

DEPARTMENT OF COMPUTER SCIENCE  
SERIES OF PUBLICATIONS A  
REPORT A-2013-5

# Dynamic Aspects of Knowledge Bases

Satu Eloranta

*To be presented, with the permission of the Faculty of Science of the University of Helsinki, for public criticism in Linus Torvalds Auditorium B123, Exactum Building, on 27 June 2013 at noon.*

UNIVERSITY OF HELSINKI  
FINLAND

**Supervisor**

Matti Nykänen, University of Eastern Finland, Kuopio, Finland

**Pre-examiners**

Tomi Janhunen, Aalto University, Finland

Johan Lilius, Åbo Akademi, Finland

**Opponent**

Antti Valmari, Tampere University of Technology, Finland

**Custos**

Esko Ukkonen, University of Helsinki, Finland

**Contact information**

Department of Computer Science  
P.O. Box 68 (Gustaf Hällströmin katu 2b)  
FI-00014 University of Helsinki  
Finland

Email address: [info@cs.helsinki.fi](mailto:info@cs.helsinki.fi)

URL: <http://www.cs.helsinki.fi/>

Telephone: +358 9 1911, telefax: +358 9 191 51120

Copyright © 2013 Satu Eloranta

ISSN 1238-8645

ISBN 978-952-10-8989-3 (paperback)

ISBN 978-952-10-8990-9 (PDF)

Computing Reviews (1998) Classification: I.2.3, I.2.4

Helsinki 2013

# Dynamic Aspects of Knowledge Bases

Satu Eloranta

Department of Computer Science  
P.O. Box 68, FI-00014 University of Helsinki, Finland  
satu.eloranta@cs.helsinki.fi

PhD Thesis, Series of Publications A, Report A-2013-5  
Helsinki, June 2013, 123 pages  
ISSN 1238-8645  
ISBN 978-952-10-8989-3 (paperback)  
ISBN 978-952-10-8990-9 (PDF)

## Abstract

A knowledge base is considered a system that is told information about an external world and that answers questions about this world. Our goal here is to outline knowledge bases that involve both knowledge and beliefs. In previous studies, various kinds of belief change have been studied in isolation, but we want to tie them together. We aim at knowledge bases that could carry the epistemic states of agents, that is, the knowledge and the beliefs that an agent has at any one moment in time.

The difference between knowledge and belief is that while knowledge increases monotonically with time, beliefs may at some later point in time turn out to be false. Beliefs may change for various reasons: in belief revision, beliefs are changed when receiving new information about a world that has not changed, while in belief update a change in the world is to be recorded. Different types of change call for different treatments. In belief-change studies, various change types have been characterized by rationality criteria set on each type. The main principles in these criteria are maintaining consistency of beliefs and minimality of change.

When dealing with belief change, our approach is to take knowledge as an integrity constraint that should always hold, and we describe how the rationality criteria should be modified accordingly. In our refined rationality criteria, beliefs that are inconsistent with the knowledge of the knowledge base will never be allowed to enter into the knowledge base.

In the rationality criteria, a common assumption is that the most recent information is the most reliable, and it has therefore been prioritized over the old beliefs. However, this may not be the case in all circumstances. In order to complete the collection of belief-change types, we propose a new, commutative type of change for entering competing evidence into the knowledge base.

The representation theorems that have been given for belief revision indicate that belief revision involves an ordering of disbelief on possible alternative situations, or equivalently, an epistemic entrenchment on logical formulas. A formula less entrenched is more easily given up when eliminating inconsistencies. In view of the changes in the rationality criteria, we also refine the representation theorems.

We introduce two finite representations for knowledge bases, one with a finite ordered set of propositional formulas that are satisfiable but pairwise inconsistent with each other, and the other with a finite list of pairwise inconsistent propositional formulas. Both representations involve dynamic orderings of disbelief that have arisen out of the previous change operations.

We show that for the knowledge base to satisfy the rationality criteria given for belief revision, the dynamic ordering of disbelief in the knowledge base is vital. The representations and the operators that we introduce in this thesis demonstrate how this ordering of disbelief could be dealt with in various operations.

## **Computing Reviews (1998) Categories and Subject**

### **Descriptors:**

I.2.3 Deduction and Theorem Proving

I.2.4 Knowledge Representation Formalisms and Methods

### **General Terms:**

theory

### **Additional Key Words and Phrases:**

non-prioritized belief revision, belief update, knowledge base

# Acknowledgements

I wish to express my deep gratitude to Professor Esko Ukkonen for mentoring. I would like to thank Professor Matti Nykänen and Professor Seppo Sippu for the supervision of this thesis. I am most grateful to the pre-examiners Docent Tomi Janhunen and Professor Johan Lilius. I especially wish to thank Docent Tomi Janhunen for his elaborate comments on the manuscript of this dissertation.

I am grateful the members of the Protean group: Dr. Raul Hakli, Professor Matti Nykänen, and Olli Niinivaara, for cooperation.

I am ever so grateful to my family and friends for their encouragement and support.

I am grateful to the Department of Computer Science for the facilities. I wish to thank Department Secretary Pirjo Kokkonen and my other colleagues for the well-being at the Department. I wish to thank the system administrators of the Department for help with the computing facilities.



# Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
<b>2</b>	<b>Preliminaries</b>	<b>7</b>
2.1	Propositional logics . . . . .	7
2.2	Model set semantics . . . . .	8
2.3	The concept of the knowledge base . . . . .	9
2.4	Epistemic states . . . . .	10
2.5	Types of epistemic change . . . . .	11
2.6	Conditionals . . . . .	12
2.7	Summary of the assumptions on the knowledge base . . . . .	12
<b>3</b>	<b>Examples of belief-change policies</b>	<b>15</b>
3.1	Syntactically-oriented belief-revision policies . . . . .	15
3.2	Semantically-oriented belief-revision policies . . . . .	18
3.3	Semantically-oriented belief-update policies . . . . .	20
3.4	Ordering-oriented belief-revision policies . . . . .	22
3.5	Grading-oriented belief-revision policies . . . . .	24
3.6	Comparison of belief-change policies . . . . .	26
<b>4</b>	<b>Rationality criteria for belief revision and update</b>	<b>29</b>
4.1	Postulates for belief revision . . . . .	29
4.2	Representation theorems for belief revision . . . . .	31
4.3	Postulates for belief contraction . . . . .	33
4.4	Postulates for belief update . . . . .	35
4.5	A representation theorem for belief update . . . . .	36
4.6	Postulates for belief erasure . . . . .	36
4.7	Analysis of the postulates for belief revision . . . . .	37
4.8	Analysis of some operators . . . . .	40
4.9	On integrity constraints . . . . .	43
4.10	Related work . . . . .	43

<b>5</b>	<b>Revised rationality criteria for belief and knowledge change</b>	<b>49</b>
5.1	Static rationality criteria on epistemic states . . . . .	50
5.2	Refined postulates for belief revision . . . . .	50
5.3	Refined postulates for belief contraction . . . . .	52
5.4	Rationality criteria for knowledge expansion . . . . .	54
5.5	Rationality criteria for competing evidence . . . . .	55
5.6	Refined postulates for belief update and erasure . . . . .	56
5.7	Related work . . . . .	57
<b>6</b>	<b>New operator collections</b>	<b>59</b>
6.1	Operators on epistemic bases . . . . .	59
6.2	Operators on epistemic functions . . . . .	71
6.3	Discussion . . . . .	79
<b>7</b>	<b>Modelling knowledge expansion and belief revision</b>	<b>81</b>
7.1	Knowledge change . . . . .	81
7.2	Orderings of disbelief . . . . .	82
7.3	Grove's system of spheres . . . . .	82
7.4	Modelling belief revision . . . . .	91
7.5	Modelling knowledge expansion . . . . .	93
7.6	Elementary epistemic states . . . . .	94
7.7	Discussion . . . . .	96
<b>8</b>	<b>Modelling belief update</b>	<b>101</b>
8.1	Finite proposition fields . . . . .	101
8.2	Adjusting the postulates for belief update . . . . .	102
8.3	A representation theorem for belief update . . . . .	102
8.4	Modifying update operators . . . . .	106
<b>9</b>	<b>Conclusion and future work</b>	<b>109</b>
	<b>References</b>	<b>113</b>



## List of basic notations

$\mathcal{L}$	a propositional language
$Voc(\mathcal{L})$	the set of atomic formulas in $\mathcal{L}$
$W$	the power set of $Voc(\mathcal{L})$ , the set of all logically possible models
$Voc(A)$	the set of atomic formulas in formula $A$
$T_B$	the belief set of knowledge base $T$
$T_K$	the knowledge set of knowledge base $T$
$\llbracket T_K \rrbracket$	the possible models of $T$
$\llbracket T \rrbracket$	the most plausible models of $T$ , a short form of $\llbracket T_B \rrbracket$
$\llbracket A \rrbracket$	the models of formula $A$
$w \Delta w'$	the symmetric difference of $w$ and $w'$



# Chapter 1

## Introduction

A *knowledge base* can be viewed as an object that is given information about an external world and that answers queries about that world. The problem is that some of the information may later turn out to be false. Such pieces of information are actually called *beliefs*, not knowledge [Hin62, chapter 2].

If one wants to accept a new piece of information that is inconsistent with the old beliefs, some of the old beliefs ought to be given up in order to maintain consistency of beliefs. Furthermore, one may have different choices for performing the removal, in other words the change may be ambiguous [FUV83]. Let us exemplify these problems in a context, where a collection of logical formulas represents the set of beliefs.

**Example 1.1.** Let  $T$  denote a set of formulas  $T = \{a, b, a \wedge b \rightarrow c\}$ . The propositional formula  $c$  is logically entailed by  $T$ , thus adding the formula  $\neg c$  to  $T$  makes the set of formulas inconsistent. How should one remove from  $T$  the formula  $c$ , which is not one of the formulas in  $T$ , but can be derived from them, that is, the formula is contained in the closure of  $T$ ? Removing any of the formulas from  $T$  results in removing the formula  $c$  from the closure. Thus each of the sets  $T_1 = \{a, b\}$ ,  $T_2 = \{a, a \wedge b \rightarrow c\}$ , and  $T_3 = \{b, a \wedge b \rightarrow c\}$  is a candidate for the result of the removal.

Beliefs may change for various reasons, and different types of input [AbG85] call for different treatments [KeW85]. When new information is obtained about a static world, the beliefs are *revised*. When a change in the external world is to be recorded, the beliefs are *updated*. Let us exemplify the difference.

**Example 1.2.** Imagine you are walking in a park. There are two kiosks in the park, say A and B. When you come across a person eating ice cream, you then believe that at least one of the kiosks is open. Your information

is incomplete and that is why you consider three alternatives as plausible: both the kiosks are open, only A is open, or only B is open. When you arrive at kiosk A and learn that it has been closed all day, you revise your beliefs. You discard the two alternatives having kiosk A open. In the only alternative left, kiosk B is open, thus you now believe that kiosk B is open.

Had you found out that kiosk A was closed just a moment ago, you would have updated your beliefs. Then you would have considered changing each of the alternatives separately. If both kiosks were open, then B would still be open, otherwise both kiosks would be closed. After the update you would have believed that kiosk A was closed, but you would have had no information about whether kiosk B is open or not<sup>1</sup>.

Belief change has been studied in database theory [FUV83], in philosophy [AGM85], and in artificial intelligence [McC87]. In databases, view updates cause problems in choosing an unambiguous operation that, when applied to the underlying relations, produces the desired change in the view [BaS81]. Other problems are the validation of integrity constraints [ToA91] and updating incomplete information [Gra91b].

Belief change is an essential component in many applications in artificial intelligence. One is problem-solving agents, which reason and modify their beliefs as a result of messages sent by their perceptors or other agents. Another application is detecting errors in a system, whose behaviour is inconsistent with its specification. The theory then contains the specification and the observations of the system behaviour. The goal is to find out which are the components responsible for the misbehaviour of the system [Rei87]. An early application of theory change was Doyle's Truth Maintenance System (TMS) [Doy79], a problem solving subsystem in the field of design.

Hypothetical "what if" queries are practical in many applications in artificial intelligence [Gin86]. Conditionals formalize hypothetic reasoning. The truth value of a conditional "if  $A$  then  $B$ " can be determined by the *Ramsey test*: "if  $A$  then  $B$ " is true in a knowledge base  $T$ , if and only if inserting  $A$  into  $T$  results in a knowledge base in which  $B$  is believed [Lew73].

The philosophers Alchourrón, Gärdenfors, and Makinson [AGM85] have proposed a set of rationality criteria for belief revision; these criteria are known as the *AGM-postulates*. Every belief-revision operator that satisfies the AGM-postulates is associated with an ordering, called *epistemic entrenchment*, upon the beliefs [GäM88]. A belief more entrenched is

---

<sup>1</sup>We do not always know whether there is a change in the world or not. In our example that could be the situation, if there were no opening hours at hand.

less willingly given up. An additional set of postulates for belief revision [DaP94, DaP97] has been proposed in order to rule out operators that give unintuitive results in iterated revisions. Belief update [KaM91a] has its own, slightly different set of postulates. The main principles in these sets of postulates are maintaining consistency of the beliefs and minimality of change.

The original AGM-postulates involve revising closed sets of formulas in an arbitrary language with few restrictions. The language may or may not contain conditionals. However, if the language contains both conditionals and a revision operator that satisfies the AGM-postulates, then using the operator in the Ramsey test results in *triviality of the logic* [Gär88, chapter 7], that is, it is impossible to have three satisfiable formulas such that their pairwise conjunctions are unsatisfiable.

The AGM-postulates consider the input as the most recent and the most reliable piece of information. As such, the input is always accepted. This type of belief revision is called *prioritized belief revision*. However, the knowledge base may have some information that it will refuse to give up at any situation. In computer science this information might be called integrity constraints, in philosophy knowledge [Hin62]. In belief revision literature the term core beliefs has also been used [Han99]. Because of these core beliefs or knowledge, the input could sometimes be rejected [Mak97]. This framework is called *non-prioritized belief revision* (see [Han99] for a survey). In this study the term *knowledge* is used for irrefusable information.

Let us now turn to the major three questions that motivated this thesis. First of all, the earliest formulation of the postulates for iterated belief revision [DaP94] assumed that knowledge bases could be represented by single propositional formulas. This assumption caused the AGM-postulates and the postulates for iterated belief revision to be inconsistent with each other [FrL94, Elo95, Elo97] and resulted in triviality of logic. The problem has been fixed [DaP97] since then, but we will discuss it and show that ruling out contradictory input is not enough to make the early formulations of the postulates consistent.

The second question pertains to the AGM-postulates accepting beliefs known to be false, or even self-contradictory beliefs. Should not knowledge affect belief change? In this thesis, the postulates are refined to reject beliefs known to be false.

The third question is due to belief revision and update having been discussed in isolation. What if we had a series of changes including both revisions and updates? What is the effect of an update on the epistemic

entrenchment in a knowledge base? We propose update operators that deal with the epistemic entrenchment.

In this thesis, the knowledge base is considered as a component of an independent agent recording the epistemic state of the agent, which [Gär88, chapter 1] contains the beliefs (and knowledge) of the agent at any one moment. The knowledge base itself has no intentionality. The knowledge base is told propositional formulas; we will leave the classification of the input to the agent as well as the communication with other agents.

Our goal is to define an abstract data type called “knowledge base”. Our contribution includes

- modifications to the rationality criteria for belief and knowledge change, including a new type of commutative change called “competing evidence”,
- new representations of knowledge bases along with their change operators, and
- refined representation theorems for belief revision, knowledge expansion, and belief update.

The rest of the thesis is organized as follows. In the preliminaries in Chapter 2, our assumptions on the knowledge base are fixed. These hypotheses allow both belief and knowledge change, revision and update, finite and infinite languages.

In Chapter 3, various belief-change policies are reviewed. In Chapter 4 we review the rationality criteria proposed for belief change and discuss the first one of the motivating questions, how to avoid the inconsistency between the AGM-postulates and the postulates for iterated belief revision.

In Chapter 5 the second one of the motivating questions is addressed and the rationality criteria for belief change are refined to include the effect of knowledge on belief change. The classification of change is also discussed and a new, commutative type of commutative belief change is proposed.

In Chapter 6 we will address the third question and propose two implementations of the knowledge base: two different sets of change operators along with suitable finite representations of knowledge bases. These sets of operators cover various change types and they satisfy our rationality criteria for the change types in question.

In Chapter 7 refined representation theorems for belief revision and knowledge change are proved. The epistemic entrenchment is considered as part of the knowledge base, and the finiteness of the ordering is analyzed.

In Chapter 8 we will prove our refined representation theorem for belief update. In the final chapter, Chapter 9, a conclusion of the thesis is given along with some directions for further work.





# Chapter 2

## Preliminaries

We will start this chapter by recalling propositional logic and its set semantics. We will then recall our definition of the knowledge base as an object that the agent can use to record its epistemic state. We will next shortly review epistemic states and classify their change operators. At the end of the chapter we shall sum up our assumptions.

### 2.1 Propositional logics

Let us first recall propositional logic. *Propositional formulas* are built of atomic formulas using connectives  $\neg$  (not),  $\vee$  (or),  $\wedge$  (and),  $\rightarrow$  (material implication), and  $\leftrightarrow$  (equivalence). No assumptions are made about the atomic formulas. *Propositional symbols* are abstractions of atomic formulas.

The meaning of the formulas can be provided by fixing a context in which each atomic formula can be interpreted as true or false. Such contexts are called *interpretations*. The partition of the atomic formulas into true formulas and false formulas is called a truth distribution. Formally, a *truth distribution*  $v$  is a function that maps every propositional symbol to a truth value  $t$  (true) or  $f$  (false). In propositional logic it is sufficient to consider truth distributions instead of interpretations. We use the symbol  $\perp$  as a constant in the language to denote a *contradiction*, a formula that is not true in any truth distribution, and the symbol  $\top$  to denote a *tautology*, a formula that is true in every truth distribution.

If a formula  $A$  is valued true in a truth distribution assigned by an interpretation  $w$ , we say that  $w$  is a *model* of  $A$ ,  $w \models A$ . Given a set of formulas  $S$ , then  $w \models S$ , if and only if  $w \models A$  for all  $A \in S$ . A formula  $A$  is *logically entailed* by a set of formulas  $S$ ,  $S \models A$ , if and only if for every truth distribution  $w$  such that  $w \models S$  holds, also  $w \models A$  holds.

In a propositional deduction system a formula  $A$  is a *consequence* of a set of formulas  $S$ , denoted by  $S \vdash A$ , if and only if it can be deduced from  $S$  using the axioms and the rules of the system. *Axioms* are theory-independent, universally true formulas. An example of a rule is modus ponens. *Modus ponens* states that, given formulas  $A$  and  $A \rightarrow B$ , we can deduce the formula  $B$ . Propositional deduction systems are *sound*, that is,  $S \vdash A$  implies  $S \models A$ , and *complete*, that is,  $S \models A$  implies  $S \vdash A$  [RyS92]. The *closure* of a set  $S$ , denoted  $Cn(S)$ , is defined  $Cn(S) = \{A \in \mathcal{L} \mid S \vdash A\}$ . A set  $S$  is called *closed*, if  $S = Cn(S)$ , otherwise it is called *open*.

## 2.2 Model set semantics

Given a propositional language, let us construct model set semantics for the language.

Let  $\mathcal{L}$  denote a propositional language, and let  $Voc(\mathcal{L})$  denote the set of all the atomic propositional formulas of  $\mathcal{L}$ . We say that the language is infinite, if the set  $Voc(\mathcal{L})$  is infinite. We construct a set of all truth distributions for  $\mathcal{L}$  as  $W = \mathcal{P}(Voc(\mathcal{L}))$ , the power set of  $Voc(\mathcal{L})$ . The elements in  $W$  can be taken as interpretations by defining for each  $p \in Voc(\mathcal{L})$  and  $w \in W$ ,  $w \models p$  if and only if  $p \in w$ . Thus the power set  $W$  has one element for each possible truth distribution of the language  $\mathcal{L}$ . Given the set of all logically possible models  $W$ , the elements of the power set  $\mathcal{P}(W)$  are called *propositions*.

In *model set semantics* the meaning of each formula is identified by a corresponding proposition. We may define for all atomic formulas  $p \in P$

$$\llbracket p \rrbracket = \{w \in W \mid p \in w\},$$

and for all nonatomic formulas  $A, B \in \mathcal{L}$  [Gär88, chapter 2],

$$\begin{aligned} \llbracket \neg A \rrbracket &= W \setminus \llbracket A \rrbracket, \\ \llbracket A \wedge B \rrbracket &= \llbracket A \rrbracket \cap \llbracket B \rrbracket, \\ \llbracket A \vee B \rrbracket &= \llbracket A \rrbracket \cup \llbracket B \rrbracket, \\ \llbracket A \rightarrow B \rrbracket &= (W \setminus \llbracket A \rrbracket) \cup \llbracket B \rrbracket, \\ \llbracket \top \rrbracket &= W, \\ \llbracket \perp \rrbracket &= \emptyset. \end{aligned}$$

Then for all  $A \in \mathcal{L}$  and  $w \in W$ ,  $w \models A$  if and only if  $w \in \llbracket A \rrbracket$ . Thus  $A \models B$  if and only if  $\llbracket A \rrbracket \subseteq \llbracket B \rrbracket$ . If  $\llbracket A \rrbracket = W$ , then  $A$  is a tautology. If  $\llbracket A \rrbracket = \emptyset$ , then  $A$  is a contradiction.

