^\rho\nabla_\mu\nabla_\nu\phi \right) \\ &- \lambda\left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + U_0\right)\Big], \end{aligned} \tag{6.25}$$

where the projector is defined by

$$P_\alpha{}^\mu = \delta_\alpha^\mu + \frac{\partial_\alpha\phi\partial^\mu\phi}{2U_0}\,. \tag{6.26}$$

Variation of the action (6.25) with respect to the Lagrange multiplier λ implies the constraint (6.4) on the scalar field ϕ. The major difference compared to the first CRG action (6.6) is that the scalar quantity $\partial^\mu\phi\partial^\nu\phi G_{\mu\nu}$ has been replaced with a projected tensor

$$P_\alpha{}^\mu P_\beta{}^\nu \left(R_{\mu\nu} - \frac{1}{2U_0}\partial_\rho\phi\nabla^\rho\nabla_\mu\nabla_\nu\phi \right). \tag{6.27}$$

The derivative operator acting on this tensor is the same one as in the original CRG action. We again consider that the potential U_0 is constant for simplicity.

The argument for power-counting renormalizability is made similarly as in the previous model. Once the constraint (6.4) is solved and gauge is fixed, the linearized Lagrangian contains only spatial derivatives of the perturbation $\tilde{g}_{\mu\nu}$ of the metric on Minkowski background. Then in the Lagrangian one obtains the projected tensor and the derivative operator in terms of only spatial derivatives,

$$P_i{}^\mu P_j{}^\nu \left(R_{\mu\nu} - \frac{1}{2U_0} \partial_\rho \phi \nabla^\rho \nabla_\mu \nabla_\nu \phi \right) = \frac{1}{2} \left(\partial^k \partial_i \tilde{g}_{jk} + \partial^k \partial_j \tilde{g}_{ik} - \partial^k \partial_k \tilde{g}_{ij} - \partial_i \partial_j (h_\mu{}^\mu) \right) , \tag{6.28}$$

$$\partial^\mu \phi \partial^\nu \phi \nabla_\mu \nabla_\nu - \partial_\mu \phi \partial^\mu \phi \nabla^\nu \nabla_\nu = 2U_0 \partial^k \partial_k \, . \tag{6.29}$$

Including a suitable number of these factors into the action should render the theory power-counting renormalizable.

At long distances the behaviour of the action (6.25) is supposed to be dominated by the EH part of the action, producing physics consistent with GR. At short distances and high energies the action of CRG is dominated by the higher-derivative terms with the coupling α, which enable power-counting renormalizability. However, the action (6.25) is of uncommon type and it is not at all clear what kind of physical degrees of freedom it contains. In general, higher-order time derivatives increase the number of degrees of freedom, while the constraint on the scalar field and the nonminimal couplings between the fields complicate things considerably. The presence of higher-order time derivatives of both the metric and the scalar field may imply the existence of an unstable ghost, because every extra time derivative of a variable translates into an extra physical degree of freedom, provided the number of constraints in the system is unaltered, and such a higher-order degree of freedom carries an energy with opposite sign compared to the corresponding lower-order degree of freedom. Therefore a close inspection of the action is required in order to see whether it defines a healthy theory in the first place.

6.2.1 First-order ADM representation of the action

We obtain the ADM representation of the action (6.25) similarly to the first case of CRG. We shall directly employ the foliation of spacetime adapted to ϕ for simplification (see around (6.10)). The scalar projector (6.26) becomes the orthogonal projector (3.2) onto the spatial hypersurface Σ_t. First we obtain the projected tensor (6.27) in the action as

$$h^\mu{}_\alpha h^\nu{}_\beta \left(R_{\mu\nu} - \nabla_n K_{\mu\nu} + a_\mu a_\nu \right) = {}^{(3)}R_{\alpha\beta} + K K_{\alpha\beta} - D_\alpha a_\beta \, . \tag{6.30}$$

The vector a_μ vanishes for the chosen foliation. Decomposing the second-order covariant derivative $(\nabla_\mu \nabla_\nu)$ of the tensor (6.30) with respect to the foliation of spacetime is no small task. In the end, for any tensor $A_{\alpha\beta}$ tangent to Σ_t, we obtain a simple result for the scalar combination in the action

$$A^{\alpha\beta} \left(n^\mu n^\nu \nabla_\mu \nabla_\nu + \nabla^\mu \nabla_\mu \right) A_{\alpha\beta} = A^{ij} \left(D^k D_k A_{ij} - K \frac{1}{N} \mathcal{L}_{Nn} A_{ij} + 2 K K_i{}^k A_{kj} + 2 K_{ik} K^{kl} A_{lj} \right) . \tag{6.31}$$

In the action, $A_{\alpha\beta}$ is given by (6.30). Since the action still contains second-order time derivatives, we introduce an extra field related to the metric variables as $\zeta_{ij} = {}^{(3)}R_{ij} + KK_{ij}$.[2] Then the action takes the desired form (2.4) as

$$S_{\mathrm{CRG}} = \int_{t_1}^{t_2} \mathrm{d}t \int_{\Sigma_t} N\sqrt{h} \left[\frac{K_{ij}K^{ij} - K^2 + {}^{(3)}R}{2\kappa} - 2\alpha U_0 h^{ik} h^{jl} \zeta_{kl} \right.$$
$$\times \left(D^k D_k \zeta_{ij} - K \frac{1}{N} \mathcal{L}_{Nn} \zeta_{ij} + 2KK_i{}^k \zeta_{kj} + 2K_{ik}K^{kl}\zeta_{lj} \right)$$
$$\left. - \alpha\lambda^{ij} \left(\zeta_{ij} - {}^{(3)}R_{ij} - KK_{ij} \right) \right], \quad (6.32)$$

When $\zeta_{ij} = 0$, the action reduces to an ultralocal special case of HL gravity. In order to find more general dynamics we assume that at least two components of ζ_{ij} are nonzero, while the rest of the components can attain any values. In principle, it makes no difference which of the components are chosen to be nonzero, but we choose one of them to be the trace of ζ_{ij} due to notational elegance.

6.2.2 Hamiltonian formalism

The canonical momenta conjugate to N, N^i, h_{ij}, ζ_{ij} and λ^{ij} are denoted by p_N, p_i, p^{ij}, p_ζ^{ij} and p_{ij}^λ, respectively. Since the action (6.32) is independent of the time derivatives of N, N^i and λ^{ij}, we have primary constraints:

$$p_N \approx 0\,, \qquad p_i(\boldsymbol{x}) \approx 0\,, \qquad p_{ij}^\lambda(\boldsymbol{x}) \approx 0\,. \quad (6.33)$$

The momenta canonically conjugate to h_{ij} and ζ_{ij} are defined by

$$p^{ij} = \sqrt{h} \left[\frac{K^{ij} - h^{ij}K}{2\kappa} + \frac{\alpha U_0}{N} h^{ij} \left(\dot{\zeta}_{kl} - \mathcal{L}_{\vec{N}} \zeta_{kl} \right) h^{km} h^{ln} \zeta_{mn} \right. \quad (6.34)$$
$$- 2\alpha U_0 \left(h^{ij}K^{kl} + h^{ik}K^{jl} + h^{jk}K^{il} + h^{ik}h^{jl}K \right) \zeta_{km}\zeta_{ln} h^{mn}$$
$$\left. + \frac{\alpha}{2} \left(h^{ij}\lambda^{kl}K_{kl} + \lambda^{ij}K \right) \right],$$
$$p_\zeta^{ij} = 2\alpha U_0 \sqrt{h} K h^{ik} h^{jl} \zeta_{kl}\,. \quad (6.35)$$

We shall adopt a convention where the trace component of a tensor or a tensor density is denoted without indices and the traceless component is denoted with the bar accent. For instance according to this convention we decompose the variables ζ_{ij} and p_ζ^{ij} as

$$\zeta_{ij} = \bar{\zeta}_{ij} + \frac{1}{3} h_{ij}\zeta\,, \qquad p_\zeta^{ij} = \bar{p}_\zeta^{ij} + \frac{1}{3} h_{ij} p_\zeta\,. \quad (6.36)$$

[2] There is a lot of freedom in choosing the variables associated with the higher-order time derivatives. Other possible choices are for example: a scalar variable $\zeta = ({}^{(3)}R^{lj} + KK^{lj})({}^{(3)}R_{lj} + KK_{lj})$ or the extrinsic curvature $\zeta_{ij} = K_{ij}$ (see paper **VI** [20] for details).