For any (possibly infinite) set of formulas  $S$ , the model set of  $S$  can be defined  $\llbracket S \rrbracket = \bigcap \{\llbracket A \rrbracket \mid A \in S\}$ . Thus  $\llbracket S \rrbracket$  maps the theory to an element in

the power set  $\mathcal{P}(W)$ . A formula  $B$  is then entailed by  $S$ , that is,  $S \models B$ , if and only if  $\llbracket S \rrbracket \subseteq \llbracket B \rrbracket$ .

**Example 2.1.** Let us formalize our kiosk example. Let  $a$  denote an abstraction of the sentence “kiosk  $A$  is open”, and let  $b$  denote an abstraction of the sentence “kiosk  $B$  is open”. Let  $T = \{a \vee b, \neg a\}$  be our theory. If  $a$  and  $b$  are the atomic formulas in the language, then  $W = \mathcal{P}(a, b) = \{\emptyset, \{b\}, \{a\}, \{a, b\}\}$ . Let  $w_0, w_1, w_2$ , and  $w_3$  denote the sets accordingly. Thus  $\llbracket a \rrbracket = \{w_2, w_3\}$  and  $\llbracket b \rrbracket = \{w_1, w_3\}$ . The set  $\llbracket T \rrbracket$  can be calculated  $\llbracket T \rrbracket = \llbracket a \vee b \rrbracket \cap \llbracket \neg a \rrbracket = (\llbracket a \rrbracket \cup \llbracket b \rrbracket) \cap (W \setminus \llbracket a \rrbracket) = \{w_1, w_2, w_3\} \cap \{w_0, w_1\} = \{w_1\}$ . Because  $\llbracket T \rrbracket = \{w_1\} \subseteq \{w_1, w_3\} = \llbracket b \rrbracket$ ,  $T \models b$  holds.

Let  $W$  denote the set of all logically possible models. The members of the power set  $\mathcal{P}(W)$  are called *propositions*. A complete *field of propositions* is a non-empty set of subsets of  $W$  closed under complementation and arbitrary union and intersection [Spo88]. Let  $\mathcal{F}$  denote a field of propositions. A proposition is an *atom* of  $\mathcal{F}$ , if it is a minimal nonempty element of  $\mathcal{F}$ . Formally,  $X$  is an atom of  $\mathcal{F}$ , if and only if  $X \in \mathcal{F}$ ,  $X \neq \emptyset$ , and there is no  $Y \in \mathcal{F}$  with  $\emptyset \subset Y \subset X$ .

**Example 2.2.** Let us consider a language  $\mathcal{L}_P$  with  $P = \{a, b\}$ , and let  $W = \mathcal{P}(a, b)$  denote the set of all logically possible models. We write  $W = \{w_0, w_1, w_2, w_3\}$ , where  $w_0 = \emptyset$ ,  $w_1 = \{b\}$ ,  $w_2 = \{a\}$ , and  $w_3 = \{a, b\}$ . The set  $\mathcal{P}(W)$  itself is one example of a proposition field, another example of a field of propositions is its subset  $\mathcal{F}_1 = \{\emptyset, \{w_0, w_1\}, \{w_2, w_3\}, \{w_0, w_1, w_2, w_3\}\}$ . The atoms of  $\mathcal{F}_1$  are propositions  $\{w_0, w_1\}$  and  $\{w_2, w_3\}$ , the atoms of  $\mathcal{P}(W)$  are propositions  $\{w_0\}, \{w_1\}, \{w_2\}$  and  $\{w_3\}$ .

Let  $n = |\text{Voc}(\mathcal{L})|$  denote the cardinality of the set of the atomic formulas in the language. Then the cardinality  $|W| = 2^n$ , and  $|\mathcal{P}(W)| = 2^{2^n}$ . Note that if  $n$  is infinite, then none of the elements in  $W$  can be addressed by the language. However, we can address some of the elements in  $\mathcal{P}(W)$ .

## 2.3 The concept of the knowledge base

Recall that we will consider the knowledge base as an object that is given information about the external world and that answers queries about that world. We do not consider knowledge base as an autonomous agent. Instead, the knowledge base is taken as an object that the agent can use to record its epistemic state.

When communicating with the knowledge base, various methods can be used. The methods of the object include accessories to answer questions as well as methods to change the epistemic state.

The knowledge base is given information using propositional formulas. It is the agent that performs the classification of a change required and then selects the appropriate method to perform the change. The methods used to change the epistemic state are hereafter called *operators*.

Because the knowledge base is an object, its implementation is hidden. It can be accessed only by using its methods.

## 2.4 Epistemic states

According to Gärdenfors [Gär88, chapter 1], the epistemic state contains pieces of information, called epistemic elements. Each element is associated with an epistemic attitude that expresses how reliable that particular piece of information is considered to be.

There are two paradigms for representing epistemic states [HaV91]. When the epistemic state of an agent is a collection of logical formulas, a fact is considered to be known by the agent, if it can be proved using the formulas. The problems encountered in this case are the representation of the knowledge in an applicable language and theorem proving. Another way is to represent the epistemic state by some information structure that represents some semantical model. In this paradigm theorem proving is replaced by verifying the truth of the formula in the model [HaV91].

**Example 2.3.** Let us consider again our kiosk example  $T = \{a \vee b, \neg a\}$  with  $a$  denoting the sentence “kiosk A is open”, and  $b$  denoting the sentence “kiosk B is open”. Then by using the resolution rule, we can derive  $b$  from the two formulas in  $T$ .

When the information in an epistemic state is incomplete, several states of affairs are considered possible. In the case of knowledge, these alternatives are called *epistemic alternatives*, in the case of beliefs, *doxastic alternatives* [Hin62, chapter 3].

Let  $X$  denote a subset of the set of all logically possible alternatives (models)  $W$ . We say that  $X$  is the *set of possible models*, if it is the smallest subset of  $W$  that is known to contain the true situation of the external world, that is, it is the set of the epistemic alternatives. We say that a set  $Y$  is the *set of the most plausible models*, if it is the smallest subset of  $W$  that is believed to contain the true situation of the external world, that is, it is the set of the doxastic alternatives of the knowledge base.

**Example 2.4.** Let us recall our kiosk example 2.1 with  $T = \{a \vee b, \neg a\}$ ,  $W = \mathcal{P}(a, b) = \{\emptyset, \{b\}, \{a\}, \{a, b\}\}$ , and  $\llbracket T \rrbracket = \{\{b\}\}$ . Because no knowledge is involved here, all the logically possible models are epistemic alternatives,

thus the set  $W$  equals the set of possible models. There is only one doxastic alternative,  $\{b\}$ , thus the beliefs are complete.

In an epistemic state, a formula is *known*, if and only if it is true in all the epistemic alternatives of the state, and *believed*, if and only if it is true in all of the doxastic alternatives of the state. A formula is considered *possible*, if and only if it is true in at least one of the epistemic alternatives, and *compatible*, if and only if it is true in at least one of the doxastic alternatives. If the information in an epistemic state is incomplete, a propositional formula, say  $A$ , is either considered true, or its negation  $\neg A$  is considered true, or neither of the two formulas is considered true.

A knowledge base is called *competent*, if it is logically omniscient [Lev84]. *Full logical omniscience* means that knowledge is closed under logical entailment and *full logical omnibelievance* means that beliefs are closed under logical entailment. In the presence of logical omniscience, a propositional formula  $A$  is not known, if and only if its negation  $\neg A$  is considered possible, that is, consistent with the knowledge of the knowledge base. In the presence of logical omnibelievance, a formula is not believed if and only if its negation is considered compatible [Hin62, chapter 3], that is, consistent with the beliefs of the knowledge base. It is reasonable to assume that the knowledge base is competent, because the AGM-postulates explicitly insist on having full logical omnibelievance anyway.

Given a knowledge base  $T$ , we will usually use  $T_K$  to denote the collection of all propositional formulas known in  $T$  and  $T_B$  to denote the collection of all propositional formulas believed in  $T$  at any one moment. Thus we assume that that these set are closed, that is,  $cn(T_K) = T_K$  and  $cn(T_B) = T_B$ . The set  $T_K$  is called the *knowledge set* of the knowledge base  $T$ , and  $T_B$  is called the *belief set* of the knowledge base  $T$ . Because according to Hintikka [Hin62, chapter 3] one believes what one knows, we will assume that  $T_K \subseteq T_B$ , that is,  $\llbracket T_B \rrbracket \subseteq \llbracket T_K \rrbracket$ . We will later make this assumption a static constraint on epistemic states.

## 2.5 Types of epistemic change

When an epistemic input is received, it is first classified.

For those cases in which new information is received about a static world, Alchurrón, Gärdenfors and Makinson [AGM85] have named three types of change: expansion, revision, and contraction. An *expansion* is a monotonic, consistent insertion to a theory. A *revision* is an insertion of a formula to a theory inconsistent with the formula, with the result of a

consistent theory. A *contraction* is a retraction of a formula from the closure of the theory.

For those cases in which a change in the world is to be recorded, two types of change have been named [KaM91a]: an *update* is an insertion of a formula to a theory, and an *erasure* is a deletion of a formula from the closure of the theory.

We will reserve the term expansion for knowledge expansion, because it is knowledge that (due to its definition) increases monotonously. We will consider belief revision, contraction, update, and erasure as mentioned above, but we will also consider a new type of belief revision, namely *competing evidence*, which is non-prioritized, commutative belief revision. We will introduce competing evidence in detail later.

## 2.6 Conditionals

Although we do not consider learning conditional formulas, we may have conditionals in epistemic states, and maybe even accessories for querying them. The truth values of conditional formulas are determined by the Ramsey test. A conditional “if  $A$ , then  $B$  is true in an epistemic state  $T$ , if inserting  $A$  to  $T$  results in an epistemic state  $T'$ , in which  $B$  is believed.

Those conditionals whose semantics is tied to a revision operator in the Ramsey test will here be called *doxastic conditionals*. An example of a doxastic conditional might be “if I were to believe that John didn’t break the window, I would believe that Mary did”.

In counterfactuals the change takes place in the external world. A *counterfactual* is a conditional “would  $B$  be true, if  $A$  were true” in a situation where  $A$  is false. The sentence “if I had a pair of oars, I could cross the river” in a situation where one has a boat but no oars at hand exemplifies a counterfactual [Gra91a]. Counterfactuals should be evaluated by using update operators in the Ramsey test, therefore the triviality result by Gärdenfors cannot be applied to counterfactuals [Gra91a]. Counterfactuals say “should the world change”; doxastic conditionals say “should the beliefs change”. In doxastic conditionals the change takes place at the epistemic level.

## 2.7 Summary of the assumptions on the knowledge base

We will sum up our assumptions about the knowledge base as follows:

1. A knowledge base is considered as an object that is given information about an external world and that answers queries about that world. The knowledge base is a component of an agent, containing the epistemic state of the agent along with a collection of methods to change and access the state.
2. The knowledge base that is told propositional formulas; the set of the atomic formulas in the language may be infinite.
3. The knowledge base is competent but possibly incomplete and inaccurate.
4. There may be several tellers, some with, some without complete information about the world. The agent chooses one out of several various operators to change the epistemic state of the knowledge base.
5. There is no initial knowledge or belief about the external world, or about what the knowledge base will be told.
6. The external world may change at any moment without the knowledge base noticing it.
7. The epistemic state of the knowledge base depends only on the series of propositional formulas told to the base and the operators used when doing so. The epistemic state of the knowledge base does not change by itself as time passes or when changes take place in the external world. The epistemic state of the knowledge base changes only when the knowledge base is told something.
8. What is known in the epistemic state is believed in the epistemic state.

Because the series of formulas told to the knowledge base is finite, there is a point of time, when the knowledge base receives its first piece of information. We will use  $\tau$  to denote the epistemic state of the knowledge base before that moment. Then due to our assumptions, we then have  $\tau_K \equiv \tau_B \equiv \top$ .





# Chapter 3

## Examples of belief-change policies

In this chapter we shall review some examples of belief-change policies and belief-change operators. The operators involve only belief change, no knowledge is involved in these considerations. The main principles is these operators are maintaining consistency of beliefs and minimality of change. They are all examples of prioritized belief change, that is, the input is always prioritized over the old beliefs.

All the operators try to minimize the change in the epistemic state, but there are various policies as to what exactly is being minimized: the change in the set of formulas describing the world, the change in the set of doxastic alternatives, or the change in the epistemic entrenchment.

When trying to change the original theory as little as possible, *syntactically-oriented* operators minimize the change in the set of formulas, that is, in the description of the world. More semantical versions of these policies concern changing deductively closed sets of formulas, *closed theories*. *Semantically-oriented* operators minimize the change in the models of the theory. Some operators work on epistemic states with orderings or gradings of possible models and produce a new ordering or grading as a result of change. We will call such operators *ordering-oriented* and *grading-oriented* operators, correspondingly.

### 3.1 Syntactically-oriented belief-revision policies

Let us consider syntactically-oriented operators working on epistemic states represented by open sets of propositional formulas, that is, the sets of formulas that need not contain all formulas logically entailed by them. A formula is then believed, if it is logically entailed by the set of formulas, that is, given an epistemic state  $T$ , we have  $T_B = cn(T)$ . The state is

inconsistent, if the set of formulas is inconsistent.

Given a consistent (open) set of formulas, a theory  $T$ , and a satisfiable and nontautological formula  $A$ , a theory  $S$  accomplishes the addition of  $A$  to  $T$ , if  $A \in S$ , and it accomplishes the deletion of the formula  $A$  from the theory  $T$ , if  $S \not\models A$ . A theory accomplishes the addition or the deletion of a formula minimally, if there is no theory, which accomplishes it with fewer changes. If  $T \cup \{A\}$  is consistent, then it accomplishes the addition of  $A$  and is a solution. If  $T \not\models A$ , then  $T$  need not be changed when deleting  $A$ .

There may be several theories that minimally accomplish an addition or a deletion. If  $T \models A$ , then the minimal solutions to contracting  $T$  with  $A$  can be found among the theories that have been contracted with one formula from each proof of  $A$ . Let us define

$$T \downarrow A = \{B \subseteq T \mid B \not\models A \text{ and for all } C \subseteq T, \text{ if } B \subset C, \text{ then } C \models A\}.$$

The set  $T \downarrow A$  consists of those maximal subsets of  $T$  that do not entail  $A$ . Each of these is a solution to contracting the formula  $A$  from the theory  $T$ . If  $A$  is not a tautology, then the set  $T \downarrow A$  is nonempty. The minimal solutions to adding a satisfiable formula  $A$  to  $T$  can be found among the theories  $\{S \cup \{A\} \mid S \in T \downarrow \neg A\}$ .

There are various ways to deal with the ambiguity of having several minimal solutions. Disjunctive and intersective methods produce a new theory out of alternative solutions, while flock policy keeps all the alternatives separate. One policy uses prioritization of formulas as a basis of selection.

Let  $S_1, \dots, S_n$  be the theories that minimally accomplish an addition to a theory  $T$  or a deletion from a theory  $T$ . We then believe that the real world is represented by one of the truth distributions in the set  $\bigcup_{i=1}^n \llbracket S_i \rrbracket$ . Thus a theory  $T'$  is a result of the addition, if  $\llbracket T' \rrbracket = \bigcup_{i=1}^n \llbracket S_i \rrbracket$  [FUV83]. This property can be achieved by using the disjunction of the alternative theories [FUV83]. The *disjunction* of a finite set of theories  $S_1, \dots, S_n$  is defined [FUV83]  $\bigvee_{i=1}^n S_i = \{s_1 \vee \dots \vee s_n \mid s_i \in S_i, 1 \leq i \leq n\}$ . In the disjunctive method the result of contracting  $A$  from  $T$ ,  $T \bullet_d A$ , and the result of revising  $T$  by  $A$ ,  $T \circ_d A$  could be defined

$$\begin{aligned} T \bullet_d A &= \bigvee (T \downarrow A), \\ T \circ_d A &= \bigvee \{S \cup \{A\} \mid S \in T \downarrow \neg A\}. \end{aligned}$$

The method produces theories with long formulas hard to handle. The proposal is not meant to be a practical solution, but a solution with the ideal model set.

A conservative way to solve the problem of ambiguity is to define the result to be the *intersection* of the alternatives [Gin86, Rei87], that is

$$\begin{aligned} T \bullet_c A &= \bigcap (T \downarrow A), \\ T \circ_c A &= \bigcap \{S \cup \{A\} \mid S \in T \downarrow \neg A\}. \end{aligned}$$

Intersection of theories loses all information that is not certain. Such an operator may be suited for some applications, such as diagnosis [Rei87].

The idea of a *flock of theories* is to keep all the alternatives separate, so that the result of an addition or a deletion is the set of all the alternatives [Fag86]. In a singleton flock, after an addition or a deletion the model set is the same as that in a disjunctive method [Fag86]. The model sets may, however, differ in the long run as shown in the following example.

**Example 3.1.** [Fag86] Let us delete from the theory  $\{a, b\}$  the formula  $a \wedge b$ , which is in the closure of the theory. Both theories  $\{a\}$  and  $\{b\}$  accomplish the deletion with minimal changes. The intersection of these is an empty theory, whereas the disjunctive method produces the theory  $\{a \vee b\}$ . If the formula  $a \wedge b$  is deleted from a flock  $\{\{a, b\}\}$ , the new flock is  $\{\{a\}, \{b\}\}$ . If we delete from the flock  $\{\{a\}, \{b\}\}$  first the formula  $a$  and then the formula  $b$ , we get the flock  $\{\{\}\}$ . We then believe nothing but tautologies. Deleting first the formula  $a$  and then the formula  $b$  does not change the theory  $\{a \vee b\}$ .

Fagin, Ullman, and Vardi [FUV83] have considered representing the beliefs in a knowledge base by a set of *priorized formulas*. Each formula is assigned a *rank*. The rank zero is reserved for the integrity constraints. Alternative solutions that change formulas of higher priority are not considered minimal. In the proposal the prioritization of formulas is static and is left to the knowledge base administrator.

**Example 3.2.** Let  $T = \{(0, a \wedge b \rightarrow c), (1, b), (2, a)\}$ . The formula  $c$  is deleted from the theory. The new theory is  $T' = \{(0, a \wedge b \rightarrow c), (1, b)\}$ .

Then even though prioritization cuts off some alternatives, it does not completely solve the problem of ambiguity. We will later consider dynamic prioritization.

Filosofers [AGM85] have considered revising deductively closed sets of formulas, that is, given a theory  $T$ , we have  $T = cn(T) = T_B$ . The theory may be inconsistent. There is only one inconsistent belief set: it contains all formulas of the language.

Let  $T \bullet A$  denote the contraction of a theory  $T$  with a propositional formula  $A$ . Alchurrón, Gärdenfors and Makinson [AGM85] characterize contraction operators  $\bullet$  as follows:

$$T \bullet A = \begin{cases} \bigcap g(T \downarrow A), & \text{if } T \downarrow A \neq \emptyset \\ T, & \text{if } T \downarrow A = \emptyset, \end{cases}$$

where  $g$  is a selection function that selects only the most ‘important’ theories from the set  $T \downarrow A$ . In *full-meet contraction* all the theories of the set are chosen, in *maxichoice* only one of them, otherwise we have a *partial-meet contraction* determined by  $g$ .

In closed theories, all information has been made explicit. Then partial-meet contraction is necessary, otherwise  $\neg A \in T$  would imply  $T \circ A = Cn(\{A\})$ . Thus the original theory would be lost completely. This defect is known as the full meet contraction symptom [AGM85, FUV83]. Maxichoice has also its drawbacks. If we had a maxichoice operator, then revising a theory by a formula that is inconsistent with it always results in a complete theory [AGM85].

**Example 3.3.** Consider a theory  $T = Cn(\{a, b\})$ , where  $a$  and  $b$  are not tautologies, and full-meet contraction. Contracting the formula  $b$  from the theory gives a theory  $Cn(\emptyset)$ , that is,  $a$  is lost. The reason for this is that  $T$  includes the formula  $\neg a \vee b$  that is in the closure of  $b$ . The formula together with  $a$  implies  $b$ . Thus the set  $T \downarrow b$  includes one theory with the formula  $a$  but without the formula  $\neg a \vee b$ , and another theory with  $\neg a \vee b$  but without  $a$ . Thus  $a$  is not included in the intersection of the theories.