The traceless and trace components of the momentum p_ζ^{ij} are involved in further primary constraints, which are defined by

$$\bar{\Pi}^{ij} = \bar{p}_\zeta^{ij} - h^{ik}h^{jl}\bar{\zeta}_{kl}\frac{p_\zeta}{\zeta} \approx 0\,. \tag{6.37}$$

The extrinsic curvature K_{ij} can now be solved in terms of the canonical variables. Its trace is given by the trace of (6.35) as

$$K = \frac{p_\zeta}{2\alpha U_0\sqrt{h}\zeta}\,. \tag{6.38}$$

The traceless components $\bar{K}_{ij}$ can be obtained by solving

$$\bar{P}^{ij} = \frac{\sqrt{h}}{2\kappa}F^{ijkl}\bar{K}_{kl}\,, \tag{6.39}$$

where we have defined

$$\bar{P}^{ij} = \bar{p}^{ij} + \frac{5}{3}\left(\zeta^i{}_k\zeta^{jk} - \frac{1}{3}h^{ij}\zeta_{kl}\zeta^{kl}\right)\frac{p_\zeta}{\zeta} - \frac{\bar{\lambda}^{ij}}{4U_0}\frac{p_\zeta}{\zeta} \tag{6.40}$$

and

$$F^{ijkl} = \frac{1}{2}\left(h^{ik}h^{jl} + h^{il}h^{jk}\right) - 8\kappa\alpha U_0\left(h^{m(i}h^{j)(k}h^{l)n} - \frac{1}{3}h^{ij}h^{km}h^{ln}\right)\zeta_{mo}\zeta_n{}^o\,. \tag{6.41}$$

The inverse F^{-1}_{ijkl} to F^{ijkl} can be constructed as a power series. The series appears to be infinite. It does not converge unconditionally. However, for any given spacetime, there exists a range of values for the coupling α such that the series expression of F^{-1}_{ijkl} is absolutely convergent. Here we simply assume that F^{-1}_{ijkl} exists. Then no more primary constraints are needed

The total Hamiltonian is obtained as

$$H = \int_{\Sigma_t}\left(N\mathcal{H}_0 + N^i\mathcal{H}_i + v_N p_N + v^i p_i + v_\lambda^{ij}p^\lambda_{ij} + \bar{v}_{ij}\bar{\Pi}^{ij}\right), \tag{6.42}$$

where v_N, v^i, v_λ^{ij} and $\bar{v}_{ij}$ are Lagrange multipliers and we have defined

$$\begin{aligned}
\mathcal{H}_0 = \frac{1}{\sqrt{h}}\Bigg[&4\kappa p^{ij}F^{-1}_{ijkl}\bar{P}^{kl} + \frac{p}{3\alpha U_0}\frac{p_\zeta}{\zeta} - 2\kappa F^{-1}_{ijkl}\bar{P}^{kl}h^{im}h^{jn}F^{-1}_{mnop}\bar{P}^{op} \\
&+ \frac{1}{12\kappa\alpha^2U_0^2}\left(\frac{p_\zeta}{\zeta}\right)^2 + \frac{4}{9\alpha U_0}\zeta_{ij}\zeta^{ij}\left(\frac{p_\zeta}{\zeta}\right)^2 \\
&+ \frac{20\kappa}{3}\zeta^{ij}\zeta^k{}_jF^{-1}_{iklm}\bar{P}^{lm}\frac{p_\zeta}{\zeta} + 16\kappa^2\alpha U_0\zeta^{ij}F^{-1}_{iklm}\bar{P}^{lm}\zeta^n{}_jh^{ko}F^{-1}_{nopq}\bar{P}^{pq} \\
&- \frac{\kappa}{U_0}\lambda^{ij}F^{-1}_{ijkl}\bar{P}^{kl}\frac{p_\zeta}{\zeta} - \frac{\lambda}{12\alpha U_0^2}\left(\frac{p_\zeta}{\zeta}\right)^2\Bigg] \\
&- \sqrt{h}\left[\frac{^{(3)}R}{2\kappa} - 2\alpha U_0\zeta^{ij}D^kD_k\zeta_{ij} - \alpha\lambda^{ij}\left(\zeta_{ij} - {}^{(3)}R_{ij}\right)\right],
\end{aligned} \tag{6.43}$$

$$\mathcal{H}_i = -2h_{ij}D_kp^{jk} - 2\zeta_{ij}\partial_kp_\zeta^{jk} - \left(2\partial_j\zeta_{ik} - \partial_i\zeta_{jk}\right)p_\zeta^{jk}\,. \tag{6.44}$$

The Hamiltonian depends on the traceless components $\bar{p}_\zeta^{ij}$ of the momentum canonically conjugate to ζ_{ij} only through the primary constraint (6.37). That means $\bar{p}_\zeta^{ij}$ does not appear in any other constraint.

Then we introduce the secondary constraints

$$\Phi_0 = \int_{\Sigma_t} \mathcal{H}_0 \approx 0\,, \qquad \mathcal{H}_i(\boldsymbol{x}) \approx 0\,, \tag{6.45}$$

$$\Psi_{ij}(\boldsymbol{x}) = \left\{p_{ij}^\lambda(\boldsymbol{x}), \Phi_0\right\} \approx 0\,, \qquad \bar{\Pi}_{II}^{ij}(\boldsymbol{x}) = \left\{\bar{\Pi}^{ij}(\boldsymbol{x}), \Phi_0\right\} \approx 0\,, \tag{6.46}$$

which ensure that the primary constraints p_N, p_i, p_{ij}^λ and $\bar{\Pi}^{ij}$ are preserved in time, respectively. The smeared version (3.36) of the momentum constraint $\mathcal{H}_i$ is the generator of spatial diffeomorphisms for the variables $h^{ij}, p^{ij}, \zeta_{ij}, p_\zeta^{ij}$. It can be extended to a generator of spatial diffeomorphisms for all variables with the primary constraints (6.33). The global Hamiltonian constraint Φ_0 and the smeared momentum constraint $\Phi\big[\vec{X}\big]$ again satisfy similar Poisson brackets as in projectable HL theories, namely (4.40). The explicit forms of the constraints Ψ_{ij} and $\bar{\Pi}_{II}^{ij}$ are quite complicated, hence we shall not present them here. Preservation of the secondary constraints in time does not imply further constraints. The Lagrange multipliers v_λ^{ij} and $\bar{v}_{ij}$ are solved in order to ensure the consistency of the constraints Ψ_{ij} and $\bar{\Pi}_{II}^{ij}$ in time.

Let us seek to identify the physical degrees of freedom. There exists four propagating physical degrees of freedom, similarly to both the first version of CRG and the modified $F(R)$ HL gravity. Now we have 42 space-dependent canonical variables (N^i, h_{ij}, ζ_{ij}, λ^{ij} and their conjugated momenta), six local first-class constraints (p_i, Φ_i) and 22 local second-class constraints (p_{ij}^λ, $\bar{\Pi}^{ij}$, Ψ_{ij}, $\bar{\Pi}_{II}^{ij}$). Let us try to further understand the nature of the extra propagating modes. The constraints of the theory enable us to regard some of the canonical variables as being dependent on other variables. The Dirac bracket can be defined in the standard way (2.18). Then we can set the local second-class constraints p_{ij}^λ, $\bar{\Pi}^{ij}$, Ψ_{ij} and $\bar{\Pi}_{II}^{ij}$ to zero. The following variables are regarded as dependent variables: (i) the momenta p_{ij}^λ can be set to zero, (ii) the traceless component of λ^{ij} can be solved from $\bar{\Psi}_{ij} = 0$, (iii) $\Psi = 0$ can be regarded to fix the conformal factor of the metric h_{ij}, and (iv) $\bar{\Pi}^{ij} = 0$ and $\bar{\Pi}_{II}^{ij} = 0$ can be regarded to define the traceless variables $\bar{\zeta}_{ij}$ and $\bar{p}_\zeta^{ij}$ in terms of independent variables. The remaining independent canonical variables are eleven variables in h_{ij}, p^{ij}, the trace component λ, and the trace components ζ and p^ζ. We could not remove λ from the set of independent variables even though the variables λ^{ij} had an auxiliary role in the action (6.32), albeit it is possible to write the Hamiltonian without λ due to the constraints (see below).