## 3.2 Semantically-oriented belief-revision policies

The semantically-oriented operators assume that the beliefs in the epistemic state can be represented by a propositional formula. The operators search for a new belief set whose models are those models of the new formula that are ‘closest’ to the models of the old belief set. The semantically-oriented operators we shall consider have been defined in various languages and environments, but we shall consider them in classical propositional logic.

Let  $T$  denote the propositional formula representing the belief set of the epistemic state. Given the set of all logically possible models  $W$ , let  $\llbracket T \rrbracket \subseteq W$  denote the models of  $T$ , that is, the set of the most plausible models. For each logically possible model  $w$  and  $w'$ , we define the *difference*  $w \Delta w'$  as the set of the atomic formulas having a different truth value in  $w$  and  $w'$ , that is  $w \Delta w' = (w \setminus w') \cup (w' \setminus w)$ . These sets are compared either by using the subset relation or the cardinalities of the sets. In the first case the difference  $w \Delta w_1$  is strictly smaller than the difference  $w \Delta w_2$ , if  $w \Delta w_1 \subset w \Delta w_2$ . In the second case the difference  $w \Delta w_1$  is strictly smaller than the difference  $w \Delta w_2$ , if  $|w \Delta w_1| < |w \Delta w_2|$ , that is, if the set  $w \Delta w_1$  has less elements than the set  $w \Delta w_2$ .

Let us review some definitions for minimal difference. Within a set  $X \subseteq Y$ , an element  $x$  is *minimal* according to the ordering  $\leq$  on  $Y$ , if  $x \in X$  and there is no  $x' \in X$  such that  $x' < x$ . We define

$$\min(X, \leq) = \{x \in X \mid \text{for all } x' \in X, x' \not< x\}.$$

Let  $A$  and  $T$  denote propositional formulas and let  $w$  and  $w'$  denote logically possible models. The following denotations will be used later on to find the 'closest' models of  $A$ :

$$\begin{aligned} \text{diff}(T, A) &= \min(\{w \Delta w' : w \in \llbracket T \rrbracket, w' \in \llbracket A \rrbracket\}, \subseteq), \\ \text{dist}(T, A) &= \min(\{|w \Delta w'| : w \in \llbracket T \rrbracket, w' \in \llbracket A \rrbracket\}, \leq), \\ p\text{-diff}(w, A) &= \min(\{w \Delta w' : w' \in \llbracket A \rrbracket\}, \subseteq), \\ p\text{-dist}(w, A) &= \min(\{|w \Delta w'| : w' \in \llbracket A \rrbracket\}, \leq). \end{aligned}$$

When determining the minimal difference, *diff* and *p\_diff* use the subset relation in comparison, *dist* and *p\_dist* compare the cardinalities of the sets. The first two of the functions search for the minimal differences between two model sets, while the last two functions pointwise compare one model to a set of models.

We shall look into four semantically-oriented belief-revision operators: Dalal's [Dal88] operator  $\circ_D$ , Satoh's [Sat88] operator  $\circ_S$ , Weber's [Web85] operator  $\circ_W$ , and Borgida's [Bor85] operator  $\circ_B$ . For all the operators, we define  $\llbracket T \circ A \rrbracket = \llbracket A \rrbracket$  whenever  $\llbracket T \rrbracket = \emptyset$ , otherwise the operators are defined as follows [KaM91b]:

$$\begin{aligned} \llbracket T \circ_D A \rrbracket &= \{w \in \llbracket A \rrbracket : \exists w' \in \llbracket T \rrbracket, |w \Delta w'| = \text{dist}(T, A)\}, \\ \llbracket T \circ_S A \rrbracket &= \{w \in \llbracket A \rrbracket : \exists w' \in \llbracket T \rrbracket, w \Delta w' \in \text{diff}(T, A)\}, \\ \llbracket T \circ_W A \rrbracket &= \{w \in \llbracket A \rrbracket : \exists w' \in \llbracket T \rrbracket, w \Delta w' \subseteq \bigcup \text{diff}(T, A)\}, \\ \llbracket T \circ_B A \rrbracket &= \begin{cases} \llbracket T \wedge A \rrbracket, & \text{if } T \wedge A \text{ is satisfiable,} \\ \bigcup_{w \in \llbracket T \rrbracket} \{w' \in \llbracket A \rrbracket : w \Delta w' \in p\text{-diff}(w, A)\} & \text{otherwise.} \end{cases} \end{aligned}$$

The operators define rules to produce the new set of models, but they do not define the outcome of the addition as a formula. A formula  $T'$  may be the result of the revision  $T \circ A$ , if  $\llbracket T' \rrbracket = \llbracket T \circ A \rrbracket$ .

Let us compare the operators in the next examples.

**Example 3.4.** Let us revise a theory  $\neg a \wedge \neg b \wedge \neg c$  by the formula  $(a \wedge b) \vee c$ . Let  $W = \mathcal{P}(\{a, b, c\})$ , that is,  $W = \{w_0, w_1, \dots, w_7\}$  with  $w_0 = \emptyset$ ,  $w_1 = \{c\}$ ,  $w_2 = \{b\}$ ,  $w_3 = \{b, c\}$ ,  $w_4 = \{a\}$ ,  $w_5 = \{a, c\}$ ,  $w_6 = \{a, b\}$ , and  $w_7 = \{a, b, c\}$ . Thus  $\llbracket \neg a \wedge \neg b \wedge \neg c \rrbracket = \{w_0\}$ , and  $\llbracket (a \wedge b) \vee c \rrbracket = \{w_1, w_3, w_5, w_6, w_7\}$ . We calculate:

$$\begin{aligned} \text{dist}(\neg a \wedge \neg b \wedge \neg c, (a \wedge b) \vee c) &= |w_0 \Delta w_1| = 1, \\ \llbracket (\neg a \wedge \neg b \wedge \neg c) \circ_D ((a \wedge b) \vee c) \rrbracket &= \{w_1\}. \end{aligned}$$

$$\begin{aligned} \text{diff}(\neg a \wedge \neg b \wedge \neg c, (a \wedge b) \vee c) &= \{w_0 \Delta w_1, w_0 \Delta w_6\} = \{\{c\}, \{a, b\}\}, \\ \llbracket (\neg a \wedge \neg b \wedge \neg c) \circ_S ((a \wedge b) \vee c) \rrbracket &= \{w_1, w_6\}. \end{aligned}$$

$$\begin{aligned} \bigcup \text{diff}(\neg a \wedge \neg b \wedge \neg c, (a \wedge b) \vee c) &= \{a, b, c\}, \\ \llbracket (\neg a \wedge \neg b \wedge \neg c) \circ_W ((a \wedge b) \vee c) \rrbracket &= \{w_1, w_3, w_5, w_6, w_7\}. \end{aligned}$$

**Example 3.5.** To compare the operators of Satoh and Borgida, let us consider a theory  $\neg a \wedge \neg b$ . We will add to the theory the formula  $(a \wedge b) \vee (b \wedge c)$ . Given the set  $W = \mathcal{P}(\{a, b, c\})$  of possible models as before,  $\llbracket \neg a \wedge \neg b \rrbracket = \{w_0, w_1\}$  and  $\llbracket (a \wedge b) \vee (b \wedge c) \rrbracket = \{w_3, w_6, w_7\}$ . We calculate the revision as follows:

$$\begin{aligned} \text{diff}(\neg a \wedge \neg b, (a \wedge b) \vee (b \wedge c)) &= \{w_1 \Delta w_3\} = \{\{b\}\}, \\ \llbracket (\neg a \wedge \neg b) \circ_S ((a \wedge b) \vee (b \wedge c)) \rrbracket &= \{w_3\}. \end{aligned}$$

$$\begin{aligned} p\text{-diff}(w_0, (a \wedge b) \vee (b \wedge c)) &= \{w_0 \Delta w_3, w_0 \Delta w_6\} = \{\{b, c\}, \{a, b\}\}, \\ p\text{-diff}(w_1, (a \wedge b) \vee (b \wedge c)) &= \{w_1 \Delta w_3\} = \{\{b\}\}, \\ \llbracket (\neg a \wedge \neg b) \circ_B ((a \wedge b) \vee (b \wedge c)) \rrbracket &= \{w_3, w_6\} \cup \{w_3\} = \{w_3, w_6\}. \end{aligned}$$

When Satoh's operator is used, the new theory can be expressed by a formula  $\neg a \wedge b \wedge c$ , when Borgida's operator is used, the result is  $(\neg a \wedge b \wedge c) \vee (a \wedge b \wedge \neg c)$ .

Belief-change operators can be compared using their permissiveness. An operator  $\circ_1$  is more *permissive* [Win88a] than an operator  $\circ_2$ , if for all propositional formulas  $T$  and  $A$ ,  $\llbracket T \circ_2 A \rrbracket \subseteq \llbracket T \circ_1 A \rrbracket$  [Win88a]. The more permissive the operator is, the larger is the set of the most plausible models of the result. Semantically-oriented operators that use the subset relation are more permissive than those using cardinalities to determine orderings among models. Thus Satoh's operator is more permissive than Dalal's operator. The operators that determine distances pointwise are more permissive than corresponding operators that do not; that is why Borgida's operator is more permissive than the operator by Satoh. Weber's operator is also more permissive than Satoh's operator.

### 3.3 Semantically-oriented belief-update policies

We shall now review some semantically-oriented belief-update operators. Again, the operators define the rules to produce the new set of models, but

do not define the outcome of the addition as a formula. We shall look into the following semantically-oriented belief-update operators: The operator  $\diamond_F$  defined by Forbus [For89], Winslett's [Win86] *minimal-change update operator*  $\diamond_W$ , Winslett's [Win90, chapter 3] *standard update operator*  $\diamond_s$ , and its version  $\diamond_{s\downarrow}$  defined by Herzog and Rifi [HeR98].

Let  $A$  and  $T$  denote propositional formulas, and  $Voc(A)$  the set of the atomic formulas of  $A$ . The operators are defined as follows [KaM91b]:

$$\begin{aligned} \llbracket T \diamond_F A \rrbracket &= \bigcup_{w \in \llbracket T \rrbracket} \{w' \in \llbracket A \rrbracket : |w \Delta w'| = p\_dist(w, A)\}, \\ \llbracket T \diamond_W A \rrbracket &= \bigcup_{w \in \llbracket T \rrbracket} \{w' \in \llbracket A \rrbracket : w \Delta w' \in p\_diff(w, A)\}, \\ \llbracket T \diamond_s A \rrbracket &= \bigcup_{w \in \llbracket T \rrbracket} \{w' \in \llbracket A \rrbracket : w \Delta w' \subseteq Voc(A)\}, \text{ and} \\ \llbracket T \diamond_{s\downarrow} A \rrbracket &= \bigcup_{w \in \llbracket T \rrbracket} \{w' \in \llbracket A \rrbracket : w \Delta w' \subseteq \bigcup diff(\neg A, A)\}. \end{aligned}$$

A formula  $T'$  can serve as the result of an update  $T \diamond A$ , if  $\llbracket T' \rrbracket = \llbracket T \diamond A \rrbracket$ . The theory may be expressed by a formula in disjunctive normal form. Del Val [DVa92] among others have given algorithms to accomplish the update using the Winslett minimal-change update operator and disjunctive normal forms.

**Example 3.6.** Let us update the theory  $\neg a \wedge \neg b$  by the formula  $(a \wedge b) \vee c$ . Let  $W = \mathcal{P}(\{a, b, c\})$  denote the set of all logically possible models as before. Given formulas  $\neg a \wedge \neg b$  and  $(a \wedge b) \vee c$ , we have  $\llbracket \neg a \wedge \neg b \rrbracket = \{w_0, w_1\}$  and  $\llbracket (a \wedge b) \vee c \rrbracket = \{w_1, w_3, w_5, w_6, w_7\}$ . We calculate:

$$\begin{aligned} p\_dist(w_0, (a \wedge b) \vee c) &= |w_0 \Delta w_1| = 1, \\ p\_dist(w_1, (a \wedge b) \vee c) &= |w_1 \Delta w_1| = 0, \\ \llbracket (\neg a \wedge \neg b) \diamond_F ((a \wedge b) \vee c) \rrbracket &= \{w_1\} \cup \{w_1\} = \{w_1\}. \end{aligned}$$

$$\begin{aligned} p\_diff(w_0, (a \wedge b) \vee c) &= \{w_0 \Delta w_1, w_0 \Delta w_6\} = \{\{c\}, \{a, b\}\}, \\ p\_diff(w_1, (a \wedge b) \vee c) &= \{w_1 \Delta w_1\} = \{\emptyset\}, \\ \llbracket (\neg a \wedge \neg b) \diamond_W ((a \wedge b) \vee c) \rrbracket &= \{w_1, w_6\} \cup \{w_1\} = \{w_1, w_6\}. \end{aligned}$$

When using  $p\_dist$ , the model  $w_1$  was incidentally closest to both  $w_0$  and  $w_1$ .

**Example 3.7.** Let us consider updating the theory  $\neg a \wedge \neg b$  by equivalent formulas  $a \vee (c \wedge (b \vee \neg b))$  and  $a \vee c$ . Let  $W = \mathcal{P}(\{a, b, c\})$  denote the set of possible models as before. Then  $\llbracket a \vee (c \wedge (b \vee \neg b)) \rrbracket = \llbracket a \vee c \rrbracket = \{w_1, w_3, w_4, w_5, w_6, w_7\}$ . We calculate as follows:

$$\begin{aligned} Voc(a \vee (c \wedge (b \vee \neg b))) &= \{a, b, c\}, \\ \llbracket (\neg a \wedge \neg b) \diamond_s (a \vee (c \wedge (b \vee \neg b))) \rrbracket &= \{w_1, w_3, w_4, w_5, w_6, w_7\}, \\ Voc(a \vee c) &= \{a, c\}, \\ \llbracket (\neg a \wedge \neg b) \diamond_s (a \vee c) \rrbracket &= \{w_1, w_4, w_5\}. \end{aligned}$$

Thus even though  $a \vee c \equiv a \vee (c \wedge (b \vee \neg b))$ ,  $\llbracket (\neg a \wedge \neg b) \diamond_s (a \vee c) \rrbracket \neq \llbracket (\neg a \wedge \neg b) \diamond_s (a \vee (c \wedge (b \vee \neg b))) \rrbracket$ .

**Example 3.8.** Both ordering- and grading-oriented operators dynamically construct a new ordering for the new epistemic state. Let us update the theory  $\neg a \wedge \neg b$  by the formula  $a \vee (c \wedge (b \vee \neg b))$  using the operator defined by Herzig and Rifi. Given the set  $W = \mathcal{P}(\{a, b, c\})$  of possible models as before, we calculate:

$$\begin{aligned} \bigcup \text{diff}(a \vee (c \wedge (b \vee \neg b)), \neg(a \vee (c \wedge (b \vee \neg b)))) &= \{a, c\}, \\ \llbracket (\neg a \wedge \neg b) \diamond_{s\downarrow} (a \vee (c \wedge (b \vee \neg b))) \rrbracket &= \{w_1, w_4, w_5\}, \\ \llbracket (\neg a \wedge \neg b) \diamond_{s\downarrow} (a \vee c) \rrbracket &= \{w_1, w_4, w_5\}. \end{aligned}$$

Winslett's minimal-change update-operator is more permissive than the operator by Forbus, because it uses the subset relation for comparison. Winslett's standard update operator is the most permissive update operator. Also the version by Herzig and Rifi is permissive: it is aimed to allow to enter disjunctive input into the knowledge base.

### 3.4 Ordering-oriented belief-revision policies

Ordering-oriented belief-revision operators work on epistemic states that carry a total pre-order on the set of all logically possible models. A pre-order  $\leq$  on a set  $Y$  is a reflexive and transitive relation, a subset of  $Y \times Y$ . It is *total*, if for all  $x, y \in Y$ , either  $x \leq y$  or  $y \leq x$  holds. A *partial* pre-order is a pre-order that need not be total. For all  $x, y \in Y$ , if  $x \leq y$  but  $y \not\leq x$ , we say  $x < y$ . If  $x \leq y$  and  $y \leq x$ , we say  $x \sim y$ . Recall that we defined  $\text{min}(X, \leq) = \{x \in X \mid \text{for all } x' \in X, x' \not< x\}$  This definition can be applied even to partial pre-orders.

The set of the most plausible models of an epistemic state is the set of the models minimal in the ordering. As a result of revising an epistemic state, the ordering-oriented belief-revision operators produce the ordering for the new epistemic state. These operators are called memory operators because of the ordering carrying some 'memory' in the epistemic state. This memory is not contained in the belief set of the epistemic state, but can be expressed by doxastic conditionals. We assume that in the initial state  $\tau$ , we have  $\leq_\tau = W \times W$ .

Let  $W$  denote the set of all logically possible models, and let  $\leq_T$  denote the ordering in an epistemic state  $T$ . Then  $\llbracket T \rrbracket = \text{min}(W, \leq_T)$  whenever  $\llbracket T \rrbracket \neq \emptyset$ . There are two orderings involved in these two memory operators: a dominating ordering  $\leq_A$  determined by the new formula, and the



complementing ordering  $\leq_T$  contained in the epistemic state. The operators differ in the way the dominating ordering is created. In any case, the ordering in the state  $T \circ A$  is defined as follows:

$$w \leq_{T \circ A} w' \text{ if and only if } w <_A w', \text{ or } w \sim_A w' \text{ and } w \leq_T w'.$$

The *basic memory revision operator*  $\circ_{bm}$  [KoP01] has a straightforward way to determine the ordering for a propositional formula  $A$ . A total pre-order  $\leq_A^b$  on  $W$  is defined as follows [KoP01]:

$$w \leq_A^b w' \text{ if and only if } w \models A \text{ or } w' \not\models A.$$

The *Dalal memory operator*  $\text{circ}_{dm}$  [KoP01] uses pointwise distance  $p\_dist$  to create the ordering for a propositional formula  $A$ : [KoP01]:

$$w \leq_A^d w' \text{ if and only if } p\_dist(w, A) \leq p\_dist(w', A).$$

**Example 3.9.** Let us revise the theory  $\neg a \wedge \neg b \wedge \neg c$  by the formula  $(a \wedge b) \vee c$ . Given the set of all logically possible models  $W = \{w_0, w_1, \dots, w_7\}$  with  $\llbracket a \rrbracket = \{w_4, w_5, w_6, w_7\}$ ,  $\llbracket b \rrbracket = \{w_2, w_3, w_6, w_7\}$ , and  $\llbracket c \rrbracket = \{w_1, w_3, w_5, w_7\}$  as before, we have  $\llbracket \neg a \wedge \neg b \wedge \neg c \rrbracket = \{w_0\}$  and  $\llbracket (a \wedge b) \vee c \rrbracket = \{w_1, w_3, w_5, w_6, w_7\}$ . We will consider state  $T = \tau \circ_{bm} (\neg a \wedge \neg b \wedge \neg c)$ , where  $\tau$  denotes the initial state with  $\leq_\tau = W \times W$ . Thus in the orderings  $\leq_{\neg a \wedge \neg b \wedge \neg c}^b$  and  $\leq_T$ , the model  $w_0$  is minimal, while  $w_1, \dots, w_7$  are maximal. We write  $w_0 <_T w_1, w_2, w_3, w_4, w_5, w_6, w_7$  for short. Orderings  $\leq_{(a \wedge b) \vee c}^b$  and  $\leq_{T'}$  for the state  $T' = T \circ_{bm} ((a \wedge b) \vee c)$  are calculated as follows:

$$w_1, w_3, w_5, w_6, w_7 <_{(a \wedge b) \vee c}^b w_0, w_2, w_4,$$

$$w_1, w_3, w_5, w_6, w_7 <_{T'} w_0 <_{T'} w_2, w_4.$$

Thus  $\llbracket T \circ_{bm} (a \wedge b) \vee c \rrbracket = \{w_1, w_3, w_5, w_6, w_7\} = \llbracket (a \wedge b) \vee c \rrbracket$ .

**Example 3.10.** Given the set  $W = \mathcal{P}(\{a, b, c\})$  as in the previous examples, let us revise the state  $T = \tau \circ_{dm} \neg a \wedge \neg b \wedge \neg c$  by the formula  $(a \wedge b) \vee c$ . Let  $T'$  denote the state  $T \circ_{dm} (a \wedge b) \vee c$ . We calculate the orderings as follows:

$$w_0 <_T w_1, w_2, w_4 <_T w_3, w_5, w_6 <_T w_7,$$

$$w_1, w_3, w_5, w_6, w_7 <_{(a \wedge b) \vee c}^d w_0, w_2, w_4,$$

$$w_1 <_{T'} w_3, w_5, w_6 <_{T'} w_7 <_{T'} w_0 <_{T'} w_2, w_4.$$

We get  $\llbracket T \circ_{dm} (a \wedge b) \vee c \rrbracket = \{w_1\} = \llbracket \neg a \wedge \neg b \wedge \neg c \rrbracket$ .