The fact that a given higher derivative theory of gravity possesses extra degrees of freedom is always alarming, because such modes are often ghosts or otherwise pathological. We can see that the kinetic part of the Hamiltonian contains terms involving the momentum p_ζ which can attain arbitrarily negative values on the constraint surface. This is not quite as evident as it would seem at first sight because of the complexity of the constraints. Preferably, we would have liked to obtain a diagonalized form of the Hamiltonian constraint, and possibly solve some of the constraints as well. Unfortunately, that turned out to be next to impossible due to the complicated forms of $\mathcal{H}_0$ and the constraints. However, a slightly better view of the Hamiltonian is achieved by taking out a linear combination of the constraints Ψ_{ij} from $\mathcal{H}_0$ in the global Hamiltonian constraint. We obtain the sum

$$\begin{aligned}\mathcal{H}_0 + \lambda^{ij}\Psi_{ij} = \frac{1}{\sqrt{h}}\Bigg[&4\kappa p^{ij} F^{-1}_{ijkl}\bar{\mathcal{P}}^{kl} + \frac{p}{3\alpha U_0}\frac{p_\zeta}{\zeta} - 2\kappa F^{-1}_{ijkl}\bar{\mathcal{P}}^{kl} h^{im} h^{jn} F^{-1}_{mnop}\bar{\mathcal{P}}^{op} \\ &+ \frac{\kappa}{8U_0^2} F^{-1}_{ijkl}\bar{\lambda}^{kl} h^{im} h^{jn} F^{-1}_{mnop}\bar{\lambda}^{op}\left(\frac{p_\zeta}{\zeta}\right)^2 + \frac{1}{12\kappa\alpha^2 U_0^2}\left(\frac{p_\zeta}{\zeta}\right)^2 \\ &+ \frac{4}{9\alpha U_0}\zeta_{ij}\zeta^{ij}\left(\frac{p_\zeta}{\zeta}\right)^2 + \frac{20\kappa}{3}\zeta^{ij}\zeta^k{}_j F^{-1}_{iklm}\bar{\mathcal{P}}^{lm}\frac{p_\zeta}{\zeta} \\ &+ 16\kappa^2\alpha U_0 \zeta^{ij} F^{-1}_{iklm}\bar{\mathcal{P}}^{lm}\zeta^n{}_j h^{ko} F^{-1}_{nopq}\bar{\mathcal{P}}^{pq} \\ &- \frac{\kappa^2\alpha}{U_0} F^{-1}_{iklm}\bar{\lambda}^{lm}\zeta^n{}_j h^{ko} F^{-1}_{nopq}\bar{\lambda}^{pq}\left(\frac{p_\zeta}{\zeta}\right)^2 \\ &- \frac{\kappa}{U_0}\bar{\lambda}^{ij} F^{-1}_{ijkl}\bar{\lambda}^{kl}\left(\frac{p_\zeta}{\zeta}\right)^2\Bigg] - \sqrt{g}\left(\frac{R}{2\kappa} - 2\alpha U_0 \zeta^{ij} D^k D_k \zeta_{ij}\right), \end{aligned} \tag{6.47}$$

where we denote

$$\bar{\mathcal{P}}^{kl} = \bar{p}^{kl} + \frac{5}{3}\left(\zeta^k{}_m \zeta^{lm} - \frac{1}{3}h^{kl}\zeta_{mn}\zeta^{mn}\right)\frac{p_\zeta}{\zeta}. \tag{6.48}$$

One can also demonstrate certain properties of the kinetic part of the Lagrangian with a simple toy model (see paper **VI** [20].) These considerations suggests that there exists a degree of freedom that carries negative energy. As is usual in higher time derivative theories, energy of the higher-order degree of freedom has opposite sign compared to the corresponding lower order degree of freedom. We emphasize that the physical Hamiltonian is a constraint that vanishes everywhere on the constraint surface.[3] That means the kinetic and potential terms must cancel each other at all times. When interacting positive and negative energy degrees of freedom are present, any state (including “empty space”) can decay into compensating positive and negative energy excitations. This makes the theory unstable. The only way this could be avoided are the constraints. Unfortunately, the Hamiltonian constraint is global and therefore it does not constrain local physics. We believe the momentum constraint does not protect the stability either, since the extra degree of freedom cannot be removed by a spatial diffeomorphism. Thus this theory cannot be considered to be a realistic description of gravity, albeit it might possess favourable renormalization characteristics. In this respect it is similar to the generally covariant higher derivative gravity [9, 11, 10], whose action is of the type (1.3). However, the discovered

[3]Generally the physical Hamiltonian is interpreted as total energy, which is given by the surface terms in a gravitational theory (see Sec. 3.3). We do not study the total gravitational energy in CRG, but rather concentrate on the propagating degrees of freedom.

Hamiltonian structure of new CRG is more complicated than that of generally covariant (renormalizable) higher derivative gravity [76, 137, 138], which will be discussed in Chapter 7.

We expect the CRG actions corresponding to higher values of the critical exponent $z > 3$ to exhibit a similar ghost problem as the case $z = 3$. The number of time derivatives present in the ADM representation of the action grows with z. The cases $z = 4, 5, 6$ most likely to reproduce the same problem but in an even more complicated form than in the case $z = 3$. CRG actions corresponding to sufficiently high z will necessarily be unstable, once the order of time derivatives is greater than the number of available constraints.

In more general perspective, we conjectured that generally covariant higher derivative theories of gravity which aim to achieve power-counting renormalizability via spontaneous (constraint induced) Lorentz and/or diffeomorphism symmetry breaking will in general have to face a similar challenge with ghosts as CRG. There are two ways to avoid the problem. Either the higher time derivatives totally disappear (cancel out) from the ADM representation of the given action or the constraints available after symmetry breaking conspire to protect the stability of the higher-order degrees of freedom.

Recently we have begin to suspect that in this type of theory it might be better to treat the normal n^μ as a genuine dynamical variable with constraints restricting it to unit norm and zero vorticity. This way a more a general treatment might be achieved. In covariant form the action would be defined as

$$\begin{aligned} S_{\mathrm{CRG}} = \int_{\mathcal{M}} \mathrm{d}^4x\sqrt{-g}\,\Bigg[\frac{R}{2\kappa} &- \alpha\left({}^{(3)}R^{\alpha\beta} + KK^{\alpha\beta} - D^\alpha a^\beta\right) \\ &\times \left(n^\mu n^\nu \nabla_\mu \nabla_\nu + \nabla^\mu \nabla_\mu\right)\left({}^{(3)}R_{\alpha\beta} + KK_{\alpha\beta} - D_\alpha a_\beta\right) \\ &+ \lambda\left(n_\mu n^\mu + 1\right) + B^{\mu\nu}\mathcal{F}_{\mu\nu} + M_{\mu\nu\rho\sigma}B^{\mu\nu}B^{\rho\sigma}\Bigg], \end{aligned} \tag{6.49}$$

where the vorticity for n_μ is

$$\mathcal{F}_{\mu\nu} = h^\rho{}_\mu h^\sigma{}_\nu \nabla_{[\rho} n_{\sigma]} \tag{6.50}$$

and λ, $B^{\mu\nu}$ and $M_{\mu\nu\rho\sigma}$ are Lagrange multiplier fields. Variations of the action with respect to the Lagrange multipliers yields

$$n_\mu n^\mu = -1\,, \qquad B^{\mu\nu} = 0\,, \qquad \mathcal{F}_{\mu\nu} = 0\,. \tag{6.51}$$

The normal would be associated with the scalar field ϕ by

$$n_\mu = -N\nabla_\mu\phi \tag{6.52}$$

and choosing it to be the time $\phi = t$ (ADM gauge choice) yields the ADM formulation of the theory. This approach would be similar to the covariant formulation of HL gravity [139] (also see [117, 118]), but with a much more complicated action including higher-order time derivatives and several extra kinetic terms. The only advantage of CRG over HL gravity is the spontaneous breaking of Lorentz invariance in the high energy regime. The idea of achieving renormalizable gravity via spontaneous symmetry breaking is certainly appealing. But it is hardly worth the price of accepting such a complicated Hamiltonian structure, let alone an unstable extra degree of freedom.

Chapter 7

Hamiltonian analysis of generally covariant higher-derivative gravity

Hamiltonian formulation and canonical quantization of generally covariant gravitational theories whose actions (1.3) contain quadratic curvature terms were originally studied in [76, 137, 138]. These classic works have been followed and elaborated ever since (see e.g. [78]). Based on a work in progress, we present some results of our Hamiltonian analysis of the potentially renormalizable cases of curvature-squared gravity, including the highly interesting case of conformally invariant Weyl gravity.

There are a few reasons to revisit this subject. The analysis [76] includes all the cases of curvature-squared gravity, but the chosen action produces an unnecessarily complicated structure of constraints.[1] The analysis [137] is considerably simpler due to a better choice of the action. However, it defines the canonical variables associated with the second-order time derivatives in the Lagrangian in a way that interchanges the role of fields and momenta quite confusingly. We have discovered that the projection of Weyl tensor onto the hypersurfaces of the foliation of spacetime contains a fully traceless component, which was missed in [137]. The first Hamiltonian analysis [138] considered the case of conformally invariant Weyl gravity in the strong-coupling approximation.

In addition, there are some recent developments that require scrutiny. Critical points of the gravitational action with curvature squared terms were recently studied in [140], where it is claimed that the massive spin-2 mode can become massless for a certain choice of coupling constants: $\beta = -\alpha/3 = (4\kappa^2\Lambda)^{-1}$ in (1.3). The relation of such critical gravity to conformal gravity in four-dimensional spacetime has been studied in [141]. It has been proposed that one can

[1] In addition, the standing of cosmological constant when added into conformally invariant Weyl gravity has not been understood properly. However, we shall not consider this problem here, since our analysis is still incomplete.

obtain solutions of four-dimensional Einstein gravity with cosmological constant by introducing a simple Neumann boundary condition into conformally invariant Weyl gravity [142]. In a somewhat similar fashion it has been argued that one can obtain ghost-free four-dimensional massive gravity by introducing Dirichlet boundary conditions into curvature-squared gravity on an asymptotically de Sitter spacetime [143]. If these claims are true generally, we should discover a corresponding change in the Hamiltonian structure of the given theory.

7.1 The action and its ADM representation

We consider a gravitational action of the form [137]

$$S_{C^2} = \int_{\mathcal{M}} \mathrm{d}^4x\sqrt{-g}\left[\Lambda + \frac{R}{2\kappa} - \frac{\alpha}{4}C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + \frac{\beta}{8}R^2 + \gamma G\right], \tag{7.1}$$

where κ, α, β and γ are coupling constants. $C_{\mu\nu\rho\sigma}$ is the Weyl tensor

$$\begin{aligned} C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} &- \frac{2}{(D-2)}\left(g_{\mu[\rho}R_{\sigma]\nu} - g_{\nu[\rho}R_{\sigma]\mu}\right) \\ &+ \frac{2}{(D-1)(D-2)}g_{\mu[\rho}g_{\sigma]\nu}R\,, \end{aligned} \tag{7.2}$$

where D is the dimension of spacetime ($D = 4$). The Weyl tensor is by definition the traceless part of the Riemann tensor. In the last part of the action (7.1), G is the Gauss-Bonnet-Chern curvature term,

$$G = R_{\alpha\beta\gamma\delta}R_{\mu\nu\rho\sigma}\varepsilon^{\alpha\beta\mu\nu}\varepsilon^{\gamma\delta\rho\sigma} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2\,. \tag{7.3}$$

In four-dimensional spacetime, its integral over spacetime becomes a topological invariant which is proportional to the Euler characteristic of the spacetime manifold. Since we consider smooth variations of spacetime, which do not change its topology, the Gauss-Bonnet-Chern part of the action can be regarded as a constant, and hence we can drop it. Note that a nonlinear function of G in a Lagrangian of type $R + f(G)$ would imply an extra scalar degree of freedom, similarly as the total derivative in the scalar curvature (3.19) when a nonlinear Lagrangian $f(R)$ is considered.

In order to have a variational principle that is consistent with the desired equations of motion, no boundary surface terms are required in the action (7.1), unlike in GR (3.20). This is because now the minimal boundary conditions impose the variations of both the metric and extrinsic curvature to zero. However, whenever the EH part of the action is included ($\kappa^{-1} \neq 0$), we choose to include the boundary term $\frac{1}{\kappa}\oint_{\partial\mathcal{M}}\mathrm{d}^3x\sqrt{|\gamma|}K$, so that the same total surface contribution is obtained as in GR, namely (3.23). We again assume orthogonality of the Cauchy surfaces Σ_t and the timelike boundary $\mathcal{B}$ for simplicity. In case the hypersurfaces Σ_t and $\mathcal{B}$ would intersect nonorthogonally, we could include a surface term according to (3.24).

Let us then obtain the ADM representation of the action by decomposing it with respect to a foliation of spacetime. The projection relations for the Weyl tensor (7.2) can be written in terms of the ADM variables as

$$
\begin{aligned}
{}_\perp C_{\boldsymbol{injn}} &= -\frac{1}{2}\left(\delta^k_i\delta^l_j - \frac{1}{3}h_{ij}h^{kl}\right)\left(\mathcal{L}_n K_{kl} - {}^{(3)}R_{kl} - K_{kl}K - \frac{1}{N}D_kD_lN\right), \\
{}_\perp C_{ijk\boldsymbol{n}} &= 2D_{[i}K_{j]k} + D_lK^l{}_{[i}h_{j]k} - D_{[i}Kh_{j]k}\,, \\
{}_\perp C_{ijkl} &= \mathcal{K}_{ijkl} + h_{ik\,\perp}C_{\boldsymbol{j}\boldsymbol{n}\boldsymbol{l}\boldsymbol{n}} - h_{il\,\perp}C_{\boldsymbol{jnkn}} - h_{jk\,\perp}C_{\boldsymbol{inln}} + h_{jl\,\perp}C_{\boldsymbol{inkn}}\,,
\end{aligned}
\tag{7.4}
$$

where the traceless quadratic extrinsic curvature tensor $\mathcal{K}_{ijkl}$ is defined as

$$
\begin{aligned}
\mathcal{K}_{ijkl} = {} & K_{ik}K_{jl} - K_{il}K_{jk} - h_{ik}\left(K_{jl}K - K_{jm}K^m{}_l\right) \\
& + h_{il}\left(K_{jk}K - K_{jm}K^m{}_k\right) + h_{jk}\left(K_{il}K - K_{im}K^m{}_l\right) \\
& - h_{jl}\left(K_{ik}K - K_{im}K^m{}_k\right) \\
& + \frac{1}{2}\left(h_{ik}h_{jl} - h_{il}h_{jk}\right)\left(K^2 - K_{mn}K^{mn}\right).
\end{aligned}
\tag{7.5}
$$

Our result (7.4) for the component of Weyl tensor that is fully tangent to the spatial hypersurface differs from the one given in [137] by the presence of $\mathcal{K}_{ijkl}$. In [137], it was obtained that the traceless part of ${}_\perp C_{ijkl}$ is solely the vanishing three-dimensional Weyl tensor, but in fact there exists a nonvanishing traceless part, namely (7.5). However, this correction has no effect on the Hamiltonian formulation of the theory (7.1), but it can affect more exotic theories that couple Weyl tensor to something else than itself. Weyl tensor squared is obtained as

$$
C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} = 8{}_\perp C_{\boldsymbol{injn}\,\perp}C^{i}{}_{\boldsymbol{n}}{}^{j}{}_{\boldsymbol{n}} - 4{}_\perp C_{ijk\boldsymbol{n}\,\perp}C^{ijk}{}_{\boldsymbol{n}}\,. \tag{7.6}
$$

In obtaining the result (7.6), we used the fact that $\mathcal{K}_{ijkl}$ is traceless and its square is zero,

$$
\mathcal{K}_{ijkl}\mathcal{K}^{ijkl} = 2\mathcal{K}_{ikjl}K^{ij}K^{kl} = -6P(K)^i{}_jK^j{}_i = 0\,, \tag{7.7}
$$

where the characteristic polynomial $P(\lambda)$ for the tensor $K^i{}_j$ with the tensor itself as the argument is identically zero due to the tensor form of the Cayley-Hamilton theorem, $P(K)^i{}_j = 0$.

Then the action (7.1) can be written in the first-order form (2.4) as

$$
\begin{aligned}
S_{C^2} = \int_{t_1}^{t_2}\mathrm{d}t\int_{\Sigma_t} N\Bigg\{\sqrt{h}\Bigg[\Lambda + \frac{1}{2\kappa}\left({}^{(3)}R + K_{ij}K^{ij} - K^2\right) \\
- 2\alpha{}_\perp C_{\boldsymbol{injn}\,\perp}C^{i}{}_{\boldsymbol{n}}{}^{j}{}_{\boldsymbol{n}} + \alpha{}_\perp C_{ijk\boldsymbol{n}\,\perp}C^{ijk}{}_{\boldsymbol{n}} + \frac{\beta}{8}R^2\Bigg] \\
+ \lambda^{ij}\left(\mathcal{L}_n h_{ij} - 2K_{ij}\right)\Bigg\} + S_{\text{surface}}\,,
\end{aligned}
\tag{7.8}
$$

where the independent variables are N, N^i, h_{ij}, K_{ij} and λ^{ij}. The surface contribution S_{surface} is the same as in GR (3.23) if EH action is included, and otherwise zero.

7.2 Hamiltonian formulation

Canonical momenta conjugate to N, N^i, h_{ij}, K_{ij} and λ^{ij} shall be denoted by p_N, p_i, p^{ij}, $\mathcal{P}^{ij}$ and p^{λ}_{ij}, respectively. The primary constraints are

$$p_N \approx 0\,, \qquad p_i \approx 0\,, \qquad p^{\lambda}_{ij} \approx 0\,, \qquad \Pi^{ij} = p^{ij} - \lambda^{ij} \approx 0\,. \tag{7.9}$$

The auxiliary variables λ^{ij} and p^{λ}_{ij} can be eliminated by imposing the second-class constraints Π^{ij} and p^{λ}_{ij} to zero strongly as explained in Sec. 2.2.2, where the general formalism for constrained higher derivative theories was introduced. The momentum canonically conjugate to K_{ij} is defined as

$$\mathcal{P}^{ij} = \sqrt{h}\left(2\alpha{}_\perp C^{i}{}_{\boldsymbol{n}}{}^{j}{}_{\boldsymbol{n}} + \frac{\beta}{2}h^{ij}R\right). \tag{7.10}$$

The number and nature of constraints and physical degrees of freedom depends on the values of the coupling constants. Therefore we shall treat the different cases separately. We consider only cases with $\alpha \neq 0$, which are the only ones that can improve UV behavior. The cases with $\alpha = 0$ include only GR and the quadratic case of $f(R)$ gravity, with or without the cosmological constant, which are well known and understood. For a review of Hamiltonian formulations of $f(R)$ gravity, see [144]. First we shall consider the conformally invariant Weyl gravity, which will serve as the reference theory to which the other cases are compared to.