### 3.5 Grading-oriented belief-revision policies

Spohn [Spo88] has considered epistemic states as mappings from a set of possible models to the class of ordinals. The ordinal assigned to a possible model expresses the *disbelief grade* of the model.

Spohn [Spo88] introduces his ordinal conditional functions in the context of complete fields of propositions on logically possible models. A complete *field of propositions* is a non-empty set of subsets of  $W$  closed under complementation and arbitrary union and intersection [Spo88]. Let  $\mathcal{F}$  denote a field of propositions. A proposition is an *atom* of  $\mathcal{F}$ , if it is a minimal nonempty element of  $\mathcal{F}$ .

Let  $W$  denote the set of all logically possible models and let  $\mathcal{F}$  be a complete field of propositions on  $W$ . Let  $\kappa$  denote a function from  $W$  into the class of ordinals. Spohn [Spo88] calls  $\kappa$  a  *$\mathcal{F}$ -measurable ordinal conditional function*, if and only if  $\{w \in W \mid \kappa(w) = 0\} \neq \emptyset$  and for all atoms  $X \in \mathcal{F}$  and all  $w, w' \in X$ ,  $\kappa(w) = \kappa(w')$ . That is, the set of models mapped to the ordinal zero must be nonempty, and all models in a same atom of the proposition field must always be mapped to the same ordinal. The set of the most plausible models in the state  $\kappa$  is the set  $\kappa^{-1}(0) = \{w \in W \mid \kappa(w) = 0\}$ . A propositional formula  $A$  is then believed in state  $\kappa$ , if  $\kappa^{-1}(0) \subseteq \llbracket A \rrbracket$ .

In Spohn's epistemic state the disbelief grade of a collection of models is the smallest disbelief grade of the models in it, that is, for any  $X \in \mathcal{F} \setminus \{\emptyset\}$ ,  $\kappa(X) = \min\{\kappa(w) \mid w \in X\}$ . Then, for any  $X \in \mathcal{F} \setminus \{\emptyset, W\}$ ,  $\kappa(X) = 0$  or  $\kappa(W \setminus X) = 0$  or both, and for all  $X, Y \in \mathcal{F} \setminus \{\emptyset, W\}$ ,  $\kappa(X \cup Y) = \min\{\kappa(X), \kappa(Y)\}$ . A formula  $A$  is more plausible than a formula  $B$ , if  $\kappa(\llbracket A \rrbracket) < \kappa(\llbracket B \rrbracket)$  or  $\kappa(W \setminus \llbracket A \rrbracket) > \kappa(W \setminus \llbracket B \rrbracket)$ .

To keep things simple, we restrict our functions  $\kappa$  to mappings from possible models to natural numbers and call them *ranking functions* [Spo99]. Then correspondingly, grades are called *ranks*.

Spohn [Spo88] has defined  $\alpha$ -conditionalization for changing the epistemic state represented by ordinal conditional functions. Let  $A$  denote a formula that is not a contradiction nor a tautology. In  $\alpha$ -conditionalization  $\kappa_{A,\alpha}$  the ranking function  $\kappa$  is changed so that the rank of the set  $\llbracket A \rrbracket$  is set to zero and the rank of the set  $\llbracket \neg A \rrbracket$  is set to  $\alpha$ :

$$\kappa_{A,\alpha}(w) = \begin{cases} \kappa(w) - \kappa(\llbracket A \rrbracket) & \text{if } w \in \llbracket A \rrbracket, \\ \kappa(w) - \kappa(\llbracket \neg A \rrbracket) + \alpha & \text{if } w \notin \llbracket A \rrbracket. \end{cases}$$

In the new epistemic state the ordering among the models in  $\llbracket A \rrbracket$  has not changed, neither has the ordering among the models that are not in the set  $\llbracket A \rrbracket$ , but these two collections of models have been shifted in the ordering compared to each other.

The change may be cancelled, if the previous rank of the proposition is known [Spo88]. When the rank of a proposition is set to zero, we have a case of contraction.

Ranking functions resemble probabilistic modelling. In probability theory, epistemic attitudes are modelled by probabilities or probability ranges. The total probability mass does not vary; it only gets redistributed when changes occur. The probability of a set of models is the sum of the probabilities of the models in it, whereas when ranking functions are used, the rank of a set of models is the best rank of the models in the proposition [Spo88].

Darwiche and Pearl [DaP94] have defined a belief-revision operator using Spohn's  $\alpha$ -conditionalization: if the formula  $A$  is believed in the state  $T$ , then there is no change, otherwise the rank of  $\neg A$  is set to 1. Let  $\circ_r$  denote the revision operator by Darwiche and Pearl. Thus, if  $\kappa$  represents the state  $T$ , and  $\kappa'$  represents the state  $T \circ_r A$ , then

$$\kappa' = \begin{cases} \kappa & \text{if } \kappa(\llbracket \neg A \rrbracket) > 0, \\ \kappa_{A,1} & \text{if } \kappa(\llbracket \neg A \rrbracket) = 0. \end{cases}$$

The set of the most plausible models of the revised state is  $\{w \in W \mid \kappa'(w) = 0\}$ .

Revision may be irreversible when using the operator by Darwiche and Pearl, because the plausibility grade is always set to 1.

The following example shows us that the operator distinguishes between the states  $(\tau \circ a) \circ b$  and  $\tau \circ (a \wedge b)$  as the syntactically oriented operators distinguish between the theories  $\{a, b\}$  and  $\{a \wedge b\}$ . Here  $\tau$  denotes the initial state, the state of total ignorance. The function  $\kappa_\tau$  maps all possible models to the rank 0.

**Example 3.11.** Let us consider revising epistemic states  $(\tau \circ_r a) \circ_r b$  and  $\tau \circ_r (a \wedge b)$  by a formula  $\neg a$ . Let  $W = \{w_0, w_1, w_2, w_3\}$  with  $\llbracket a \rrbracket = \{w_2, w_3\}$  and  $\llbracket b \rrbracket = \{w_1, w_3\}$ . The ranks are calculated as follows:

	$w_0$	$w_1$	$w_2$	$w_3$
$\tau$	0	0	0	0
$\tau \circ_r a$	1	1	0	0
$(\tau \circ_r a) \circ_r b$	2	1	1	0
$((\tau \circ_r a) \circ_r b) \circ_r \neg a$	1	0	2	1
$\tau$	0	0	0	0
$\tau \circ_r (a \wedge b)$	1	1	1	0
$(\tau \circ_r (a \wedge b)) \circ_r \neg a$	0	0	2	1

The set of most plausible models in  $((\tau \circ_r a) \circ_r b) \circ_r \neg a$  is  $\{w_1\}$ . The set of most plausible models in  $(\tau \circ_r (a \wedge b)) \circ_r \neg a$  is  $\{w_0, w_1\}$ . Thus  $((\tau \circ_r a) \circ_r b) \circ_r \neg a$  and  $(\tau \circ_r (a \wedge b)) \circ_r \neg a$  are not equivalent, even though the sets of most plausible models of the states  $(\tau \circ_r a) \circ_r b$  and  $\tau \circ_r (a \wedge b)$  are identical.

### 3.6 Comparison of belief-change policies

Let us review the characteristics of the belief-change policies.

When revising open theories, the change depends on whether a formula is explicitly in the theory or implied by it [Win88a]. The formulas implied by the theory are always automatically revised. The result of the revision may depend on the syntax. The results of contracting theories  $\{a, b\}$  and  $\{a \wedge b\}$  with  $a$  differ: in the latter case we no longer believe that  $b$ . Having the formula  $a \wedge b$  in the theory expresses dependence of  $a$  and  $b$  on each other.

When using semantically-oriented operators, the effect of a rule is transient. In order to make a rule persist, integrity constraints are needed, but then again, they are completely persistent. The semantically-oriented belief-revision operators have been criticized [Bre91] for treating in the same way the formulas explicitly told to the knowledge base and the formulas implied by them. On one hand, what we deduced based on false beliefs should be revised along the false beliefs. On the other hand, that may not be the case in update: what we deduced may still hold even if changes had taken place. That explains why the operators for update are semantically oriented.

Both ordering- and grading-oriented operators dynamically construct a new ordering for the new epistemic state. Example 3.11 showed that the grading-oriented belief-revision operator by Darwiche and Pearl distinguishes between the states  $(\tau \circ a) \circ b$  and  $\tau \circ (a \wedge b)$  as the syntactically-oriented operators distinguish between the theories  $\{a, b\}$  and  $\{a \wedge b\}$ . Let us next exemplify how this grading-oriented operator after having revised the state  $\tau \circ (a \wedge b)$  by the formulas  $\neg a$  and  $a$  reaches the original state unlike the syntactically-oriented operators when adding the formulas  $\neg a$  and  $a$  to the theory  $\{a \wedge b\}$ .

**Example 3.12.** Let  $W = \{w_0, w_1, w_2, w_3\}$  denote the set of all logically possible models with  $\llbracket a \rrbracket = \{w_2, w_3\}$  and  $\llbracket b \rrbracket = \{w_1, w_3\}$ . Let us revise the state  $\tau \circ_r (a \wedge b)$  by the formulas  $\neg a$  and  $a$ :

	$w_0$	$w_1$	$w_2$	$w_3$
$\tau \circ_r (a \wedge b)$	1	1	1	0

$$\begin{array}{l} (\tau \circ_r (a \wedge b)) \circ_r \neg a \quad \quad \quad 0 \quad 0 \quad 2 \quad 1 \\ ((\tau \circ_r (a \wedge b)) \circ_r \neg a) \circ_r a \quad \quad 1 \quad 1 \quad 1 \quad 0 \end{array}$$

The resulting state is identical to the state  $\tau \circ (a \wedge b)$ . The result can be obtained by using any of the semantically oriented operators, but the syntactically oriented operators cannot regain the state.

Let us now exemplify how the operator by Darwiche and Pearl after having revised the state  $\tau \circ (a \vee b)$  by the formulas  $a$  and  $\neg a$  results in believing  $b$ . This is a “memory property” that the semantically-oriented revision operators lack but the syntactically-oriented operators usually have.

**Example 3.13.** Assume  $W$ ,  $\llbracket a \rrbracket$  and  $\llbracket b \rrbracket$  as in the previous example. Let us revise the state  $\tau \circ (a \vee b)$  by the formulas  $a$  and  $\neg a$ :

$$\begin{array}{l} \tau \circ_r (a \vee b) \quad \quad \quad w_0 \quad w_1 \quad w_2 \quad w_3 \\ (\tau \circ_r (a \vee b)) \circ_r a \quad \quad \quad 1 \quad 0 \quad 0 \quad 0 \\ ((\tau \circ_r (a \vee b)) \circ_r a) \circ_r \neg a \quad \quad 2 \quad 1 \quad 0 \quad 0 \\ ((\tau \circ_r (a \vee b)) \circ_r \neg a) \circ_r a \quad \quad 1 \quad 0 \quad 1 \quad 1 \end{array}$$

Because  $w_1 \in \llbracket b \rrbracket$ , the result logically entails the formula  $b$ .

In his work on ordinal conditional functions, Spohn [Spo88] considered belief revision policies such as the basic memory operator. He found, however, many defects in ordering-oriented revision-operators. First, in these proposals epistemic changes are not reversible. For reversibility, gaps in the ordering, that is, ranking functions are needed. Secondly, the firmness of the new belief cannot be chosen as in  $\alpha$ -conditionalization. Thirdly, epistemic changes are not commutative even in case of learning new atomic formulas.

**Example 3.14.** Let  $W = \{w_0, w_1, w_2, w_3\}$  denote the set of all logically possible models with  $\llbracket a \rrbracket = \{w_2, w_3\}$  and  $\llbracket b \rrbracket = \{w_1, w_3\}$ . Let us revise the initial state  $\tau$  by formulas  $a$  and  $b$ .

a) If we use the operator by Darwiche and Pearl, the change is commutative:

$$\begin{array}{l} \tau \circ_r a \quad \quad \quad w_0 \quad w_1 \quad w_2 \quad w_3 \\ (\tau \circ_r a) \circ_r b \quad \quad \quad 1 \quad 1 \quad 0 \quad 0 \\ \tau \circ_r b \quad \quad \quad 2 \quad 1 \quad 1 \quad 0 \\ (\tau \circ_r b) \circ_r a \quad \quad \quad 1 \quad 0 \quad 1 \quad 0 \\ (\tau \circ_r b) \circ_r a \quad \quad \quad 2 \quad 1 \quad 1 \quad 0 \end{array}$$

Here in both cases, the models of  $a \wedge \neg b$  are as plausible as the models of  $\neg a \wedge b$ .

**b)** If we would use the basic memory operator, the change would not be commutative. Let us here express the orderings in the states as ranking functions to make the comparison easier:

	$w_0$	$w_1$	$w_2$	$w_3$
$\tau \circ_{bm} a$	1	1	0	0
$(\tau \circ_{bm} a) \circ_{bm} b$	3	1	2	0
$\tau \circ_{bm} b$	1	0	1	0
$(\tau \circ_{bm} b) \circ_{bm} a$	3	2	1	0

In the state  $(\tau \circ_{bm} a) \circ_{bm} b$  the models of  $\neg a \wedge b$  are considered more plausible than any model of  $a \wedge \neg b$ , in the state  $(\tau \circ_{bm} b) \circ_{bm} a$  the models of  $a \wedge \neg b$  are considered more plausible than any model of  $\neg a \wedge b$ .

However, commutativity and reversibility are not considered to be necessary characteristics of a belief-revision operator, as we shall see, when we next look into the rationality criteria for belief change.

# Chapter 4

## Rationality criteria for belief revision and update

We shall next take a look at the rationality criteria that have been proposed on belief change. The rationality criteria act as dynamic integrity constraints on the behaviour of the knowledge base.

The criteria depend on the type of change. We shall look into five sets of postulates: the AGM-postulates for belief revision and contraction [AGM85], the DP-postulates for iterated belief revisions [DaP94, DaP97], and the KM-postulates for belief update and erasure [KaM91a]. We will also take a look at representation theorems for some sets of postulates. The representation theorems are concrete constructions or modellings for the processes guarded by those postulates.

There are  $n$  No knowledge sets of the epistemic states are involved in these postulates. When formalizing the postulates, we will nevertheless make the distinction between an epistemic state and its belief set explicit. We will analyze the postulates for belief revision to show the importance of making such a distinction.

At the end of this chapter we shall discuss incorporating integrity constraints into belief change and some related work.

### 4.1 Postulates for belief revision

In belief revision new information is obtained about a static world. Let  $\circ$  denote a belief-revision operator,  $T$  an epistemic state with a belief set  $T_B$  and let  $A$  and  $B$  denote propositional formulas. The AGM-postulates

[AGM85] for belief revision are phrased here as follows<sup>1</sup>:

- (R1):  $(T \circ A)_B \models A$ .
- (R2): If  $T_B \not\models \neg A$ , then  $(T \circ A)_B \equiv T_B \cup \{A\}$ .
- (R3): If  $A$  is satisfiable, then  $(T \circ A)_B$  is consistent.
- (R4): If  $A \equiv B$ , then  $(T \circ A)_B \equiv (T \circ B)_B$ .
- (R5):  $(T \circ A)_B \cup \{B\} \models (T \circ (A \wedge B))_B$ .
- (R6): If  $(T \circ A)_B \not\models \neg B$ , then  $(T \circ (A \wedge B))_B \models (T \circ A)_B \cup \{B\}$ .

Postulate (R1) says that the new piece of information is accepted, that is, the insertion succeeds. Postulate (R2) says that if the new piece of information is compatible with the old beliefs, neither is any of them discarded nor is anything not entailed by the old beliefs and the new information added to the belief set. Postulate (R3) says that adding a satisfiable formula to the belief set must not make it inconsistent. Postulate (R4) calls for irrelevance of syntax.

According to Alchurrón, Gärdenfors and Makinson, an operator may be called a revision operator if it satisfies postulates of (R1)–(R4); postulates (R5) and (R6) are considered supplementary. Postulates (R5) and (R6) may be thought to guard iterated change. Together they say that if learning  $A$  does not contradict  $B$ , then learning first  $A$  and then  $B$  gives the same belief set than learning  $A \wedge B$  in the first place.

**Example 4.1.** The disjunctive belief-revision method (page 16) is not defined in the case where the original theory is inconsistent or in the case where the formula to be added or deleted is a tautology or a contradiction. But in other cases, it is easy to see that it satisfies postulates (R1)–(R4) by definition. In order to see that postulate (R5) is also satisfied, assume a consistent theory  $T$  and a nontautological, satisfiable formula  $A$ . Let  $T \downarrow \neg A = \{S_1, S_2, \dots, S_n\}$  be the set of those maximal subsets of  $T$  that do not entail  $\neg A$ . Because we assumed that  $A$  is consistent, this set is nonempty. The definition says that  $T \circ A = \bigvee_{i=1}^n (S_i \cup \{A\})$ . Let  $w \in \llbracket T \circ A \rrbracket \cap \llbracket B \rrbracket$  for some propositional formula  $B$ . Then  $w \in \llbracket S_i \rrbracket \cap \llbracket A \rrbracket \cap \llbracket B \rrbracket$  for some  $i = 1, \dots, n$ . Because  $S_i$  is a maximal subset of  $T$  such that  $S_i \not\models \neg A$ , and because  $S_i \not\models \neg B$ ,  $S_i$  is a maximal subset of  $T$  such that  $S_i \not\models \neg(A \wedge B)$ . Thus  $w \in \llbracket T \circ (A \wedge B) \rrbracket$  and (R5) holds.

**Example 4.2.** The disjunctive belief-revision method does not satisfy postulate (R6). Given  $T = \{a, b, c\}$ , we get  $T \circ (\neg a \leftrightarrow b) = \{a, c, \neg a \leftrightarrow b\} \vee$

<sup>1</sup>Note that given epistemic states  $T$  and  $T'$  we apply the equivalence and the logical entailment in the criteria only to the belief sets of the epistemic states, that is, to the sets of propositional formulas believed in the states. Thus  $T_B \equiv T'_B$  does not mean the equivalence of the states  $T$  and  $T'$ , but the equivalence of the respective belief sets.



$\{b, c, \neg a \leftrightarrow b\}$  and  $T \circ ((\neg a \leftrightarrow b) \wedge (\neg b \leftrightarrow c)) = \{a, c, (\neg a \leftrightarrow b) \wedge (\neg b \leftrightarrow c)\} \vee \{b, (\neg a \leftrightarrow b) \wedge (\neg b \leftrightarrow c)\}$ . Although  $T \circ (\neg a \leftrightarrow b) \not\models \neg(\neg b \leftrightarrow c)$ ,  $T \circ ((\neg a \leftrightarrow b) \wedge (\neg b \leftrightarrow c)) \not\models T \circ (\neg a \leftrightarrow b)$ . Postulate (R6) does not therefore hold.

Because postulate (R1) always gives priority to the input, these postulates determine what is called *prioritized belief revision*. So even if the input formula is unsatisfiable, it will be accepted.

In order to rule out operators that give unintuitive results in iterated belief revisions, Darwiche and Pearl [DaP94, DaP97] proposed the following additional postulates (DP-postulates) for iterated belief revision:

- (RR1): If  $B \models A$ , then  $((T \circ A) \circ B)_B \equiv (T \circ B)_B$ .
- (RR2): If  $B \models \neg A$ , then  $((T \circ A) \circ B)_B \equiv (T \circ B)_B$ .
- (RR3): If  $(T \circ B)_B \models A$ , then  $((T \circ A) \circ B)_B \models A$ .
- (RR4): If  $(T \circ B)_B \not\models \neg A$ , then  $((T \circ A) \circ B)_B \not\models \neg A$ .

The motivation for these postulates is as follows. According to rule (RR1), if we obtain two pieces of information with the latter being more accurate, the resulting belief set should be the same as if we had learned only the latter piece of information. If we receive two opposite pieces of information, then according to rule (RR2), the resulting belief set should again be the same as having received only the latter piece of information. According to postulate (RR3), believing  $A$  should not be prevented by learning  $A$ , if  $A$  were otherwise believed, and (RR4) says that an insertion should not cause its own negation. Postulates (R5) and (R6) may be thought as special cases of postulate (RR1).

## 4.2 Representation theorems for belief revision

Grove [Gro88] has shown how belief revision can be modelled by using a system of spheres on the logically possible models, or equivalently, by using a total preorder on the logically possible models. These modellings resemble the spheres semantics [Lew73] on counterfactuals with the distinction of having, instead of a single model, the set of the most plausible models in the center of the system of spheres. Darwiche and Pearl [DaP94, DaP97] shown how the DP-postulates relate to changing that ordering in belief revision. We will here use the formulation by Darwiche and Pearl [DaP97] for both representation theorems.