7.2.1 Weyl gravity: $\Lambda = 0$, $\kappa^{-1} = 0$, $\alpha \neq 0$, $\beta = 0$

We could also set the coupling $\alpha = 1$, but we choose not to, because keeping it will help in comparing to the other cases. There are no surface term in the action (7.8), $S_{\text{surface}} = 0$.

The momentum (7.10) consists only of the projection ${}_\perp C^{i}{}_{\boldsymbol{n}}{}^{j}{}_{\boldsymbol{n}}$ of the Weyl tensor. Since this projection is traceless, the trace of the momentum $\mathcal{P}^{ij}$ is a primary constraint,

$$\mathcal{P} \approx 0\,. \tag{7.11}$$

We again adopt notation where the trace of a quantity is denoted without indices and the traceless component is denoted by the bar accent.

We obtain the total Hamiltonian as

$$\begin{aligned} H = &\int_{\Sigma_t}\left(N\mathcal{H}_0 + N^i\mathcal{H}_i + v_N p_N + v^i p_i + v_{\mathcal{P}}\mathcal{P}\right) \\ &+ \oint_{\mathcal{B}_t} r_i\left(D_j N\mathcal{P}^{ij} - ND_j\mathcal{P}^{ij} + 2N_j p^{ij} + 2N^j\mathcal{P}^{ik}K_{jk}\right). \end{aligned} \tag{7.12}$$

where the v fields are arbitrary Lagrange multipliers. The Hamiltonian constraint is given as

$$\begin{aligned} \mathcal{H}_0 = 2p^{ij}K_{ij} - \frac{\mathcal{P}_{ij}\mathcal{P}^{ij}}{2\alpha\sqrt{h}} + \mathcal{P}^{ij\,(3)}R_{ij} + \mathcal{P}^{ij}K_{ij}K \\ + D_iD_j\mathcal{P}^{ij} - \alpha\sqrt{h}{}_\perp C_{ijk\boldsymbol{n}}{}_\perp C^{ijk}{}_{\boldsymbol{n}}\,, \end{aligned} \tag{7.13}$$

where $\mathcal{P}_{ij} = h_{ik}h_{jl}\mathcal{P}^{kl}$. The momentum constraint is defined as

$$\mathcal{H}_i = -2h_{ij}D_k p^{jk} + \mathcal{P}^{jk}D_i K_{jk} - 2D_j\left(\mathcal{P}^{jk}K_{ik}\right) \tag{7.14}$$

or more symmetrically in terms of partial derivatives as

$$\begin{aligned}\mathcal{H}_i = &-2h_{ij}\partial_k p^{jk} - \left(2\partial_j h_{ik} - \partial_i h_{jk}\right)p^{jk} \\ &- 2K_{ij}\partial_k\mathcal{P}^{jk} - \left(2\partial_j K_{ik} - \partial_i K_{jk}\right)\mathcal{P}^{jk}\,.\end{aligned} \tag{7.15}$$

The smeared momentum constraint (3.36) generates infinitesimal spacetime-dependent spatial diffeomorphism for the dynamical variables $(h_{ij}, p^{ij}, K_{ij}, \mathcal{P}^{ij})$ on the hypersurface Σ_t. We can again extend the momentum constraint to a generator of spatial diffeomorphism for all variables by absorbing certain terms into the Lagrange multipliers of the primary constraints p_N and p_i. The variables N, N^i, h_{ij} and K_{ij} behave as regular scalar or tensor fields under the spatial diffeomorphisms, while their canonically conjugated momenta behave as scalar or tensor densities of unit weight. The surface terms in the Hamiltonian (7.12) appear when we remove spatial derivatives of N and N^i via integration by parts. The first and last two surface terms appear due to integration by parts for terms involving N and N^i, respectively.

In all terms of the Hamiltonian (7.12), we could alternatively replace $\mathcal{P}^{ij}$ with its traceless components $\bar{\mathcal{P}}^{ij}$, because any term depending on a positive power of the primary constraint $\mathcal{P}$ can be absorbed into the Lagrange multiplier term $v_{\mathcal{P}}\mathcal{P}$. We choose to write all the constraints with $\mathcal{P}^{ij}$.

In order to ensure the preservation of the primary constraint in time, we impose the local secondary constraints:

$$\mathcal{H}_0 \approx 0\,, \qquad \mathcal{H}_i \approx 0\,, \tag{7.16}$$

and

$$\mathcal{Q} = 2p + \mathcal{P}^{ij}K_{ij} \approx 0\,. \tag{7.17}$$

We have only first-class constraints. The Hamiltonian is extended with $\int_{\Sigma_t} v_{\mathcal{Q}}\mathcal{Q}$. The Hamiltonian $\mathcal{H}_0$ and momentum $\mathcal{H}_i$ constraints are associated with the general covariance. They satisfy similar Poisson brackets as in GR except that the Poisson bracket between Hamiltonian constraints contains extra terms that are proportional to $\mathcal{P}$. The constraint $\mathcal{P}$ is associated with the absence of the massive scalar degree of freedom, thanks to $\beta = 0$. The constraint $\mathcal{Q}$ is associated with the conformal invariance of Weyl gravity. It indeed generates scale transformations for the canonical variables. There exists six physical degrees of freedom, counted using Dirac's formula (2.26). Two modes can be associated with the massless spin-2 graviton. Four modes are carried by the massive spin-2 ghost, which enables renormalizability.

For gauge fixing we need two more gauge conditions in addition to those (3.43) used in GR. We can choose to fix the traces of the metric and the extrinsic curvature to match those of flat space:

$$\chi_4 = \delta^{ij}h_{ij} - 3 = 0\,, \qquad K = 0\,. \tag{7.18}$$

The lapse and shift variables drop out as in GR. We are left with twelve constraint $(\mathcal{H}_0, \mathcal{H}_i, \mathcal{Q}, \chi_\mu, \mathcal{P}, K)$ among the 24 canonical variables $h_{ij}, p^{ij}, K_{ij}, \mathcal{P}^{ij}$.

Whether the introduction of boundary conditions can truly reduce the set of solutions of conformal gravity to that of GR or ghost-free massive gravity is under investigation.

7.2.2 General relativity plus Weyl gravity: $\kappa^{-1} \neq 0$, $\alpha \neq 0$, $\beta = 0$

This case includes both the action of Weyl gravity and EH action. Clearly the conformal symmetry of Weyl gravity is broken. The surface term

$$\frac{1}{\kappa}\oint_{\mathcal{B}_t} \mathrm{d}^2x N\sqrt{\sigma}\,{}^{(2)}K$$

is included into the total Hamiltonian (7.12) due to presence of the EH action.

The Hamiltonian constraint is given as

$$\begin{aligned}\mathcal{H}_0 = {} & 2p^{ij}K_{ij} - \frac{\mathcal{P}_{ij}\mathcal{P}^{ij}}{2\alpha\sqrt{h}} + \mathcal{P}^{ij\,(3)}R_{ij} + \mathcal{P}^{ij}K_{ij}K + D_iD_j\mathcal{P}^{ij} \\ & - \sqrt{h}\Lambda - \frac{\sqrt{h}}{2\kappa}\left({}^{(3)}R + K_{ij}K^{ij} - K^2\right) - \alpha\sqrt{h}{}_\perp C_{ijk\boldsymbol{n}\perp}C^{ijk}{}_{\boldsymbol{n}}\,.\end{aligned} \tag{7.19}$$

The secondary constraint $\mathcal{Q}$ now takes a different form

$$\mathcal{Q} = 2p + \mathcal{P}^{ij}K_{ij} + \frac{2}{\kappa}\sqrt{h}K \approx 0\,, \tag{7.20}$$

because of the presence of the EH part of the action. The Poisson bracket between $\mathcal{P}$ and $\mathcal{Q}$ no longer closes,

$$\{\mathcal{P}(\boldsymbol{x}), \mathcal{Q}(\boldsymbol{y})\} = \left(\mathcal{P} - \frac{6}{\kappa}\sqrt{h}\right)(\boldsymbol{y})\delta(\boldsymbol{x}-\boldsymbol{y})\,. \tag{7.21}$$

Thus $\mathcal{P}$ and $\mathcal{Q}$ are now second-class constraints. The first-class Hamiltonian constraint is the linear combination of $\mathcal{H}_0$ and $\mathcal{P}$:

$$\mathcal{H}_0 - w_{\mathcal{P}}\mathcal{P}\,, \qquad w_{\mathcal{P}} = \frac{2\kappa\Lambda}{3} + \frac{1}{2}\left({}^{(3)}R - K_{ij}K^{ij} + K^2\right). \tag{7.22}$$

We can impose the constraints $\mathcal{P}$ and $\mathcal{Q}$ to zero strongly by replacing the Poisson bracket with the Dirac bracket (2.18). Then the first-class Hamiltonian constraint is $\mathcal{H}_0$ alone. The total Hamiltonian takes the same form (7.23) as in the next case (Sec. 7.2.3), but with the simpler Hamiltonian constraint (7.19) due to $\mathcal{P} = 0$.

We can now gain insight on the generality of the critical gravity proposal [140]. In the full nonlinear theory, the value of Λ does not affect the structure of the constraints and Hamiltonian. Since there exist eight first-class constraints $(p_N, p_i, \mathcal{H}_0, \mathcal{H}_i)$ and two second-class constraints $(\mathcal{P}, \mathcal{Q})$, regardless of the presence or value of Λ, the number of local physical degrees of freedom is seven. Two modes are associated with the massless spin-2 graviton and five modes with the massive spin-2 field. This suggests that the possibility for the latter excitations to become massless is an artefact of the linearized formulation on anti-de Sitter spacetime. We already discovered a somewhat similar contrast in Chapter 6 between the linearized formulation of CRG on Minkowski spacetime and the Hamiltonian formulation of the full nonlinear theory.

7.2.3 Curvature-squared gravity without conformal invariance: $\alpha \neq 0$, $\beta \neq 0$

Then we consider the general gravitational theory without conformal invariance, whose Lagrangian is quadratic in curvature. Cosmological constant and EH action can be either included or excluded, since their presence does not affect the fundamental Hamiltonian structure of the theory, when both curvature-squared terms are included ($\alpha \neq 0, \beta \neq 0$). This is the realistic case when conformal symmetry is broken, because quantum fluctuations are bound to generate the explicit R^2 term. Indeed the condition $\beta = 0$ of Sec. 7.2.2 cannot be retained in the quantum regime, unless there exists conformal invariance for its protection.

We can now solve all the velocities $\partial_t K_{ij}$ in terms of the canonical variables, and consequently perform the Legendre transformation with the primary constraints (7.9) only. The total Hamiltonian is obtained as

$$\begin{aligned} H = &\int_{\Sigma_t} \left(N\mathcal{H}_0 + N^i\mathcal{H}_i + v_N p_N + v^i p_i\right) - \frac{1}{\kappa}\oint_{\mathcal{B}_t} N\sqrt{\sigma}\,{}^{(2)}K \\ &+ \oint_{\mathcal{B}_t} r_i\left(D_j N\mathcal{P}^{ij} - ND_j\mathcal{P}^{ij} + 2N_j p^{ij} + 2N^j\mathcal{P}^{ik}K_{jk}\right), \end{aligned} \tag{7.23}$$

with the Hamiltonian constraint

$$\begin{aligned} \mathcal{H}_0 = &\, 2p^{ij}K_{ij} - \frac{\mathcal{G}_{ijkl}\mathcal{P}^{ij}\mathcal{P}^{kl}}{2\alpha\sqrt{h}} + \mathcal{P}^{ij\,(3)}R_{ij} + \mathcal{P}^{ij}K_{ij}K + D_iD_j\mathcal{P}^{ij} \\ &- \frac{\mathcal{P}}{2}\left({}^{(3)}R - K_{ij}K^{ij} + K^2\right) - \sqrt{h}\Lambda \\ &- \frac{\sqrt{h}}{2\kappa}\left({}^{(3)}R + K_{ij}K^{ij} - K^2\right) - \alpha\sqrt{h}{}_\perp C_{ijk\boldsymbol{n}\perp}C^{ijk}{}_{\boldsymbol{n}}\,. \end{aligned} \tag{7.24}$$

The generalized DeWitt metric is given by

$$\mathcal{G}_{ijkl} = \frac{1}{2}\left(h_{ik}h_{jl} + h_{il}h_{jk}\right) - \frac{\alpha + 3\beta}{9\beta}h_{ij}h_{kl}\,. \tag{7.25}$$

The physical Hamiltonian is given by the surface terms with respect to a reference background, similarly as in GR. The contributions of the higher-derivative degrees of freedom to the surface terms are evident.

There exist eight first-class constraints ($p_N, p_i, \mathcal{H}_0, \mathcal{H}_i$) that are associated with the general covariance of spacetime and no second-class constraints. Thus the number of local physical degrees of freedom is eight. In other words, as mentioned before, the massive scalar and spin-2 modes appear, in addition to the massless spin-2 graviton. The massive scalar mode is highly relevant at the cosmological scale, as is demonstrated by the case of $f(R)$ gravity.

7.2.4 Concluding remarks

In all the cases with $\alpha \neq 0$, the Ostrogradskian form of the Hamiltonian is clearly visible in the first term $2p^{ij}K_{ij}$ of the Hamiltonian constraints. It corresponds to the first term $\sum_i P_i^1 G_i^1$ of the generic Hamiltonian (2.43). This implies the appearance of the massive spin-2 ghost. The dilemma of higher-derivative gravity (7.1) is that this ghost is required for renormalizability [9]. Only in the case of conformally invariant Weyl gravity, there exist as many constraints as there are unstable directions in phase space. Thus Weyl gravity is the only renormalizable theory of the type (7.1) that could in principle avoid the problem with ghosts, namely the lack of either stability or unitarity.[2]

[2]Recently [13], it has been argued that conformal gravity is unitary, but its Hamiltonian is non-Hermitian. However, in order to achieve this, the gravitational field $g_{\mu\nu}$ would have to be anti-Hermitian, i.e., the metric would be purely imaginary.

Chapter 8

Noncommutative gravity as a gauge theory of the twisted Poincaré symmetry

We review paper **I** [25] in this chapter. An extended treatment of the subject can be found in [145].

Since the seminal works of Utiyama [146] and Kibble [147] it has been understood that a classical gravitational theory can be derived as a gauge theory of the Poincaré symmetry. Elevating the global Poincaré symmetry of Minkowski spacetime to a local gauge symmetry, one derives the Poincaré gauge theory [147], whose simplest case is equivalent to the Einstein-Cartan theory of gravity. Geometry of spacetime differs from GR by a nonvanishing torsion tensor. Gravitational field equations relate curvature to energy-momentum, similar to GR (1.2), but with the difference that the Ricci tensor and energy-momentum tensor need not be symmetric. The torsion tensor ${T^\rho}_{\mu\nu}$ is related to the spin density tensor ${S^\rho}_{\mu\nu}$ as

$$ {T^\rho}_{\mu\nu} + \delta^\rho_\mu {T^\sigma}_{\nu\sigma} - \delta^\rho_\nu {T^\sigma}_{\mu\sigma} = \kappa {S^\rho}_{\mu\nu} \,. \tag{8.1} $$

The torsion effects are very weak in most circumstances, but become significant when density is extremely high. The contribution of spin density becomes equally important with energy-momentum, when the density of matter is $10^{47}\,\mathrm{g/cm^3}$ for electrons or $10^{54}\,\mathrm{g/cm^3}$ for nucleons [148]. Such high densities can only be encountered in the early universe and in black holes, but they are still much smaller than the Planck density $m_{\mathrm{P}}/l_{\mathrm{P}}^3 \sim 10^{94}\,\mathrm{g/cm^3}$ at which the quantum gravitational effects are expected to dominate. GR and its diffeomorphism invariance are known to emerge by gauging the group of spacetime translations [149]. Poincaré gauge theory includes Lorentz group, which adds the spin effects. Poincaré gauge theory of gravity is a viable alternative to GR. See [150] for an up to date review of gauge theories of gravity.

We seek to generalize the gauge theory of gravity approach to noncommutative spacetime. Noncommutative gravity should emerge as a gauge theory of a symmetry respected by the noncommutative spacetime. The most natural candidates for such a symmetry are the translational invariance and a noncommutative substitute for the relativistic invariance.

First we must understand what relativistic invariance could mean on a noncommutative spacetime. The noncommutative algebra of operators generated by the noncommutative coordinate operators $\hat{x}^\mu$ satisfying (1.5) can be represented on the algebra of functions on commutative spacetime. This is accomplished by replacing the pointwise product of functions by the noncommutative (and nonlocal) Moyal star-product

$$\begin{aligned} f(x) \star g(x) &= f(x) \exp\left(\frac{i}{2} \overleftarrow{\partial}_\mu \theta^{\mu\nu} \overrightarrow{\partial}_\nu\right) g(x) \\ &= f(x)g(x) + \sum_{n=1}^{\infty} \frac{1}{n!} \left(\frac{i}{2}\right)^n \theta^{\mu_1\nu_1} \dots \theta^{\mu_n\nu_n} \\ &\quad \times \partial_{\mu_1} \cdots \partial_{\mu_n} f(x)\, \partial_{\nu_1} \cdots \partial_{\nu_n} g(x)\,. \end{aligned} \tag{8.2}$$

We assume that $\theta^{\mu\nu}$ is a constant antisymmetric matrix. Hence the Lorentz symmetry is broken in noncommutative spacetime. The lack of relativistic invariance was a serious problem for noncommutative field theory, because of the lack of proper representations to describe the fields that we know to exist. Relativistic invariance was discovered to be replaced by the twisted Poincaré symmetry [23, 24]. The representation content of the twisted Poincaré algebra, however, is unaltered and identical to the representation of the usual Poincaré algebra. Spin-statistics relation is preserved in noncommutative field theories with twisted Poincaré symmetry [151].