Let  $W$  denote the set of all logically possible models of a propositional language  $\mathcal{L}$  and let  $\llbracket T \rrbracket$  denote the set of the most plausible models of the

state  $T$ . A function that maps each epistemic state  $T$  to a total pre-order  $\leq_T$  on  $W$  is called *faithful*, if the following conditions hold [DaP97]:

1. If  $w, w' \in \llbracket T \rrbracket$ , then  $w \simeq_T w'$ .
2. If  $w \in \llbracket T \rrbracket$  and  $w' \notin \llbracket T \rrbracket$ , then  $w <_T w'$ .
3. If  $T = T'$ , then  $\leq_T = \leq_{T'}$ .

Thus  $\min(W, \leq_T) = \llbracket T \rrbracket$  holds whenever  $\llbracket T \rrbracket \neq \emptyset$ .

According to the representation theorem for single belief revisions, the most plausible models of the revised state are those models of the new formula that are minimal in the ordering.

**Proposition 4.1.** *A revision operator  $\circ$  satisfies postulates (R1)–(R6), if and only if there is a faithful function that maps each epistemic state  $T$  to a total pre-order  $\leq_T$  on the logically possible models of the language and the following condition holds:*

$$(BR) : \llbracket T \circ A \rrbracket = \min(\llbracket A \rrbracket, \leq_T).$$

**Example 4.3.** For Dalal's operator (page 19), let us define a faithful function that maps a theory  $T$  to a total preorder  $\leq_T$  by defining  $w_1 \leq_T w_2$ , if and only if  $p\_dist(w_1, T) \leq p\_dist(w_2, T)$ . Dalal's operator may then be defined  $\llbracket T \circ_D A \rrbracket = \min(\llbracket A \rrbracket, \leq_T)$  [KaM91b].

The following representation theorem shows how the ordering among possible models is affected by belief revision.

**Proposition 4.2.** [DaP94] *A revision operator  $\circ$  satisfies postulates (RR1)–(RR4), if and only if there is a faithful function that maps each epistemic state  $T$  to a total pre-order  $\leq_T$  on the logically possible models of the language and the following conditions hold:*

- (O1): *If  $w_1 \models A$  and  $w_2 \models A$ , then  $w_1 \leq_T w_2$  iff  $w_1 \leq_{T \circ A} w_2$ .*
- (O2): *If  $w_1 \models \neg A$  and  $w_2 \models \neg A$ , then  $w_1 \leq_T w_2$  iff  $w_1 \leq_{T \circ A} w_2$ .*
- (O3): *If  $w_1 \models A$  and  $w_2 \models \neg A$ , then  $w_1 <_T w_2$  only if  $w_1 <_{T \circ A} w_2$ .*
- (O4): *If  $w_1 \models A$  and  $w_2 \models \neg A$ , then  $w_1 \leq_T w_2$  only if  $w_1 \leq_{T \circ A} w_2$ .*

According to the first condition, the revision should not change the ordering among the worlds that model the new formula. The second postulate says that the revision should not change the ordering among the worlds that do not model the new formula. The third condition says that if a world that models the new formula was more plausible than a world that does not model the new formula, it should remain so after the revision. According to the last condition, if a world that models the new

formula was not less disbelieved than a world that does not model the new formula, it should remain so after the revision. Thus the only way the ordering can change is that the models of the input formula can be shifted downwards in the ordering compared to other logically possible models.

**Example 4.4.** The basic memory operator  $\circ_{bm}$  (page 23) satisfies postulates (R1)–(R6) and (RR1)–(RR4) [KoP01]. By definition,  $\llbracket T \circ_{bm} A \rrbracket = \min(\llbracket A \rrbracket, \leq_T)$ , thus it satisfies postulates (R1)–(R6). Clearly the ordering among the models of the new formula remains unchanged, and so does the ordering among the possible models that do not model the new formula. Also  $w_1 <_{T \circ A} w_2$  holds for all  $w_1 \models A$  and  $w_2 \models \neg A$ . Thus the operator satisfies postulates (R1)–(R6) and (RR1)–(RR4).

### 4.3 Postulates for belief contraction

In belief contraction some belief concerning a static world is given up. Let  $\bullet$  denote a belief-contraction operator, and let  $A$  and  $B$  denote propositional formulas. We will phrase the AGM-postulates for belief contraction [AGM85] as follows:

- (C1): If  $A$  is not a tautology, then  $(T \bullet A)_B \not\models A$ , else  $(T \bullet A)_B \equiv T_B$ .
- (C2): If  $T_B \not\models A$ , then  $(T \bullet A)_B \equiv T_B$ .
- (C3):  $T_B \models (T \bullet A)_B$ .
- (C4): If  $A \equiv B$ , then  $(T \bullet A)_B \equiv (T \bullet B)_B$ .
- (C5):  $(T \bullet A)_B \cup \{A\} \models T_B$ .

According to postulate (C1) the operation is successful, if the formula to be deleted is not a tautology. Postulate (C2) says that if the formula to be deleted is not believed in the epistemic state, then the belief set of the state remains unchanged. Postulate (C4) demands irrelevance of syntax. According to postulates (C3) and (C5), the most plausible models of state  $T \bullet A$  are the the most plausible models of state  $T$  accompanied by some models of  $\neg A$ .

According to postulates (C1)–(C5), if we know whether or not contracting a state  $T$  with a formula  $A$  was effective, we can withdraw the contraction. If  $T_B \models A$ , then according to postulates (C3) and (C5)  $(T \bullet A)_B \cup \{A\} \equiv T_B$ . If  $T_B \not\models A$ , then the beliefs have remained unchanged according to postulate (C2).

Belief revision involves epistemic entrenchment, a pre-order defined on the formulas of the propositional language [GäM88]. Given formulas  $A$  and  $B$ , let  $A \leq B$  denote that the formula  $A$  is *less or equally entrenched* to the

formula  $B$ . We say  $A < B$ , if  $A \leq B$  but not  $B \leq A$ . Epistemic entrenchment must satisfy the following conditions [GäM88]:

- (EE1): If  $A \leq B$  and  $B \leq C$ , then  $A \leq C$  (transitivity)
- (EE2): If  $A \models B$ , then  $A \leq B$  (dominance)
- (EE3): For all  $A$  and  $B$ ,  $A \leq A \wedge B$  or  $B \leq A \wedge B$  (conjunctivity)
- (EE4): If  $T$  is consistent, then  $A \notin T$ , iff  $A \leq B$  for all  $B \in T$  (minimality)
- (EE5): If  $B \leq A$  for all  $B$ , then  $A$  is a tautology (maximality)

The second condition says that if  $A$  logically entails  $B$  and one of them has to be given up, then the change is smaller, if only  $A$  is abandoned. The third condition says that in order to give up the formula  $A \wedge B$ , one has to give up either the formula  $A$  or the formula  $B$ . According to the fourth condition, the formulas not included in the theory are minimally entrenched, according to the fifth, tautologies are maximally entrenched.

Assume an epistemic entrenchment defined on all the formulas of the language. In revision, formulas less entrenched are those chosen to be given up when necessary. Then the entrenchment defines a unique revision operator and vice versa as follows:

- (C•):  $B \in T \bullet A$ , iff  $B \in T$  and either  $A < (A \vee B)$  or  $A$  is a tautology,
- (C≤):  $A \leq B$ , iff  $A \notin T \bullet (A \wedge B)$  or  $A \wedge B$  is a tautology.

Assume that epistemic states are represented by propositional formulas. Now, if a revision operator  $\circ$  satisfies postulates (R1)–(R4) and we define a contraction operator  $\bullet$  by the so-called *Harper identity*

$$T \bullet A =_{\text{def}} T \vee (T \circ \neg A),$$

then the operator  $\bullet$  satisfies postulates (C1)–(C5). A corresponding dependency can be defined in the other direction. If the contraction operator satisfies postulates (C1)–(C4), and if the revision operator is defined by the *Levi identity*

$$T \circ A =_{\text{def}} (T \bullet \neg A) \wedge A,$$

then the revision operator  $\circ$  satisfies postulates (R1)–(R4) [Gär88, chapter 3]. However, when epistemic states are not representable by propositional formulas, Harper and Levi identities are not sufficient to define operators.

When changing closed theories, if the epistemic entrenchment  $\leq$  satisfies conditions (EE1)–(EE5), then the operator defined by rule (C•) satisfies the AGM-postulates and condition (C≤). If the contraction operator  $\bullet$  satisfies the AGM-postulates, then the ordering defined by rule (C≤) satisfies conditions (EE1)–(EE5) and (C•) [GäM88].

## 4.4 Postulates for belief update

In belief update, a formula that records a change in the world is inserted into a theory. Katsuno and Mendelzon [KaM91a] have proposed their KM-postulates for belief update assuming that the epistemic state  $T$  can be represented by a propositional formula. In this context, we have rephrased the postulates so that we only assume that the belief set  $T_B$  of the state can be represented by a propositional formula.

Let  $\diamond$  denote a belief-update operator, and let  $T_B$  denote the belief set of the epistemic state  $T$ . Let  $A$  and  $B$  denote propositional formulas. We phrase the rationality criteria for belief update [KaM91a] as follows:

- (U1):  $(T \diamond A)_B \models A$ .
- (U2): If  $T_B \models A$ , then  $(T \diamond A)_B \equiv T_B$ .
- (U3): If  $T_B$  and  $A$  are satisfiable, then also  $(T \diamond A)_B$  is satisfiable.
- (U4): If  $T_B \equiv T'_B$  and  $A \equiv B$ , then  $(T \diamond A)_B \equiv (T' \diamond B)_B$ .
- (U5):  $(T \diamond A)_B \wedge B \models (T \diamond (A \wedge B))_B$ .
- (U6): If  $(T \diamond A)_B \models B$  and  $(T \diamond B)_B \models A$ , then  $(T \diamond A)_B \equiv (T \diamond B)_B$ .
- (U7): If  $T_B$  is complete, then  $(T \diamond A)_B \wedge (T \diamond B)_B \models (T \diamond (A \vee B))_B$ .
- (U8): If  $T_B \equiv T'_B \vee T''_B$ , then  $(T \diamond A)_B \equiv (T' \diamond A)_B \vee (T'' \diamond A)_B$ .

Postulate (U1) says that the new piece of information should be accepted. According to postulate (U2), if the formula  $A$  is already believed in  $T$ , then the update by  $A$  has no effect on the belief set. According to postulate (U3), the result of the update should be consistent whenever postulates (U1) and (U2) permit it. Postulate (U4) calls for irrelevance of syntax. Postulate (U5) is analogous to postulate (R5).

Postulates (U1)–(U5) for belief update correspond to postulates (R1)–(R5) for belief revision. Postulate (U6) says that if updating  $T$  by a formula  $A$  entails  $B$  and vice versa, the results of the updates should be equivalent. Postulate (U7) is applied only to complete belief sets. It says that if a model is considered as one of the most plausible models both after updating the set by a formula  $A$  and after updating it by a formula  $B$ , then the model should be among the most plausible models after updating the belief set by a formula  $A \vee B$ . Katsuno and Mendelzon [KaM91a] need this postulate to prove their representation theorem for belief update, but because belief sets are seldom complete, the postulate does not seem to have much motivation otherwise [HeR99].

Postulate (U8) describes the most characteristic property of belief update: we consider the change to alternative states of affairs separately. This postulate causes the monotony of update: if  $T_B \models T'_B$ , then  $(T \diamond A)_B \models (T' \diamond A)_B$ . Belief revision is nonmonotonic even in that respect.

## 4.5 A representation theorem for belief update

Also belief update involves orderings on the truth distributions of the interpretations of the language [KaM91a]. According to the definition by Katsuno and Mendelzon [KaM91a], a function that maps a logically possible model  $w \in W$  to an ordering  $\leq_w$  on the logically possible models  $W$  of the language  $\mathcal{L}$  is *faithful*, if  $w <_w w'$  whenever  $w \neq w'$ . In other words,  $\min(W, \leq_w) = \{w\}$ .

Each model of the old epistemic state will be assigned an ordering, and the models of the new state will be the models of the new formula that are minimal in at least one of the orderings, as formalized in the following condition:

$$(BP) : \llbracket T \diamond A \rrbracket = \bigcup_{w \in \llbracket T \rrbracket} \min(\llbracket A \rrbracket, \leq_w).$$

The representation theorem for belief update says that the set of the most plausible models of the updated state can be obtained by using partial pre-orders or partial orders on the models of the new formula.

**Proposition 4.3.** [KaM91a] *Given an operator  $\diamond$ , the following conditions are equivalent:*

1. *The operator  $\diamond$  satisfies the postulates (U1)–(U8).*
2. *There is a faithful function that maps each model  $w$  of  $T_B$  to a partial pre-order  $\leq_w$  on  $W$  such that condition (BP) holds.*
3. *There is a faithful function that maps each model  $w$  of  $T_B$  to a partial order  $\leq_w$  on  $W$  such that condition (BP) holds.*

It is not necessary for the ordering involved in belief update to be total. A partial pre-order or a partial order are both sufficient.

**Example 4.5.** For Forbus's operator  $\diamond_F$  (page 21), a faithful function can be defined to map each model  $w \in \llbracket T \rrbracket$  to a total preorder  $\leq_w$  such that  $w_1 \leq_w w_2$ , if and only if  $|w \Delta w_1| \leq |w \Delta w_2|$ . The operator can then be defined  $\llbracket T \diamond_F A \rrbracket = \bigcup_{w \in \llbracket T \rrbracket} \min(\llbracket A \rrbracket, \leq_w)$ .

## 4.6 Postulates for belief erasure

In belief erasure, a formula is deleted from a theory as a result of a change in the world. Let  $\blacklozenge$  denote a belief-erasure operator, and let  $T \blacklozenge A$  denote

the epistemic state with a formula  $A$  erased from state  $T$ . The postulates for erasure have been defined [KaM91a] as follows<sup>2</sup>:

- (E1): If  $T_B$  is satisfiable and  $A$  is not a tautology, then  $(T\blacklozenge A)_B \not\models A$ .
- (E2): If  $T_B \models \neg A$ , then  $(T\blacklozenge A)_B \equiv T_B$ .
- (E3):  $T_B \models (T\blacklozenge A)_B$ .
- (E4): If  $T_B \equiv T'_B$  and  $A \equiv B$ , then  $(T\blacklozenge A)_B \equiv (T'\blacklozenge B)_B$ .
- (E5):  $(T\blacklozenge A)_B \wedge A \models T_B$ .
- (E8): If  $T_B \equiv T'_B \vee T''_B$ , then  $(T\blacklozenge A)_B \equiv (T'\blacklozenge A)_B \vee (T''\blacklozenge A)_B$ .

According to (E1), the erasure  $T\blacklozenge A$  is successful whenever  $A$  is not a tautology and the epistemic state  $T$  is consistent. Postulate (E2) says that if the formula to be erased is not considered compatible, then the erasure has no effect. Postulates (E4) and (E8) correspond to postulates (U4) and (U8). According to postulates (E3) and (E5), the most plausible models of state  $T\blacklozenge A$  are the most plausible models of state  $T$  accompanied by some models of  $\neg A$ .

Postulates (E3) and (E5) correspond to postulates (C3) and (C5) respectively. Postulate (E2) differs from postulate (C2), because the contraction of a theory  $T$  with a formula  $A$  is effective in the case  $T_B \models A$ , but the erasure of the formula  $A$  is effective if  $T_B \not\models \neg A$ . In both cases the most plausible models of the new epistemic state are the the most plausible models of state  $T$  accompanied by some models of  $\neg A$ . Postulate (E1) differs from postulate (C1) in the case  $T_B$  is inconsistent: inconsistency cannot be eliminated by the means of belief erasure or belief update.

In case the epistemic state is represented by a propositional formula and the update operator  $\diamond$  satisfies postulates (U1)–(U4) and (U8), then the erasure operator  $\blacklozenge$  can be defined as  $T\blacklozenge A =_{def} T \vee (T \diamond \neg A)$  [KaM91a]. A *symmetric erasure* [Win90] of a formula  $A$  expresses that we have no beliefs concerning  $A$ . The result of the symmetric erasure can be defined using the formula  $(T \diamond A) \vee (T \diamond \neg A)$ , when the epistemic state  $T$  is representable by a logical formula.

## 4.7 Analysis of the postulates for belief revision

The first formulation of the postulates for iterated belief revision [DaP94] assumed that epistemic states could be represented by single propositional formulas, causing thereby triviality of logic. To avoid triviality of logic,

---

<sup>2</sup>Labels (E1) and (E3) have been swapped here for the consistency of labelling. In this way postulates (U1) and (E1) both express the conditions on the success of the operation.

we have used in the AGM-postulates the original formulation of postulate (R4) by Alchurrón, Gärdenfors and Makinson [AGM85] instead of the following postulate (R4') that formalizes the version used by Kazuno and Mendelzon [KaM91a] for epistemic states represented by single propositional formulas:

$$(R4'): \text{ If } T_B \equiv T'_B \text{ and } A \equiv B, \text{ then } (T \circ A)_B \equiv (T' \circ B)_B.$$

The use of version (R4') instead of postulate (R4) makes the joined set of the AGM- and the DP-postulates inconsistent, if we assume that the logic is not trivial. In other words, if any operator satisfies the two sets of postulates, then the logic must be trivial. A logic is *trivial*, if there are not four satisfiable formulas such that three of them are pairwise inconsistent with each other, and the fourth one is consistent with each of them.

**Example 4.6. a)** Assume a language  $\mathcal{L}$  with  $Voc(\mathcal{L}) = \{a, b\}$ . Then formulas  $a \wedge b$ ,  $a \wedge \neg b$ , and  $\neg a \wedge b$  are satisfiable but pairwise inconsistent with each other and they are all consistent with  $a \vee b$ . The language is not trivial.

**b)** Assume a language  $\mathcal{L}'$  with  $Voc(\mathcal{L}') = \{a\}$ . The only satisfiable formulas that are inconsistent with each other are  $a$  and  $\neg a$ . The language is trivial.

**Theorem 4.1.** [Elo95, Elo97] *If the language is nontrivial, then postulates (R1), (R2), (R3), and (R4') are inconsistent with postulate (RR2).*

*Proof.* Assume a nontrivial language, and let  $\circ$  denote a revision operator that satisfies postulates (R1), (R2), (R3), (R4'), and (RR2). Because the language is nontrivial, then by definition satisfiable formulas  $A$ ,  $B$ ,  $C$ , and  $D$  exist such that formulas  $A$ ,  $B$ , and  $C$  are consistent with  $D$  but pairwise inconsistent with each other. Let  $T$  denote an epistemic state with a belief set  $T_B \equiv D$ .

By (R2),  $(T \circ A)_B \equiv T_B \cup \{A\}$  and  $(T \circ B)_B \equiv T_B \cup \{B\}$ . Because  $A \wedge B$  and  $B \wedge C$  are unsatisfiable, then by (RR2),  $((T \circ A) \circ C)_B \equiv (T \circ C)_B$  and  $((T \circ B) \circ C)_B \equiv (T \circ C)_B$ . Thus  $((T \circ A) \circ C)_B \equiv ((T \circ B) \circ C)_B$  and by (R4'),  $((T \circ A) \circ C) \circ (A \vee B)_B \equiv ((T \circ B) \circ C) \circ (A \vee B)_B$ . Because  $C$  is inconsistent with  $A \vee B$ , then by (RR2)  $((T \circ A) \circ C) \circ (A \vee B)_B \equiv ((T \circ A) \circ (A \vee B))_B$  and  $((T \circ B) \circ C) \circ (A \vee B)_B \equiv ((T \circ B) \circ (A \vee B))_B$ . Because by (R1) and (R3)  $(T \circ A)_B \cup \{A \vee B\}$  and  $(T \circ B)_B \cup \{A \vee B\}$  are consistent, then, by (R2),  $((T \circ A) \circ (A \vee B))_B \equiv (T \circ A)_B \cup \{A \vee B\}$  and  $((T \circ B) \circ (A \vee B))_B \equiv (T \circ B)_B \cup \{A \vee B\}$ . Thus  $((T \circ A) \circ (A \vee B))_B \models A$  and  $((T \circ B) \circ (A \vee B))_B \models B$ .

Because  $((T \circ A) \circ C) \circ (A \vee B)_B \equiv ((T \circ B) \circ C) \circ (A \vee B)_B$  we have  $((T \circ A) \circ C) \circ (A \vee B)_B \models B$ . Because the formulas  $A$  and  $B$  were assumed to be inconsistent with each other, and by (R3) the state  $((T \circ A) \circ C) \circ (A \vee B)$  is consistent, we have a contradiction.  $\square$



We will next prove that using postulate  $(R4')$  instead of  $(R4)$  has also an other defect, namely, if the new formula is inconsistent with the belief set of the epistemic state, then the belief set of the new epistemic state will not be affected by the original epistemic state.