The Poincaré algebra $\mathcal{P}$ consists of the generators of spacetime translations P_μ and the generators of Lorentz transformations $M_{\mu\nu}$. The universal enveloping algebra $\mathcal{U}(\mathcal{P})$ of Poincaré algebra has a hidden Hopf algebra structure, which includes the coproduct

$$\Delta_0(X) = X \otimes \mathbf{1} + \mathbf{1} \otimes X\,, \qquad \Delta_0(\mathbf{1}) = \mathbf{1} \otimes \mathbf{1}\,, \qquad X \in \mathcal{P} - \{\mathbf{1}\}\,. \tag{8.3}$$

The coproduct of $\mathcal{U}(\mathcal{P})$ is twisted with the Abelian twist element

$$\mathcal{F} = e^{\frac{i}{2}\theta^{\mu\nu} P_\mu \otimes P_\nu} \tag{8.4}$$

as

$$\Delta_t(X) = \mathcal{F} \Delta_0(X) \mathcal{F}^{-1}\,. \tag{8.5}$$

Any twist element has to satisfy the twist condition

$$\mathcal{F}_{12}(\Delta_0 \otimes \mathrm{id})\mathcal{F} = \mathcal{F}_{23}(\mathrm{id} \otimes \Delta_0)\mathcal{F}\,, \qquad \mathcal{F}_{12} = \mathcal{F} \otimes \mathbf{1}\,, \qquad \mathcal{F}_{23} = \mathbf{1} \otimes \mathcal{F}\,. \tag{8.6}$$

When Poincaré algebra is twisted, the multiplication map for its representations has to be redefined. In the case of smooth fields on spacetime, with pointwise multiplication $m(f \otimes g) = fg$, the resulting multiplication of fields is the Moyal star-product (8.2):

$$m_t(f \otimes g) = m\left(\mathcal{F}^{-1}(f \otimes g)\right) = f \star g\,. \tag{8.7}$$

The action of the twisted Poincaré algebra on a star-product of fields is deformed as

$$X(f \star g) = m_t\left(\Delta_t(X)(f \otimes g)\right), \qquad X \in \mathcal{P}\,. \tag{8.8}$$

As a result both the commutator of coordinates ($x^\mu \star x^\nu - x^\nu \star x^\mu = \theta^{\mu\nu}$) and the Minkowski length ($x^2 \equiv \eta_{\mu\nu} x^\mu \star x^\nu = \eta_{\mu\nu} x^\mu x^\nu$) are left invariant by the action of the twisted Poincaré algebra. Thus noncommutative spacetime is invariant under twisted Poincaré transformations.

Formulation of local gauge symmetries on a noncommutative spacetime is a delicate issue. Most gauge groups cannot be defined on noncommutative spacetime, because they do not close under the star-product. The noncommutative unitary group $U_\star(n)$ can be defined, but with representations limited by the no-go theorem [152]. A noncommutative standard model based on the gauge groups $U_\star(n)$ has been constructed [153]. Another approach to the noncommutative gauge theories has been through the so-called Seiberg-Witten map [154], which originally related a noncommutative $U_\star(n)$ gauge theory to a commutative one, both obtained as low-energy effective limits in string theory. Seiberg-Witten map has been subsequently used to formulate noncommutative gauge theories with gauge fields valued in the enveloping algebra of $su(n)$ [155]. A corresponding noncommutative version of the standard model has been built [156]. Attempts on extending the Poincaré algebra with an internal gauge algebra and under a common twist (either the Abelian twist (8.4) [157] or a gauge invariant non-Abelian twist [158]) have not been successful. It is intriguing that the external Poincaré symmetry and an internal gauge symmetry cannot be unified under a common twist, when the Moyal spacetime structure (1.5) is retained.[1] But perhaps the twisted Poincaré symmetry itself (being an external symmetry) can be gauged?

Several noncommutative theories of gravity have been proposed (see paper **I** [25] for a brief introduction). However, none of them are fully satisfactory from the viewpoint of symmetries, and the dynamics of noncommutative gravity arising from string theory is richer than for actions written in terms of the Moyal star-product [159]. We suspect that a reason for this is the noninvariance of the star-product under spacetime diffemorphisms. Gauging the twisted Poincaré algebra may provide us a noncommutative theory of gravity that is covariantly deformed under local Poincaré transformations.

Since the global Poincaré symmetry is twisted with the Abelian twist (8.4) in the case of the flat noncommutative space-time, also the generalized Poincaré gauge symmetry on noncommutative space-time should be a quantum symmetry. A natural way to generalize the Poincaré gauge symmetry to noncommutative spacetime is to consider it as a twisted gauge symmetry, so that the global

[1] Supersymmetry may enable the construction of a noncommutative gauge theory by means of a twist due to its intrinsic internal symmetry.

twisted Poincaré symmetry is obtained in the limit of vanishing gauge fields. We introduce the vierbein ${e^a}_\mu$ and spin connection ${\omega_\mu}^{ab}$ gauge fields in order to compensate the noncovariance of the partial derivatives, similarly as in the commutative case. Partial derivatives $\partial_\mu = iP_\mu$ are replaced by covariant derivatives, which are given in the coordinate frame as

$$\nabla_\mu = i\left({e^a}_\mu(x)P_a + \frac{1}{2}{\omega_\mu}^{ab}(x)\Sigma_{ab}\right), \tag{8.9}$$

where Σ_{ab} generate a finite-dimensional representation of the Lorentz algebra. In order to obtain a theory that is covariantly deformed under the Poincaré gauge transformations, the frame-dependent translation generators P_μ have to be replaced by the covariant derivatives $-i\nabla_\mu$ in the Abelian twist element (8.4). The covariant non-Abelian twist element is of the form

$$\mathcal{T} = e^{-\frac{i}{2}\theta^{\mu\nu}\nabla_\mu\otimes\nabla_\nu + O(\theta^2)}, \tag{8.10}$$

where $O(\theta^2)$ stands for the possible additional covariant terms in higher orders of the noncommutativity parameter $\theta^{\mu\nu}$.

The multiplication of fields must be deformed as in (8.7), i.e., $m_t = m\circ\mathcal{T}^{-1}$. It is associative only if $\mathcal{T}$ satisfies the twist condition (8.6). First we consider the twist element (8.10) with only the first order term in θ in the exponent. The twist condition (8.6) turns out to contain terms that do not cancel already at second order in θ. In the left-hand side, the terms are

$$\begin{aligned}\frac{1}{2}\left(-\frac{i}{2}\right)^2\theta^{\mu\nu}\theta^{\rho\sigma}\big(&2\nabla_\mu\nabla_\rho\otimes\nabla_\nu\otimes\nabla_\sigma + 2\nabla_\mu\otimes\nabla_\nu\nabla_\rho\otimes\nabla_\sigma\\ &+ \nabla_\mu\otimes\nabla_\rho\otimes\nabla_\nu\nabla_\sigma + \nabla_\rho\otimes\nabla_\mu\otimes\nabla_\nu\nabla_\sigma\big)\end{aligned} \tag{8.11}$$

and in the right-hand side

$$\begin{aligned}\frac{1}{2}\left(-\frac{i}{2}\right)^2\theta^{\mu\nu}\theta^{\rho\sigma}\big(&2\nabla_\rho\otimes\nabla_\mu\nabla_\sigma\otimes\nabla_\nu + 2\nabla_\rho\otimes\nabla_\mu\otimes\nabla_\nu\nabla_\sigma\\ &+ \nabla_\mu\nabla_\rho\otimes\nabla_\nu\otimes\nabla_\sigma + \nabla_\mu\nabla_\rho\otimes\nabla_\sigma\otimes\nabla_\nu\big).\end{aligned} \tag{8.12}$$

These terms can not be cancelled by terms that contain the field strength tensors, namely

$$R^{ab}{}_{\mu\nu}\Sigma_{ab}\,, \qquad T^a{}_{\mu\nu}\nabla_a\,, \tag{8.13}$$

because the two indices for such tensors come from the same $\theta^{\mu\nu}$, unlike for the $\nabla\nabla$ factors in (8.11) and (8.12). This is why such terms are not included in twist element (8.10) in the first place. Other possible second order terms in (8.10) have the forms

$$\begin{aligned}&\theta^{\mu\nu}\theta^{\rho\sigma}\,1\otimes\nabla_\mu\nabla_\nu\nabla_\rho\nabla_\sigma\,, &&\theta^{\mu\nu}\theta^{\rho\sigma}\,\nabla_\mu\nabla_\nu\nabla_\rho\nabla_\sigma\otimes 1\,,\\ &\theta^{\mu\nu}\theta^{\rho\sigma}\,\nabla_\mu\otimes\nabla_\nu\nabla_\rho\nabla_\sigma\,, &&\theta^{\mu\nu}\theta^{\rho\sigma}\,\nabla_\mu\nabla_\nu\nabla_\rho\otimes\nabla_\sigma\,,\\ &\theta^{\mu\nu}\theta^{\rho\sigma}\,\nabla_\mu\nabla_\nu\otimes\nabla_\rho\nabla_\sigma\,,\end{aligned} \tag{8.14}$$

with all permutations of the indices of the covariant derivatives. We have verified that when introduced into the twist element (8.10) and consequently into the twist condition (8.6), these second orders terms can never cancel all the terms in (8.11) and (8.12). Therefore, the twist condition (8.6) cannot be fulfilled for second order in θ. Here the discussion was presented for the exponential form (8.10) of the twist, but the results are valid for any invertible covariant twist.

Since GR and its general covariance can be derived by gauging the translation symmetry, it is interesting to see whether the gauge theory of the group of translations can be consistently defined together with the twisted Poincaré symmetry. The covariant derivative for the local translations is

$$d_\mu = ie^a{}_\mu P_a \,. \tag{8.15}$$

For one-dimensional representations the covariant derivative of Poincaré gauge symmetry (8.9) reduces to (8.15). Since the covariant derivatives of the translation group do not commute,

$$[d_\mu, d_\nu] = C^\rho{}_{\mu\nu} d_\rho \,, \qquad C^\rho{}_{\mu\nu} = (e^a{}_\mu \partial_a e^b{}_\nu - e^a{}_\nu \partial_a e^b{}_\mu) e_b{}^\rho \,, \tag{8.16}$$

the covariant element

$$\mathcal{T} = e^{-\frac{i}{2}\theta^{\mu\nu} d_\mu \otimes d_\nu + O(\theta^2)} = e^{\frac{i}{2}\theta^{\mu\nu} e^a{}_\mu P_a \otimes e^b{}_\nu P_b + O(\theta^2)} \tag{8.17}$$

is not of the Abelian type (8.4). Therefore we face similar problems as with the full Poincaré gauge symmetry. The twist element (8.17) does not satisfy the twist condition (8.6), even though its form is much simpler now. Thus, regarding the validity of the non-Abelian Poincaré gauge covariant twist element (8.10), the external gauge symmetry associated with the general coordinate transformations is equally problematic as the local Lorentz symmetry.

Thus, we have obtained the result that the Poincaré gauge covariant non-Abelian element (8.10) is not a twist, and the star-product defined by it is not associative. We conclude that the twisted Poincaré symmetry cannot be gauged by generalizing the Abelian twist (8.4) to a covariant non-Abelian twist (8.10), nor by introducing a more general covariant twist element. This is a serious obstruction, since losing associativity complicates the construction of gauge theories considerably. Attempts on restoring the associativity of the star-product have been made, e.g. [26, 27, 160–164], but no fully associative star-product beyond scalar functions has been obtained. The question of unifying the external (global or local) Poincaré symmetry and the internal gauge symmetry under a common twist remains an open fundamental problem of noncommutative gauge theories. A potential way to formulate the gauge theory of noncommutative gravity would be to replace the requirement of general coordinate transformations with respect to the whole Lorentz group with the requirement of general coordinate transformations only under the residual symmetry of noncommutative field theories as argued in [165].

Chapter 9

Conclusions

The nature of the quantum structure of spacetime and gravity is undoubtedly one of the greatest problems of contemporary theoretical physics. Whether gravity should actually be quantized, e.g., in the sense of QFT or string theory, is still not understood properly. This is because of both theoretical difficulties and the lack of experimental data that could guide us. Alternatively, gravity could be an emergent phenomenon that arises from an unknown quantum theory, which does not involve gravity and possibly even spacetime at the fundamental level.

Including higher-order derivatives into a gravitational action can greatly improve its renormalization characteristics. In the class of generally covariant theories with full symmetry under diffeomorphisms of spacetime, an action including fourth-order spacetime derivatives in quadratic curvature terms is known to be renormalizable [9] and asymptotically free [10]. However, this theory includes a massive spin-2 degree of freedom, which carries negative energy and thus makes the theory unstable. Such unstable ghosts appear in all regular interacting higher derivative field theories, but constrained (gauge) theories can sometimes avoid the problem. In the renormalizable extension of GR (1.3) or (7.1), there do not exist enough constraints to rescue stability (or unitarity). In the recent years, new kinds of gravitational theories with higher-order derivatives have been proposed. HL gravity [14] introduces the idea that space and time scale anisotropically at very high energies. That enables the inclusion of higher-order spatial derivatives in order to achieve renormalizability without including higher-order time derivatives. As a result the Lorentz symmetry is broken in the UV region, but it is envisioned to be restored in the IR limit where the system flows to an isotropic state. On the other hand, CRG [17, 18] aims to accomplish a similar advantage by introducing an exotic fluid, which can be described by a constrained scalar field, and by breaking Lorentz symmetry spontaneously.

In papers **II** [15], **III** [16] and **IV** [124], we proposed and studied generalized HL theories. In particular, the modified $F(R)$ HL gravity was considered, which combines the power-counting renormalizability of HL gravity and the interesting cosmological properties of the generally covariant $f(R)$ gravity. Hamiltonian formulation of modified $F(R)$ HL gravity was found to be quite similar to the usual HL gravity, except for the existence of an extra scalar degree of freedom. That raises the number of physical degrees of freedom to four. Similar to the usual $f(R)$ gravity, the extra scalar is not a pathological ghost, provided that the coupling constants and the functional form of the Lagrangian are chosen appropriately. The IR limit of modified $F(R)$ HL gravity differs from the usual HL gravity, since the former cannot be interpreted as a partially gauge fixed form of generally covariant $f(R)$ gravity, while HL gravity can be seen as partially gauge fixed GR. Cosmological prospects of modified $F(R)$ HL gravity were found to be rich, including the ability to realize unified inflation and late-time expansion (dark energy) without extra components like inflaton and quintessence.

In papers **V** [19] and **VI** [20], the canonical structure of CRG was discovered to be involved. Hamiltonian formulation of both the original and new versions of CRG was accomplished using a foliation of spacetime defined by the constrained scalar field. The resulting Hamiltonian structure of original CRG [17] was found to be quite similar to the structure of modified $F(R)$ HL gravity, even though the theories have little resemblance in their original formulations. However, the first version of CRG does not improve UV behavior sufficiently, which is why a new version was proposed [18]. The Hamiltonian structure of the new version of CRG was found to be very complicated, even when the scalar constraint is solved. There again exists four physical degrees of freedom. We argued that one of them is a ghost that suffers from the Ostrogradskian instability – the same problem that affects generally covariant higher derivative gravity. Even if that were not the case, the advantage of spontaneously broken Lorentz symmetry is hardly worth accepting a theory with such a complicated structure.

Hamiltonian analysis of Weyl gravity and other fully diffeomorphism invariant curvature-squared gravities (7.1) was revisited in Chapter 7, based on a work in progress together with the co-authors of paper **VI** [20]. A correction to the component of Weyl tensor that is fully tangent to the spatial hypersurface was discovered in (7.4). The Ostrogradskian character of the Hamiltonian was noted in all the cases that could be renormalizable. Only the conformally invariant Weyl gravity has enough local invariances to be able to restrain the massive spin-2 ghost even in principle – the Weyl action contains the five traceless components of the time derivative of the extrinsic curvature and it possesses five local invariances, namely diffeomorphism and conformal invariance. The recent claim of obtaining a critical case of curvature-squared gravity [140], where the massive spin-2 ghost becomes massless, was concluded to be a special feature of the linearized theory on anti-de Sitter background. In the full nonlinear theory, the value of the cosmological constant does not affect the number and nature of local physical degrees of freedom, when both EH and Weyl actions are included.

We can conclude that uncovering the Hamiltonian structure of any theory with local (gauge) symmetry is necessary for understanding the fundamental structure of the theory. That is especially the case for the considered gravitational theories, which include higher-order derivatives in order to create a renormalizable theory. In locally Lorentz invariant theories, one has to face the problems with ghosts due to the presence of higher-order time derivatives. In theories with reduced symmetry (like HL gravity), the Hamiltonian constraints can become second-class constraints. That can jeopardize the consistency of a theory. In HL gravity, the solution is to extend the action and consequently the Hamiltonian constraint with spatial derivatives of the lapse function.

Lastly, we reviewed the idea of paper **I** [25] that gravitational theory on noncommutative spacetime should be obtained by gauging a (quantum) symmetry of the noncommutative spacetime. This is analogous to the well known fact that classical gravitational theories can be obtained by gauging symmetries of Minkowski spacetime. In particular, construction of a gauge theory of the twisted Poincaré symmetry was attempted by introducing a Poincaré gauge covariant twist candidate (8.10). Unfortunately, such a covariant twist cannot satisfy the twist condition (8.6). Hence gauging the twisted Poincaré symmetry with a gauge covariant twist is not possible, much like one cannot unify the external Poincaré symmetry and an internal gauge symmetry under a common twist [158]. The same result was obtained for the translational symmetry. New fundamental insight on the nature of local invariances in noncommutative field theory is required to overcome this problem.

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