**Theorem 4.2.** [Elo95, Elo97] *If a belief-revision operator satisfies postulates  $(R1)$ ,  $(R2)$ ,  $(R3)$ ,  $(R4')$  and  $(RR1)$ , and the proposition to be added to the current epistemic state is inconsistent with it, then the revised epistemic state does not depend on the current state.*

*Proof.* Let  $\circ$  denote a revision operator that satisfies postulates  $(R1)$ ,  $(R2)$ ,  $(R3)$ ,  $(R4')$  and  $(RR1)$ . Let  $A$ ,  $B$ , and  $C$  denote satisfiable formulas, and let  $T$  denote an epistemic state that is consistent with  $A$  and  $B$  but inconsistent with  $C$ . Postulates  $(R1)$ – $(R3)$  imply that  $((T \circ A) \circ (A \vee C))_B \equiv (T \circ A)_B \equiv (T \circ (A \vee C))_B$  and  $((T \circ B) \circ (B \vee C))_B \equiv (T \circ B)_B \equiv (T \circ (B \vee C))_B$ . Because  $C \models A \vee C$  and  $C \models B \vee C$ ,  $(RR1)$  implies that  $((T \circ (A \vee C)) \circ C)_B \equiv (T \circ C)_B$  and  $((T \circ (B \vee C)) \circ C)_B \equiv (T \circ C)_B$ . Thus, by  $(R4')$ ,  $((T \circ A) \circ C)_B \equiv ((T \circ B) \circ C)_B$ .  $\square$

Let us consider different formulations for the DP-postulates  $(RR1)$  and  $(RR2)$ :

$(RR1')$ : If  $B \models A$ , then  $(T \circ A) \circ B = T \circ B$ ,

$(RR2')$ : If  $B \models \neg A$ , then  $(T \circ A) \circ B = T \circ B$ .

Using identity of epistemic states instead of equivalence of the belief sets in postulates  $(RR1)$  or  $(RR2)$  would again cause triviality.

**Theorem 4.3.** *If the language is nontrivial, then postulates  $(R2)$  and  $(RR2)$  are inconsistent with postulate  $(RR1')$ .*

*Proof.* Assume a nontrivial language, and let  $\circ$  denote a revision operator that satisfies postulates  $(R2)$ ,  $(RR1')$ , and  $(RR2)$ . Because the language is nontrivial, then by definition satisfiable formulas  $A$ ,  $B$ ,  $C$ , and  $D$  exist such that formulas  $A$ ,  $B$ , and  $C$  are consistent with  $D$  but pairwise inconsistent with each other. Let  $T$  denote an epistemic state with a belief set  $T_B \equiv D$ .

By  $(R2)$ ,  $(T \circ (A \vee B))_B \equiv T_B \cup \{A \vee B\}$  and  $(T \circ A)_B \equiv T_B \cup \{A\}$ . By  $(RR1')$ ,  $((T \circ (A \vee B)) \circ A)_B \equiv T \circ A$ . Then by  $(RR2)$ ,  $((T \circ (A \vee B)) \circ \neg A)_B \equiv ((T \circ (A \vee B)) \circ A) \circ \neg A)_B \equiv ((T \circ A) \circ \neg A)_B \equiv (T \circ \neg A)_B$ . By  $(R2)$ ,  $((T \circ (A \vee B)) \circ \neg A)_B \equiv T_B \cup \{A \vee B\} \cup \{\neg A\} \models B$ , and  $(T \circ \neg A)_B \equiv T_B \cup \{\neg A\} \not\models B$ , a contradiction.  $\square$

**Theorem 4.4.** *If the language is nontrivial, then postulates  $(R1)$ – $(R4)$  are inconsistent with postulate  $(RR2')$ .*

*Proof.* Assume a nontrivial language, and let  $\circ$  denote a revision operator that satisfies postulates (R1)–(R4), and (RR2'). Because the language is nontrivial, then by definition satisfiable formulas  $A$ ,  $B$ ,  $C$ , and  $D$  exist such that formulas  $A$ ,  $B$ , and  $C$  are consistent with  $D$  but pairwise inconsistent with each other. Let  $T$  denote an epistemic state with a belief set  $T_B \equiv D$ .

By (R2),  $(T \circ A)_B \equiv T_B \cup \{A\}$  and  $(T \circ B)_B \equiv T_B \cup \{B\}$ . Because  $A \wedge C$  and  $B \wedge C$  are unsatisfiable, then by (RR2'),  $(T \circ A) \circ C = T \circ C$  and  $(T \circ B) \circ C = T \circ C$ . Thus  $((T \circ A) \circ C) \circ (A \vee B) = ((T \circ B) \circ C) \circ (A \vee B)$ . Because  $C$  is inconsistent with  $A \vee B$ , then by (RR2')  $((T \circ A) \circ C) \circ (A \vee B) = (T \circ A) \circ (A \vee B)$  and  $((T \circ B) \circ C) \circ (A \vee B) = (T \circ B) \circ (A \vee B)$ . Then by (R2),  $T_B \cup \{A\} \cup \{A \vee B\} \equiv ((T \circ A) \circ (A \vee B))_B \equiv ((T \circ B) \circ (A \vee B))_B \equiv T_B \cup \{B\} \cup \{A \vee B\}$ , a contradiction with (R3).  $\square$

According to these theorems, if the revision operator is to satisfy the postulates for iterated belief revision, it must act upon epistemic states that contain more information than mere belief sets. The epistemic states must contain doxastic conditionals.

To avoid triviality of logic with epistemic states that contain doxastic conditionals, we have restricted the postulates to guide changing only the belief sets of epistemic states. The change of doxastic conditionals is indirectly guided by the postulates for iterated belief change by means of guiding the change in the belief sets in series of revisions.

## 4.8 Analysis of some operators

Let us see how our sample operators satisfy the postulates for belief change. We have already seen that the syntactically-oriented operators on open theories do not satisfy postulate (R4') demanding irrelevance of syntax. In Examples 4.1 and 4.2 (page 30) we saw that the disjunctive method (page 16) for revising open theories satisfies (R1)–(R5), but not postulate (R6). Analogously to the disjunctive method, the intersective method satisfies postulates (R1)–(R5). The following example will demonstrate that it does not satisfy postulate (R6).

**Example 4.7.** Given  $T = \{a, b, c\}$ , we get  $T \circ (\neg a \leftrightarrow b) = \{a, c, \neg a \leftrightarrow b\} \cap \{b, c, \neg a \leftrightarrow b\} = \{c, \neg a \leftrightarrow b\}$  and  $T \circ ((\neg a \leftrightarrow b) \wedge (\neg b \leftrightarrow c)) = \{a, c, (\neg a \leftrightarrow b) \wedge (\neg b \leftrightarrow c)\} \cap \{b, (\neg a \leftrightarrow b) \wedge (\neg b \leftrightarrow c)\} = \{(\neg a \leftrightarrow b) \wedge (\neg b \leftrightarrow c)\}$ . Although  $T \circ (\neg a \leftrightarrow b) \not\models \neg(\neg b \leftrightarrow c)$ ,  $T \circ ((\neg a \leftrightarrow b) \wedge (\neg b \leftrightarrow c)) \not\models T \circ (\neg a \leftrightarrow b)$ , and postulate (R6) does not therefore hold.

The revision policy on flocks of theories (page 17) does not satisfy postulate (R2), as we can see in the following example.

**Example 4.8.** Let  $\mathbf{T}$  denote the flock  $\{\{a, \neg a \leftrightarrow b\}, \{b, \neg a \leftrightarrow b\}\}$ , which may be obtained by adding a formula  $(\neg a \leftrightarrow b)$  to the flock  $\{\{a, b\}\}$ . If we add the formula  $a$  to the flock  $\mathbf{T}$ , we get the flock  $\mathbf{T}' = \{\{a, \neg a \leftrightarrow b\}, \{a, b\}\}$ . Even though  $\mathbf{T} \not\models \neg a$ ,  $\mathbf{T}' \not\models \mathbf{T}$ , thus postulate (R2) does not hold.

The flock method resembles update in the sense that the changes are made in the alternatives independently. The revision operator might be modified for conflict situations to choose only those members of the flock, which are not contradictory to the change, if such alternatives exist [Win88a].

According to the representation theorem for belief revision, a revision operator  $\circ$  satisfies postulates (R1)–(R6), if and only if there is a faithful function that to each epistemic state  $T$  assigns a total pre-order  $\leq_T$  such that  $\llbracket T \circ A \rrbracket = \text{Min}(\llbracket A \rrbracket, \leq_T)$ . A revision operator fails to satisfy postulate (R6), if it uses a partial preorder, or if the ordering also depends on the new formula, or if the revision is carried out pointwise [KaM91b].

Dalal's semantically-oriented belief-revision operator (page 19) uses total preorder. Borgida's operator may search closest models pointwise. The operator by Satoh uses a partial order [KaM91b]. Dalal's operator  $\circ_D$  satisfies postulates (R1)–(R6) [KaM91b]. If the theory  $T$  is consistent, operators  $\circ_S$  and  $\circ_B$  satisfy postulates (R1)–(R5), but not postulate (R6) [KaM91b]. If the theory  $T$  is consistent and the formula  $A$  is satisfiable, the operator  $\circ_W$  satisfies postulates (R1)–(R4), but not postulates (R5) nor (R6) [KaM91b]. Because all these operators also satisfy postulate (R4'), then by Theorem 4.1, they do not satisfy postulate (RR2).

The revision operator by Darwiche and Pearl (page 25) satisfies postulates (R1)–(R6) and (RR1)–(RR4), but not postulate  $R4'$ , as we saw in Example 3.11. The basic memory operator (page 23) satisfies postulates (R1)–(R6) and (RR1)–(RR4) [KoP01]. The Dalal memory operator (page 23) satisfies postulates (R1)–(R6), (RR1), (RR3), and (RR4) [KoP01], but not postulate (RR2), as the following example shows.

**Example 4.9.** Let us consider revising state  $\tau$  by formulas  $a \wedge b$ ,  $\neg a \wedge \neg b$ , and  $a \vee b$ . Let  $W = \{w_0, w_1, w_2, w_3\}$ ,  $\llbracket a \rrbracket = \{w_2, w_3\}$ ,  $\llbracket b \rrbracket = \{w_1, w_3\}$ . Then

$$\begin{aligned} w_3 &<_{a \wedge b}^d w_1, w_2 <_{a \wedge b}^d w_0, \\ w_0 &<_{\neg a \wedge \neg b}^d w_1, w_2 <_{\neg a \wedge \neg b}^d w_3, \text{ and} \\ w_1, w_2, w_3 &<_{a \vee b}^d w_0. \end{aligned}$$

In revisions  $T = \tau \circ_{dm} (a \wedge b)$ ,  $T' = T \circ_{dm} (\neg a \wedge \neg b)$ , and  $T'' = T' \circ_{dm} (a \vee b)$ , we get

$$\begin{aligned}
w_3 &<_T w_1, w_2 <_T w_0, \\
w_0 &<_{T'} w_1, w_2 <_{T'} w_3, \text{ and} \\
w_1, w_2 &<_{T''} w_3 <_{T''} w_0,
\end{aligned}$$

even though when revising  $T \circ_{dm} (a \vee b)$ , we would get an ordering equal to  $\leq_T$ .

If the theory is satisfiable, the minimal-change update operator  $\diamond_W$  (page 21) satisfies postulates (R1) and (R3)–(R5), but not postulates (R2) and (R6) [KaM91b]. The fact that the operator does not satisfy postulate (R2) implies that the operator is not a revision operator. The operator is an update operator satisfying postulates (U1)–(U8) as the operator  $\diamond_F$  by Forbus [Eit93]. For the minimal-change update operator  $\diamond_W$ , a faithful function can be defined to map each model  $w \in \llbracket T \rrbracket$  to a partial order  $\leq_w$  such that  $w_1 \leq_w w_2$ , if and only if  $w \Delta w_1 \subseteq w \Delta w_2$ . The operator may then be defined by  $\llbracket T \diamond_W A \rrbracket = \bigcup_{w \in \llbracket T \rrbracket} \min(\llbracket A \rrbracket, \leq_w)$ . The operator  $\diamond_W$  also satisfies postulate (RR4) for iterated belief revision.

**Theorem 4.5.** *The minimal-change update-operator by Winslett satisfies postulate (RR4).*

*Proof.* Assume  $\diamond = \diamond_W$  and  $(T \diamond B)_B \not\models \neg A$ . We shall prove that  $((T \diamond A) \diamond B)_B \not\models \neg A$ .

Because  $(T \diamond B)_B \not\models \neg A$ , a model  $m_0$  exists such that  $m_0 \in \llbracket T \diamond B \rrbracket \cap \llbracket A \rrbracket$ . Because  $m_0 \in \llbracket T \diamond B \rrbracket$ , a model  $m_1 \in \llbracket T \rrbracket$  exists such that for all  $w \in \llbracket B \rrbracket$ ,  $m_1 \Delta w \not\subseteq m_1 \Delta m_0$  holds. If on one hand  $m_0 \in \llbracket T \diamond A \rrbracket$ , then  $m_0 \in \llbracket (T \diamond A) \diamond B \rrbracket$ . Because  $m_0 \models A$ ,  $((T \diamond A) \diamond B)_B \not\models \neg A$ . Assume on the other hand that  $m_0 \notin \llbracket T \diamond A \rrbracket$ . The assumption implies that for all  $w \in \llbracket T \rrbracket$  there is  $w' \in \llbracket A \rrbracket$  such that  $w \Delta w' \subset w \Delta m_0$ . In particular there is  $m_2 \in \llbracket T \diamond A \rrbracket$  such that  $m_1 \Delta m_2 \subset m_1 \Delta m_0$ .

If  $m_0 \in \llbracket (T \diamond A) \diamond B \rrbracket$ , then  $((T \diamond A) \diamond B)_B \not\models \neg A$ . Assume  $m_0 \notin \llbracket (T \diamond A) \diamond B \rrbracket$ . The assumption implies that for all  $w \in \llbracket T \diamond A \rrbracket$  there is  $w' \in \llbracket B \rrbracket$  such that  $w \Delta w' \subset w \Delta m_0$ . In particular, there is  $m_3 \in \llbracket (T \diamond A) \diamond B \rrbracket$  such that  $m_2 \Delta m_3 \subset m_2 \Delta m_0$ .

Because  $m_1 \Delta w \not\subseteq m_1 \Delta m_0$  holds for all  $w \in \llbracket B \rrbracket$ ,  $m_1 \Delta m_3 \not\subseteq m_1 \Delta m_0$ . If  $m_1 \Delta m_3 = m_1 \Delta m_0$ , then  $m_3 = m_0$ , a contradiction, thus  $m_1 \Delta m_3 \not\subseteq m_1 \Delta m_0$ . In this case there is an atomic formula  $x$  such that  $x \in m_1 \Delta m_3$ , but  $x \notin m_1 \Delta m_0$ . Because  $x \notin m_1 \Delta m_0$  and  $m_1 \Delta m_2 \subset m_1 \Delta m_0$ ,  $x \notin m_1 \Delta m_2$ . Thus  $x \notin m_2 \Delta m_0$ , which implies that  $x \notin m_2 \Delta m_3$ , because  $m_2 \Delta m_3 \subset m_2 \Delta m_0$ . Because  $x \notin m_1 \Delta m_2$  and  $x \notin m_2 \Delta m_3$ ,  $x \notin m_1 \Delta m_3$  holds. This gives us a contradiction  $x \in m_1 \Delta m_3$ , thus the assumption  $m_0 \notin \llbracket (T \diamond A) \diamond B \rrbracket$  was false.  $\square$

The standard update operator  $\diamond_s$  (page 21) satisfies postulates (U1), (U3), (U7), and (U8) [HeR99]. The version  $\diamond_{s\downarrow}$  satisfies postulates (U1), (U3), (U4), and (U8) [HeR99].

## 4.9 On integrity constraints

Integrity constraints are used to express those properties of a knowledge base that should always hold. Let  $IC$  denote a formula that represents the integrity constraints. Two alternative ways to determine whether or not the epistemic state satisfies the integrity constraints have been proposed [Rei88]. According to one definition, it is sufficient to have  $T_B \cup \{IC\}$  satisfiable. According to the other definition, it is required that  $T_B \models IC$ . Katsuno and Mendelzon [KaM91b] have used the latter version when defining the effect that the integrity constraints have on belief change using the condition:

$$T \circ^{IC} A =_{def} T \circ (A \wedge IC). \quad (4.1)$$

The corresponding effect on belief contraction is defined [KaM91b] by

$$T \bullet^{IC} A =_{def} T \bullet (IC \rightarrow A). \quad (4.2)$$

The definition implies that if a formula is entailed by the integrity constraints, it is impossible to contract it from the epistemic state [KaM91b]. We may analogously define the effect of integrity constraints on belief update.

According to Poole [Poo88], integrity constraints are rules used to determine whether or not the epistemic state is consistent, but they do not participate in deduction. However, postulate (R3) calls for consistency of beliefs. If we use the conjunction of the input and the integrity constraints  $A \wedge IC$  in the revision, the integrity constraints actively take part in deduction. Poole's default logic [Poo88] can maintain consistency of epistemic states with integrity constraints only under special conditions.

## 4.10 Related work

We will conclude our discussion of the AGM, DP-, and KM-postulates by reviewing some related work concerning the classification of change, critics on the postulates, and the relation between theory change and non-monotonous logic.

## On the classification of change

Grahne [Gra91b] has studied the change of incomplete databases. He classifies the changes into insertions, changes and deletions. He furthermore classifies the insertions and deletions into absolute or positive. Absolute insertions and deletions are revisions, positive insertions and deletions are updates. The naming is based on the intuition that, in revision, the absolute amount of information increases, while in positive insertions and deletions the changes take place in the amounts of positive information. A database contains only positive information. In a positive deletion, a formula that was believed to be true becomes a formula that is believed to be false.

Revesz [Rev93] has suggested arbitration as a type of revision. In *arbitration*, the latest piece of evidence is not considered to be the most reliable. Revesz uses a trial as an example to justify the new type of revision. In arbitration old and new beliefs weigh the same. Revesz introduces eight postulates for arbitration and he gives a definition of an operator by forcing the change to be commutative. The definition does not, however, manage to avoid the unbalance between old and new information. In arbitration the new formula weighs as much as the formula representing the old epistemic state. To cure the unbalance he suggests weighted changes, but that does not solve the problem. In fact there seems to be a more severe argument against arbitration: it may fail to find a consistent solution. If we consider the example of a trial, would it not be intuitive to choose a consistent solution, when such a solution exists? Arbitration therefore does not satisfy postulate (R2), which is the very postulate that, together with the corresponding characteristic postulate for arbitration, classifies the revision operators and the arbitration operators as two disjoint sets of operators. Yet postulate (R2) would have been needed in the example to give intuitive results.

## Critics on the postulates for belief revision

We have restricted the postulates for belief revision to guide changing only the belief sets of epistemic states. This restriction is consistent with the result of Grahne, Mendelzon, and Reiter [GMR92] saying that belief revision is update at the epistemic level. Their result thus shows that the AGM-postulates must not be used at the epistemic level. In doxastic conditionals the change takes place at the epistemic level, therefore changing conditionals such as “should I believe  $A$ , I would believe  $B$ ” cannot be guided by those postulates.

The inconsistency of the AGM-postulates and the DP-postulate (RR2) has been proved independently by Freund and Lehmann [FrL94], and Eloranta [Elo95, Elo97]. The proof by Freund and Lehmann implicitly assumes that epistemic states contain only propositional beliefs. The proof is based on the existence of a single inconsistent epistemic state: recovery from such a state is impossible without total amnesia. To replace the DP-postulates, Freund and Lehmann [FrL94] suggest their own postulate

$$(R7): \quad \text{If } T \models \neg A \text{ and } T' \models \neg A, \text{ then } T \circ A = T' \circ A.$$

However, we have considered this feature to be a defect. Our proof (Theorem 4.1) does not involve revisions by contradictions. It is therefore valid even if the postulates were changed so that no contradiction would be accepted.

There have been critics on postulate (RR2) on iterated belief revision, and it has been considered counterintuitive by some [KoP01]. Lehmann [Leh95] has given a theorem stating that the AGM-postulates together with postulate (RR1) imply postulates (RR3) and (RR4). He does not, however, provide a proof for the theorem, which is contradictory to the motivating examples [DaP97, page 20] given by Darwiche and Pearl. We have merged the two examples as follows.

**Example 4.10.** Assume  $W = \{w_0, w_1, w_2, w_3\}$  with  $\llbracket A \rrbracket = \{w_2, w_3\}$  and  $\llbracket B \rrbracket = \{w_1, w_3\}$ . Assume we have an epistemic state  $T$  with  $\leq_T: w_0 <_T w_2 <_T w_3 <_T w_1$  and  $T \circ A$  with  $\leq_{T \circ A}: w_2 <_{T \circ A} w_0 <_{T \circ A} w_1 <_{T \circ A} w_3$ . Then  $\llbracket T \circ B \rrbracket = \{w_3\}$  and  $\llbracket (T \circ A) \circ B \rrbracket = \{w_1\}$ .  $(T \circ B)_B \models A$ , but  $((T \circ A) \circ B)_B \models \neg A$ . The operator satisfies postulates (R1)–(R6), (RR1), and (RR2), but not postulates (RR3) and (RR4).

### Critics on the postulates of belief update

Update treats disjunction in the new formula and in the old formula in different ways. Postulate (U8) says that  $(T \vee T') \diamond A \equiv (T \diamond A) \vee (T' \diamond A)$ , but it is not possible to define an update operator that satisfies the KM-postulates such that  $T \diamond (A \vee B) \equiv (T \diamond A) \vee (T \diamond B)$ . The operator would not satisfy the postulate (U2).

The KM-postulates cause the problem of disjunctive input: updates by inclusive formulas behave just as updates by exclusive disjunctions [HeR99]. Herzig and Rifi [HeR99] have analyzed the problem, and they have proposed their own set of postulates. They totally omit postulates (U5)–(U7) and accept postulates (U1), (U4), and (U8). They weaken postulate (U2), they modify postulate (U3) to take into account integrity constraints, and they add postulate (U0) for integrity constraints as follows:

- (U0):  $(T \diamond A)_B \models IC$ .  
 (U2'):  $T_B \wedge A \models (T \diamond A)_B$ .  
 (U2''):  $(T \diamond \top)_B \equiv T_B$ .  
 (U3'): If  $T_B$  and  $A$  are consistent with  $IC$ , then  
 $(T \diamond A)_B$  is consistent with  $IC$ .

Postulate (U6) with postulate (U2'') would imply (U2). According to Herzig and Rifi [HeR99], postulate (U7) is without importance, and postulate (U5) is harmful causing the problem of disjunctive input.

### Theory change and nonmonotonic logic

Makinson and Gärdenfors [MaG91] have compared belief revision and nonmonotonic reasoning. In their comparison they used fixed background theories: “a formula  $x$  nonmonotonically entails (with regard to a theory  $T$ ) a formula  $y$ ”, more formally  $x \vdash_T y$ , corresponds to  $T \circ x \models y$  in theory revision. By using this translation, we can compare the principles of nonmonotonic reasoning and theory revision. However, we have to restrict ourselves to those principles concerning only single background theories. Thus, we cannot translate the principles that compare revisions  $T \circ x$  and  $T' \circ x$ , neither can we translate the principles of iterated theory revision.

Makinson and Gärdenfors showed that the postulates of theory revision translated to nonmonotonic reasoning either directly correspond to the deduction rules of nonmonotonic reasoning or can be derived from them. The connection can be exemplified by the postulate (T1) corresponding to  $A \vdash A$ . Also by translating the deduction rules of nonmonotonic reasoning into theory revision we get principles entailed by the AGM-postulates. The translation of the Supraclassicality rule “if  $x \vdash y$ , then  $x \vdash y$ ” is “if  $x \vdash y$ , then  $T \circ x \models y$ ”, which is directly entailed by the postulate (R1). The Unit Reciprocity rule is “if  $x \vdash y$  and  $y \vdash x$ , then  $C(x) = C(y)$ ”, where  $C(x) = \{y \mid x \vdash y\}$ . The translation “if  $T \circ x \models y$  and  $T \circ y \models x$ , then  $T \circ x \equiv T \circ y$ ” is the postulate (U6), which is a less strict version of the postulate (R6), thus every revision operator that satisfies the postulates (R1)–(R6) satisfies the principle. Makinson and Gärdefors [MaG91] showed that the only postulate lacking a corresponding rule in most techniques of nonmonotonic reasoning is postulate (R3) which says that the resulting theory is consistent.

In artificial intelligence, reasoning about action [GiS87, Win88b] is applied when reasoning how the world is changed as a result of an action performed by a robot. In such applications we have a number of frame



problems: what remains unchanged (the frame problem), what are the ramifications (the ramification problem) and when an action can be performed (the qualification problem).

Del Val and Shoham [DVS92] have compared theory update and the theory of action. They showed that neither the KM-postulates nor the way to handle integrity constraints need to be represented as postulates, instead they can be derived from the theory of action.

Del Val and Shoham considered deterministic actions in situation calculus. *Situation calculus* is based on predicate logic having terms of three types: *situations*, *actions*, and *fluents*. Properties are propositional functions from the set of situations to truth values true and false. A property function expresses whether or not in a certain situation a property holds. As an example, in the situation calculus of the blocks world, some situations have the property “the block is on the table”, some do not. Situation calculus also contains the functions *result* and *hold*. The function *result* takes action and situation as arguments giving the new situation. The predicate *holds* takes a situation and a property as arguments. The predicate gives the truth value of the property in the situation.

The effects of the actions are given as axioms. Let  $A$  denote an action,  $C$  denote the integrity constraints and let  $P_1 \dots P_n$  denote the intermediate consequences of the action given preconditions  $R_1 \dots R_n$ . The theory thus contains the causal axioms

$$\forall s(R_i(s) \rightarrow \text{holds}(P_i, \text{result}(A, s)))$$

that give the consequences of the actions, and the integrity constraints  $\forall sC(s)$ .

The rules above let us know what is changed when an action takes place. In order to define the facts that do not change, a frame axiom is defined and used in circumscription. For this technique we need unique names axioms for fluents and situations.

In the frame axiom we choose a set of properties as an extension of a frame predicate. Choosing the properties right is important. If the frame is too large, defining the actions becomes complicated. If the frame is too small, the causal axioms become too weak [Lif90]. As an example, if we, in the blocks world, where blocks are being moved and painted, omit the colour of a block from the frame, we will not be able to reason that moving a block does not change its colour. Ramifications may cause problems, if the frame contains properties that depend on each other.

**Example 4.11.** Consider a room with two vents. According to the integrity constraints, the room is stuffy, if both of the vents are covered. The property

“the room is stuffy” may be omitted from the frame, because it is totally determined by the other properties in the frame, namely by the locations of the objects in the room. If the property were in the frame, the ramification would have to be defined in causal axioms or we would not be able to reason an intuitive result of the action moving an object on a vent.

An update may be interpreted as situation calculus as follows. Let  $S_0$  denote the situation that describes the state of the knowledge base  $T_0$  before the update takes place. The new formula  $p$  is considered to cause an action  $A_p^{S_0}$  ( $A_p^{S_0}$  is an action that in the situation  $S_0$  causes  $p$ ). The result of the update is described by  $result(A_p^{S_0}, S)$ . If  $T^S$  denotes the theory and  $Comp(T^S)$  denotes the complete theory obtained by the circumscription technique, then [DVS94]

$$T_0 \diamond p \models x, \text{ if } Comp(T^S) \models holds(T_0, S_0) \rightarrow holds(x, result(A_p^{S_0}, S_0)).$$

Different frame actions result in different operators.

Del Val and Shoham have, using the method of translation described above, derived the KM-postulates and the way to handle integrity constraints from the theory of actions. In addition, postulate (U2) needs the frame completeness condition, according to which the properties in the frame must always uniquely identify the state, that is, the truth of all the properties.

### **A word on notation**

We shall later modify these postulates further. The postulates introduced in this chapter will throughout this work be referred by using italics font, the modified postulates will be referred by using roman font.

## Chapter 5

# Revised rationality criteria for belief and knowledge change

In this thesis, we will refine<sup>1</sup> the rationality criteria for knowledge and belief change. Our aim is to (1) reject contradictive input and (2) take into account the effect of knowledge on belief change. Throughout this chapter the effect of knowledge on belief change is considered the same as the effect of integrity constraints (equations 4.1 and 4.2 on page 43) as defined by Katsuno and Mendelson [KaM91b].

We have interpreted some of the rationality criteria as static integrity constraints. We will then consider change operators as functions from propositional formulas and epistemic states satisfying these static integrity constraints to epistemic states satisfying the constraints. The rationality criteria for knowledge and belief change limit the variety of such functions.

The rationality criteria depend on the type of the change. We will start by proposing two static integrity constraints, then we will refine the postulates for belief revision and contraction. Some postulates will turn out to be redundant. We then interpret the rationality criteria for expansions as guarding knowledge expansion. By defining a small set of postulates, we will introduce a new type of change, competing evidence. *Competing evidence* is a commutative version of belief revision, where the latest piece of information is not considered to be the most reliable. We will refine the postulates for belief update and erasure, and finally we will discuss some related work.

---

<sup>1</sup>This proposal was given already in the preliminary version [Elo04, chapter 5] of this thesis.

## 5.1 Static rationality criteria on epistemic states

Some of the assumptions on the knowledge base we made in Chapter 2 may be considered as static integrity constraints on the knowledge base because they involve only one state at a time. Let  $T$  denote the epistemic state of the knowledge base with a belief set  $T_B$  and a knowledge set  $T_K$ . We propose the following two static rationality criteria on epistemic states:

- (S1):  $T_B \models T_K$ .  
 (S2):  $T_B$  is consistent.

The first constraint is motivated by Hintikka [Hin62, chapter 3], who says that what one knows, one believes. The second constraint is motivated by the AGM-postulates [AGM85]. We are going to refine the rationality criteria for belief and knowledge change in such a way that information inconsistent with the knowledge in the state will not be accepted. Together with the other rationality criteria, this makes it impossible to turn the set  $T_B$  inconsistent. Therefore, we find it reasonable to make the consistency of the beliefs in the set  $T_B$  a static integrity constraint.

## 5.2 Refined postulates for belief revision

Let us first consider the cases where new information is obtained about a static world. Let  $\circ$  denote a belief-revision operator,  $T$  an epistemic state satisfying conditions (S1) and (S2), and let  $A$  and  $B$  denote propositional formulas. We refine the joined set of postulates for belief revision (page 30) and iterated belief revision (page 31) as follows:

- (K0):  $(T \circ A)_K \equiv T_K$ .  
 (K1):  $(T \circ A)_B \models (T \circ A)_K$ .  
 (R0): If  $T_K \models \neg A$ , then  $(T \circ A)_B \equiv T_B$ .  
 (R1): If  $T_K \not\models \neg A$ , then  $(T \circ A)_B \models A$ .  
 (R2): If  $T_B \not\models \neg A$ , then  $(T \circ A)_B \equiv T_B \cup \{A\}$ .  
 (R3):  $(T \circ A)_B$  is consistent.  
 (RR1): If  $B \models A$  and  $T_K \not\models \neg B$ , then  $((T \circ A) \circ B)_B \equiv (T \circ B)_B$ .  
 (RR2): If  $B \models \neg A$  and  $T_K \not\models \neg B$ , then  $((T \circ A) \circ B)_B \equiv (T \circ B)_B$ .  
 (RR3): If  $(T \circ B)_B \models A$ , then  $((T \circ A) \circ B)_B \models A$ .  
 (RR4): If  $(T \circ B)_B \not\models \neg A$ , then  $((T \circ A) \circ B)_B \not\models \neg A$ .  
 (RR5): If  $T_B \not\models A$  and  $(T \circ B)_B \not\models \neg A$ , then  $((T \circ A) \circ B)_B \models A$ .

We propose a new postulate (K0). According to the postulate, belief revision does not affect knowledge. Postulates (K1) and (R3) say that

the static integrity constraints (S1) and (S2) must be satisfied after the revision. In postulates (R0) and (R1), an input that is inconsistent with the knowledge is rejected, even though it is the latest piece of information and would therefore be otherwise acceptable. Because the original AGM-postulates assume that we detect inconsistencies between the old and the new beliefs, we may assume that we are able to detect inconsistencies between the knowledge and the new beliefs as well. Postulate (R2) has not been changed. In postulates (RR1) and (RR2) we rule out those formulas that are inconsistent with the knowledge in the state. Postulate (RR5) is new. It says that if learning  $B$  does not contradict  $A$ , then learning  $A$  will result in an epistemic state where learning  $B$  does not make us give up believing  $A$ .

If a belief-revision operator satisfies the above-mentioned collection of postulates, it also satisfies the following postulates:

- (R4): If  $A \equiv B$ , then  $(T \circ A)_B \equiv (T \circ B)_B$ .
- (R5):  $(T \circ A)_B \cup \{B\} \models (T \circ (A \wedge B))_B$ .
- (R6): If  $(T \circ A)_B \not\models \neg(A \wedge B)$ , then  $(T \circ (A \wedge B))_B \models (T \circ A)_B \cup \{B\}$ .

Postulates (R4) and (R5) have not been changed, only postulate (R6) has been refined. These postulates are not included in our proposal because their redundancy.

**Theorem 5.1.** *If a belief-revision operator satisfies postulates (R0)–(R3), and (RR1), then it satisfies postulate (R4).*

*Proof.* Let  $\circ$  be a revision operator that satisfies postulates (R0)–(R3) and (RR1). Let  $T$  denote an epistemic state that satisfies integrity constraints (S1) and (S2), and let  $A$  and  $B$  denote propositional formulas such that  $A \equiv B$ .

If  $T_K \models \neg A$ , then  $T_K \models \neg B$  and (R0) gives us  $(T \circ A)_B \equiv T_B \equiv (T \circ B)_B$ . If  $T_K \not\models \neg A$ , then  $T_K \not\models \neg B$  and (R1) gives us  $(T \circ A)_B \models A$ . Thus  $(T \circ A)_B \models B$  and  $(T \circ A)_B \equiv (T \circ A)_B \cup \{B\}$ . By (R3),  $(T \circ A)_B \not\models \neg B$ , and (R2) gives us  $((T \circ A) \circ B)_B \equiv (T \circ A)_B \cup \{B\}$ . (RR1) finally gives us  $((T \circ A) \circ B)_B \equiv (T \circ B)_B$ .  $\square$

**Theorem 5.2.** *If a belief-revision operator  $\circ$  satisfies postulates (K0), (K1) (R0)–(R2), and (RR1), then it satisfies postulates (R5) and (R6).*

*Proof.* Let  $\circ$  denote a revision operator that satisfies postulates (K0), (K1), (R0)–(R2), and (RR1). Let  $T$  denote an epistemic state that satisfies integrity constraints (S1) and (S2), and let  $A$  and  $B$  denote propositional formulas. If  $(T \circ A)_B \models \neg B$ , then (R5) and (R6) hold trivially.

Assume  $(T \circ A)_B \not\models \neg B$ . Then by (K1),  $(T \circ A)_K \not\models \neg B$ , and by (K0),  $T_K \not\models \neg B$ .

If  $T_K \models \neg A$ , then (R0) gives us  $(T \circ A)_B \equiv T_B \equiv (T \circ (A \wedge B))_B$ . By (S1),  $T_B \models \neg A$ , thus  $(T \circ A)_B \models \neg A$  and (R6) holds trivially. Finally,  $(T \circ A)_B \cup \{B\} \equiv T_B \cup \{B\} \models T_B \equiv (T \circ (A \wedge B))_B$ . Thus, (R5) holds.

Assume  $T_K \not\models \neg A$ . Postulate (R1) then says that  $(T \circ A)_B \models A$ , thus  $(T \circ A)_B \not\models \neg(A \wedge B)$  and  $(T \circ A)_B \cup \{B\} \equiv (T \circ A)_B \cup \{A \wedge B\}$ . Then (R2) gives us  $((T \circ A) \circ (A \wedge B))_B \equiv (T \circ A)_B \cup \{A \wedge B\}$ . Because  $(T \circ A)_B \not\models \neg(A \wedge B)$ , (K1) gives us  $(T \circ A)_K \not\models \neg(A \wedge B)$ , and by (K0),  $T_K \not\models \neg(A \wedge B)$ . (RR1) then gives us  $((T \circ A) \circ (A \wedge B))_B \equiv (T \circ (A \wedge B))_B$ , that is, (R5) and (R6) hold.  $\square$

Let us next motivate our new postulate (RR5) by the following example.

**Example 5.1.** Assume a situation, in which a new family has moved next door. You have not seen them yet, but you have learned that they have three daughters, Eleanor, Marianne, and Margaret, born in that order. You therefore believe that Eleanor is taller than Marianne. You have no beliefs concerning whether or not Margaret has blue eyes, and should you learn that Marianne is taller than Eleanor, you would still have no beliefs concerning the colour of Margaret's eyes. You then first see Margaret and learn that she has got brown eyes, and after that you meet the two older daughters and see that Marianne is taller than Eleanor. Would it not be unintuitive to lose then one's belief concerning Margaret?

Our postulates for belief revision suggest that we may express knowledge using doxastic conditionals.

**Theorem 5.3.** *Given an epistemic state  $T$  that satisfies constraints (S1) and (S2), a revision operator  $\circ$  that satisfies postulates (R0)–(R3), and a propositional formula  $A$ , then  $T_K \models A$  if and only if  $(T \circ \neg A)_B \models A$ .*

*Proof.* Assume an epistemic state  $T$  that satisfies constraints (S1) and (S2), a revision operator  $\circ$  that satisfies postulates (R0)–(R3), and a propositional formula  $A$ .

If  $T_K \models A$ , then by (S1),  $T_B \models A$ , and by (R0),  $(T \circ \neg A)_B \models A$ . If  $T_K \not\models A$ , then by (R1),  $(T \circ \neg A)_B \models \neg A$ , and by (R3),  $(T \circ \neg A)_B \not\models A$ .  $\square$

### 5.3 Refined postulates for belief contraction

Next we will refine the postulates for contraction, and we will also propose a new set of postulates for the interaction between contraction and revision.

Let  $\circ$  denote a belief-revision operator,  $T$  an epistemic state satisfying conditions (S1) and (S2), and let  $A$  and  $B$  denote propositional formulas.

Let us consider a contraction  $T \bullet A$ , where the beliefs in an epistemic state  $T$  satisfying constraints (S1) and (S2) are contracted with a formula  $A$  by using an operator  $\bullet$ . For contractions (page 33), we propose the following collection of rationality criteria, which also involve a belief revision operator  $\circ$ :

- (K0):  $(T \bullet A)_K \equiv T_K$ .
- (K1):  $(T \bullet A)_B \models (T \bullet A)_K$ .
- (C0): If  $T_K \models A$ , then  $(T \bullet A)_B \equiv T_B$ .
- (C1): If  $T_K \not\models A$ , then  $(T \bullet A)_B \not\models A$ .
- (C2): If  $T_B \not\models A$ , then  $(T \bullet A)_B \equiv T_B$ .
- (C3):  $T_B \models (T \bullet A)_B$ .
- (CR1): If  $B \models A$  and  $T_K \not\models \neg B$ , then  $((T \bullet A) \circ B)_B \equiv (T \circ B)_B$ .
- (CR2): If  $B \models \neg A$  and  $T_K \not\models \neg B$ , then  $((T \bullet A) \circ B)_B \equiv (T \circ B)_B$ .
- (CR3): If  $(T \circ B)_B \models \neg A$ , then  $((T \bullet A) \circ B)_B \models \neg A$ .
- (CR4): If  $(T \circ B)_B \not\models A$ , then  $((T \bullet A) \circ B)_B \not\models A$ .

Postulate (K0) says that belief contraction does not affect knowledge. Postulate (K1) says that constraint (S1) should hold after revision. If constraint (S2) is satisfied before a contraction, then by (C3) it remains so after the contraction. By postulates (C0) and (C1), the contraction succeeds only if the input is not knowledge. Postulates (C2) and (C3) have not been changed. Postulates (CR1)–(CR4) correspond to postulates (RR1)–(RR4) for iterated revisions, and their motivation is analogous to those.

If a belief-contraction operator satisfies the above-mentioned collection of postulates, it also satisfies the following postulates:

- (C4): If  $A \equiv B$ , then  $(T \bullet A)_B \equiv (T \bullet B)_B$ .
- (C5):  $(T \bullet A)_B \cup \{A\} \models T_B$ .

**Theorem 5.4.** *If operator  $\bullet$  satisfies postulates (C1)–(C3), (CR1), and (CR2), then it satisfies postulate (C4).*

*Proof.* Let  $\bullet$  denote a contraction operator that satisfies postulates (C1)–(C3), (CR1), and (CR2), and let  $\circ$  denote a revision operator that satisfies postulates (R2) and (R4). Let  $T$  denote an epistemic state that satisfies integrity constraints (S1) and (S2), and let  $A$  and  $B$  denote propositional formulas such that  $A \equiv B$ .

If  $T_K \models A$ , then by (C0), the case is trivial. If  $T_B \not\models A$ , then by (C2), the case is trivial.

Assume  $T_K \not\models A$  and  $T_B \models A$ . Then by (S2),  $T_B \not\models \neg A$ , and by (S1),  $T_K \not\models \neg A$ .

Postulate (C1) gives us  $(T \bullet A)_B \not\models A$ , and by (C3),  $(T \bullet A)_B \not\models \neg A$ . Then postulates (R2) and (CR1) give us  $(T \bullet A)_B \cup \{A\} \equiv ((T \bullet A) \circ A)_B \equiv (T \circ A)_B$  and postulates (R4), (CR1), and (R2) give us  $(T \circ A)_B \equiv (T \circ B)_B \equiv ((T \bullet B) \circ B)_B \equiv (T \bullet B)_B \cup \{B\} \equiv (T \bullet B)_B \cup \{A\}$ . Postulates (R2), (CR2), and (R4) give us  $(T \bullet A)_B \cup \{\neg A\} \equiv ((T \bullet A) \circ \neg A)_B \equiv (T \circ \neg A)_B \equiv (T \circ \neg B)_B \equiv ((T \bullet B) \circ \neg B)_B \equiv (T \bullet B)_B \cup \{\neg B\} \equiv (T \bullet B)_B \cup \{\neg A\}$ . Thus  $(T \bullet A)_B \equiv (T \bullet B)_B$ .  $\square$

**Theorem 5.5.** *If operator  $\bullet$  satisfies postulates (C2), (C3), (CR1) and (CR2), then it satisfies postulate (C5).*

*Proof.* Let  $\circ$  denote a revision operator that satisfies postulate (R2), and let  $\bullet$  denote a contraction operator that satisfies postulates (C2), (C3), (CR1), and (CR2). Let  $T$  denote an epistemic state that satisfies integrity constraints (S1) and (S2), and let  $A$  denote a propositional formula.

If  $T_B \not\models A$ , then by (C2),  $(T \bullet A)_B \equiv T_B$  and (C5) holds.

Assume  $T_B \models A$ . Then by (S2),  $T_B \not\models \neg A$ , and by (C3),  $(T \bullet A)_B \not\models \neg A$ . Then by (R2),  $(T \bullet A)_B \cup \{A\} \equiv ((T \bullet A) \circ A)_B$ . By (S1),  $T_K \not\models \neg A$ , and by (CR1),  $((T \bullet A) \circ A)_B \equiv (T \circ A)_B$ . Then by (R2),  $(T \circ A)_B \equiv T_B \cup \{A\} \equiv T_B$ . Thus (C5) holds.  $\square$

## 5.4 Rationality criteria for knowledge expansion

According to Hintikka [Hin62, chapter 2], knowledge is something we are ready to defend no matter what we might learn. In addition to that, knowledge is always true. Thus the only change we may have concerning knowledge is monotonous expansion.

Let  $\oplus$  denote a knowledge-expansion operator,  $T$  an epistemic state satisfying conditions (S1) and (S2), and let  $A$  and  $B$  denote propositional formulas that are consistent with  $T_K$ . We propose the following postulates as rationality criteria for knowledge expansion:

- (K1):  $(T \oplus A)_B \models (T \oplus A)_K$ .
- (K2): If  $T_K \not\models \neg A$ , then  $(T \oplus A)_K \equiv T_K \cup \{A\}$ .
- (R2): If  $T_B \not\models \neg A$ , then  $(T \oplus A)_B \equiv T_B \cup \{A\}$ .
- (R3):  $(T \oplus A)_B$  is consistent.
- (RR1): If  $B \models A$  and  $T_K \not\models \neg B$ , then  $((T \oplus A) \oplus B)_B \equiv (T \oplus B)_B$ .

Note that we have assumed that here we only consider input that is consistent with the old knowledge. Postulate (K2) is analogous to postulate (R2). Postulates (K1) and (R3) say that (S1) and (S2) should be satisfied after the expansion.



Postulate (K1) together with postulate (K2) suggests that beliefs may have to be changed when knowledge changes. Therefore any knowledge-expansion operator must also satisfy the postulates for belief revision. It is, however, sufficient to have only postulates (R2), (R3), and (RR1). Together postulates (K1) and (K2) imply (R1). Postulate (RR2) is not applicable here, and in those cases when (RR3), (RR4), or (RR5) could be applicable, they are implied by (K2) together with (K1) and (R3).

## 5.5 Rationality criteria for competing evidence

We will now give a set of postulates for a new type of belief change that we call competing evidence. It relates to belief revision: in both cases we get new information about a static world. The difference lies in the epistemic attitude towards the new piece of evidence: in competing evidence the input has no priority over the old beliefs.

Let  $T$  denote an epistemic state that satisfies constraints (S1) and (S2),  $*$  a competing-evidence operator,  $\circ$  a revision operator,  $A$  and  $B$  propositional formulas. We propose the following postulates for competing evidence:

- (K0):  $(T * A)_K \equiv T_K$ .
- (K1):  $(T * A)_B \models (T * A)_K$ .
- (R0): If  $T_K \models \neg A$ , then  $(T * A)_B \equiv T_B$ .
- (R2): If  $T_B \not\models \neg A$ , then  $(T * A)_B \equiv T_B \cup \{A\}$ .
- (R3):  $(T * A)_B$  is consistent.
- (NP1):  $((T * A) * B)_B \equiv ((T * B) * A)_B$ ,
- (NR1): If  $B \models A$  and  $T_K \not\models \neg B$ , then  $((T * A) \circ B)_B \equiv (T \circ B)_B$ .
- (NR2): If  $B \models \neg A$  and  $T_K \not\models \neg B$ , then  $((T * A) \circ B)_B \equiv (T \circ B)_B$ .
- (NR3): If  $(T \circ B)_B \not\models \neg A$ , then  $((T * A) \circ B)_B \models A$ .

Postulate (K0) states that competing evidence does not affect knowledge, and according to postulate (R0), input that is contradictory to knowledge will not be accepted. The belief-revision postulate (R2) is needed also in competing-evidence change: a consistent solution is strived for, whenever such a solution exists. Postulates (K1) and (R3) say that the static integrity constraints (S1) and (S2) must be satisfied in the resulting epistemic state. Postulate (NP1) calls for commutativity. Postulates (NR1)–(NR3) correspond to postulates (RR1), (RR2), and (RR6).

## 5.6 Refined postulates for belief update and erasure

Let us consider belief updates, that is, inserting to a theory a formula that records a change in the world. We will refine the KM-postulates to reject beliefs known to be false.

Let  $\diamond$  denote a belief-update operator,  $T$ ,  $T'$  and  $T''$  epistemic states satisfying constraints (S1) and (S2), and let  $A$  and  $B$  denote propositional formulas. We do not assume that epistemic states are represented by single propositional formulas, but we do assume that their knowledge sets and belief sets can be represented by propositional formulas, and thus they denote here propositional formulas. The postulates for belief update (page 35) are refined as follows:

- (K0):  $(T \diamond A)_K \equiv T_K.$
- (K1):  $(T \diamond A)_B \models (T \diamond A)_K.$
- (U0): If  $T_K \models \neg A$ , then  $(T \diamond A)_B \equiv T_B.$
- (U1): If  $T_K \not\models \neg A$ , then  $(T \diamond A)_B \models A.$
- (U2): If  $T_B \models A$ , then  $(T \diamond A)_B \equiv T_B.$
- (U3):  $(T \diamond A)_B$  is consistent.
- (U4): If  $T_K \equiv T'_K$ ,  $T_B \equiv T'_B$ , and  $A \equiv B$ , then  $(T \diamond A)_B \equiv (T' \diamond B)_B.$
- (U5):  $(T \diamond A)_B \cup \{B\} \models (T \diamond (A \wedge B))_B.$
- (U6): If  $(T \diamond A)_B \models B$  and  $(T \diamond B)_B \models A$ , then  $(T \diamond A)_B \equiv (T \diamond B)_B.$
- (U7): If  $T_B$  is complete, then  $(T \diamond A)_B \cup (T \diamond B)_B \models (T \diamond (A \vee B))_B.$
- (U8): If  $T_K \equiv T'_K \equiv T''_K$  and  $T_B \equiv (T'_B \vee T''_B)$ , then  $(T \diamond A)_B \equiv (T' \diamond A)_B \vee (T'' \diamond A)_B.$

Postulate (K0) says that belief change does not affect the knowledge in the state. In (U0) and (U1) we refuse to accept beliefs that are inconsistent with the knowledge in the state; that is why we do not get inconsistent beliefs into our epistemic state. In (U4) and (U8) we have again taken the knowledge into account. Postulates (K1) and (U3) say that constraints (S1) and (S2) must be satisfied after the update.

Let us next consider an update  $T \blacklozenge A$ , where a formula  $A$  is erased from a theory  $T$ . We refine the KM-postulates [KaM91a] for belief update (page 37) as follows:

- (K0):  $T_K \equiv (T \blacklozenge A)_K.$
- (K1):  $(T \blacklozenge A)_B \models (T \blacklozenge A)_K.$
- (E0): If  $T_K \models A$ , then  $(T \blacklozenge A)_B \equiv T_B.$
- (E1): If  $T_K \not\models A$ , then  $(T \blacklozenge A)_B \not\models A.$
- (E2): If  $T_B \models \neg A$ , then  $(T \blacklozenge A)_B \equiv T_B.$
- (E3):  $T_B \models (T \blacklozenge A)_B.$

- (E4): If  $T_K \equiv T'_K$ ,  $T_B \equiv T'_B$ , and  $A \equiv B$ , then  $(T \blacklozenge A)_B \equiv (T' \blacklozenge B)_B$ .  
 (E5):  $(T \blacklozenge A)_B \cup \{A\} \models T_B$ .  
 (E8): If  $T_K \equiv T'_K \equiv T''_K$  and  $T_B \equiv (T'_B \vee T''_B)$ ,  
 then  $(T \blacklozenge A)_B \equiv (T' \blacklozenge A)_B \vee (T'' \blacklozenge A)_B$ .

We have added postulate (K0) saying that belief change does not affect the knowledge in the state. In (E1) we refuse to erase beliefs that are known in the state. In (E4) and (E8) we have again taken the knowledge into account. Postulates (K1) and (E3) guarantee that the static integrity constraints (S1) and (S2) are satisfied after the erasure.

## 5.7 Related work

In the former studies it has not always been made explicit whether belief or knowledge change has been considered [FrH96]. We have taken the AGM-postulates [AGM85] and the DP-postulates [DaP97] as rationality criteria for belief revision. According to our interpretation, the set of postulates suggested by Gärdenfors [Gär88, chapter 3] for monotonous expansions guard knowledge change, while the KM-postulates [KaM91a] guard belief change.

We have modified the AGM-, DP-, and KM-postulates to reject beliefs known to be false, and we have strengthened postulates (R3) and (U3) by introducing constraint (S2). Unsatisfiable beliefs have also been rejected or omitted in the studies of Friedman and Halpern [FrH94], Spohn [Spo88], and Fagin et al. [FUV83].

In belief revision, our postulate (RR5) resembles the following postulate of independence by Yi and Thielscher [YiT07]:

$$(RR6): \quad \text{If } (T \circ B)_B \not\models \neg A, \text{ then } ((T \circ A) \circ B)_B \models A.$$

In the case  $T_B \models A$ , postulate (RR5) does not imply (RR3) nor (RR4), so it is complementary to them, whereas postulate (RR6) implies postulates (RR3), (RR4), and (RR5). We will consider postulate (RR6) only optional, because it prevents idempotency.

Our Theorem 5.3, saying  $T_K \models A$  if and only if  $(T \circ \neg A)_B \models A$ , resembles the theorem  $K\alpha \leftrightarrow B^{-\alpha}\alpha$  by Lamarre and Shoham [LaS94]. The theorem can be read “knowing  $\alpha$  is equivalent to believing  $\alpha$  given evidence  $\neg\alpha$ ”. However, Lamarre and Shoham, unlike us, accept inconsistent beliefs. Then  $B^{-\alpha}\alpha$  says that beliefs under the hypothesis  $\neg\alpha$  are not consistent.

In knowledge expansion, if we were to give up the assumption of knowledge always being true, we would have something we might call certainty [LaS94] or convictions [Nyk11] instead of knowledge. Then the

issue of inconsistency of input might arise. To deal with the possible inconsistency, we might alter the postulates so that input that is inconsistent with the old convictions would result in the state of total ignorance:

$$(K0): \quad \text{If } T_K \models \neg A, \text{ then } T \oplus A = \tau.$$

For the motivation of this rationality criterion, imagine a person has just been convinced that some piece of information he has considered as irrevocable might not be true after all. He would then utter: "I no longer know what to believe in".

In competing evidence, if we want the new formula to be believed after the change (postulate (R1)), or even if we only want the formula to be considered compatible, we cannot expect commutativity of change. Therefore, if we call for commutativity, postulate (R1) has to be given up. Our rationality criteria on competing evidence differ from those on arbitration introduced by Revesz [Rev93], because we have chosen to keep postulate (R2), not postulate (R1).

Those variations of belief revision in which postulate (R1) does not hold due to rejecting some unbelievable input have been called 'non-prioritized belief revision' [Han99]. In those studies, however, believable inputs have still been prioritized over the old beliefs. We would rather call these variations (including that of ours) as 'prioritized belief revision', because it is our competing evidence, which is truly non-prioritized.

# Chapter 6

## New operator collections

In this chapter we will now introduce two sets of operators for knowledge and belief change. The aim is to demonstrate that finite, concrete implementations of knowledge bases exist such that the rationality criteria proposed for belief and knowledge change are satisfied.

We shall consider operators on two different representations of epistemic states. One is a finite ordered nonempty set of satisfiable propositional formulas, which we will call *an epistemic base*. The other is a finite set of ranked formulas, which we will call *an epistemic function*. Epistemic bases are syntactic versions of Spohn's [Spo88] simple conditional functions, epistemic functions are syntactic versions of Spohn's ranking functions (page 24). Both simple conditional functions and ranking functions carry the ordering of disbelief, but unlike ranking functions, simple conditional functions (and our epistemic bases) cannot carry gaps in the ordering.

The operators have been presented in the preliminary version of this thesis [Elo04].

### 6.1 Operators on epistemic bases

We shall first give a formal definition of an epistemic base and then introduce a small collection of change operators on epistemic bases: a knowledge-expansion operator  $\oplus$ , a belief-revision operator  $\circ$ , a belief-contraction operator  $\bullet$ , and belief-update operators  $\diamond$  and  $\diamond'$ .

#### Epistemic bases

**Definition 6.1.** An epistemic base  $T$  is a linearly ordered structure  $T = \langle S, R \rangle$ , where  $S \subseteq \mathcal{L}$  is a finite nonempty set of satisfiable propositional

formulas that are pairwise inconsistent with each other, and  $R \subseteq S \times S$  is a linear ordering on  $S$ .

Because  $R$  is a linear ordering,  $R$  is transitive, and for all  $s, s' \in S$ , exactly one of the three alternatives  $sRs'$ ,  $s = s'$ , and  $s'R s$  holds [End77]. Because  $S$  is finite, any of its subsets has a minimal element in any linear ordering on  $S$ . Thus  $R$  is a well-ordering and  $\min(S, R)$  is the minimal element in  $S$ .

The intuition behind epistemic bases goes like this. The ordering  $R$  is an ordering of disbelief. The minimal element in the ordering is the most plausible one of all the formulas in  $S$ , and so on. A propositional formula  $A$  is then believed in the epistemic state, if it is entailed by the most plausible formula in  $S$ , and it is known in the epistemic state, if it is entailed by all the formulas in  $S$ . Thus the minimal element in  $S$  represents the belief set, whereas the disjunction of all the elements in  $S$  represents the knowledge set, that is,

$$T_B \equiv \min(S, R) \quad (6.1)$$

and

$$T_K \equiv \bigvee S. \quad (6.2)$$

Then by definition, both  $T_B \models T_K$  and  $T_B \not\models \perp$  hold, which means that every epistemic base satisfies constraints (S1) and (S2). The state of total ignorance can be represented by an epistemic base  $\tau = \langle \{\top\}, \emptyset \rangle$ .

Given an epistemic base  $T = \langle S, R \rangle$ , then we define for each  $s \in S$

$$\text{ord}(s) =_{\text{def}} |\{s' \in S : (s', s) \in R\}|. \quad (6.3)$$

If  $|S| = n$ , then we may equivalently refer to  $T$  by using a list formulation

$$T = [T^0, T^1, \dots, T^{n-1}], \quad (6.4)$$

where each  $T^i$ ,  $0 \leq i < n$ , is a formula  $s \in S$  such that  $\text{ord}(s) = i$ . At the top of the list is the most plausible formula and so on. Thus  $T_B \equiv T^0$  and  $T_K \equiv \bigvee_{i=0}^{n-1} T^i$ . By definition,  $(T^i, T^j) \in R$ ,  $T^i \not\models \perp$ ,  $T^j \not\models \perp$ , and  $T^i \wedge T^j \models \perp$  for all  $0 \leq i < j < n$ .

**Example 6.1.** Let us consider an epistemic base  $T = \langle \{a \wedge b, a \wedge \neg b, \neg a \wedge c\}, \{(a \wedge b, a \wedge \neg b), (a \wedge b, \neg a \wedge c), (a \wedge \neg b, \neg a \wedge c)\} \rangle$ , or equivalently,  $T = [T^0, T^1, T^2]$ , where  $T^0 = a \wedge b$ ,  $T^1 = a \wedge \neg b$ , and  $T^2 = \neg a \wedge c$ . Then  $T_B \equiv a \wedge b$  and  $T_K \equiv (a \wedge b) \vee (a \wedge \neg b) \vee (\neg a \wedge c) \equiv a \vee c$ .

Epistemic bases  $T = [T^0, T^1, \dots, T^{n-1}]$  and  $U = [U^0, U^1, \dots, U^{m-1}]$  are defined to be *equivalent*, if and only if  $n = m$  and  $T^i \equiv U^i$  holds for all  $i$ ,

$0 \leq i < n$ . It is easy to see that this relation on epistemic bases is reflexive, symmetric, and transitive, that is, an equivalence relation.

Epistemic bases (as well as epistemic functions) carry the epistemic entrenchment involved in belief revision. A formula  $A$  is less or equally entrenched compared to a formula  $B$ , if either  $T_K \models B$ , or  $\neg A$  is consistent with some formula  $T^i$ ,  $0 \leq i < n$  and  $\neg B$  is inconsistent with all formulas  $T^j$ ,  $0 \leq j < i$ .

### A knowledge-expansion operator

Our knowledge-expansion operator  $\oplus$  is used to insert propositional formulas of knowledge into epistemic bases. When the input is consistent with  $T_K$ , the operator cuts down the number of possible models by rejecting those possible models that do not model the new formula.

**Definition 6.2** (Operator  $\oplus$ ). Given an epistemic base  $T = [T^0, T^1, \dots, T^{n-1}]$  and a propositional formula  $A$ , then  $T \oplus A =_{def} \langle \{\top\}, \emptyset \rangle$ , if  $T_K \models \neg A$ . Otherwise, we define for all  $i$ ,  $0 \leq i < n$ , formulas  $U^i = T^i \wedge A$ . Finally, we define  $T \oplus A =_{def} \langle S', R' \rangle$  where

$$S' = \{U^i \mid 0 \leq i < n, U^i \not\equiv \perp\}, \quad (6.5)$$

and

$$R' = \{(U^i, U^j) \mid 0 \leq i < j < n\} \cap (S' \times S'). \quad (6.6)$$

**Example 6.2.** Let us consider expanding the epistemic base  $T = [a \wedge b, a \wedge \neg b, \neg a \wedge c]$  by a formula  $b$ . We calculate the new formulas as follows:

$$\begin{aligned} U^0 &\equiv T^0 \wedge b \equiv (a \wedge b) \wedge b \equiv a \wedge b \\ U^1 &\equiv T^1 \wedge b \equiv (a \wedge \neg b) \wedge b \equiv \perp \\ U^2 &\equiv T^2 \wedge b \equiv (\neg a \wedge c) \wedge b \equiv \neg a \wedge b \wedge c. \end{aligned}$$

The result is the structure  $T \oplus b \equiv \langle \{a \wedge b, \neg a \wedge b \wedge c\}, \{(a \wedge b, \neg a \wedge b \wedge c)\} \rangle$ , or equivalently,  $T \oplus b \equiv [a \wedge b, \neg a \wedge b \wedge c]$ . Thus  $(T \oplus b)_B \equiv a \wedge b$  and  $(T \oplus b)_K \equiv (a \wedge b) \vee (\neg a \wedge b \wedge c) \equiv (a \vee c) \wedge b$ .

By definition, the operator maps equivalent epistemic bases to equivalent epistemic bases. It also satisfies our postulates for knowledge expansion.

**Theorem 6.1.** *The operator  $\oplus$  is a function from epistemic bases and propositional formulas to epistemic bases and it satisfies postulates (K1), (K2), (R2), (R3), and (RR1).*

*Proof.* Let  $T = [T^0, T^1, \dots, T^{n-1}]$  (or equivalently,  $T = \langle S, R \rangle$ ) denote an epistemic base, let  $A$  denote a propositional formula, and let  $\oplus$  denote the knowledge-expansion operator defined above. If  $T_K \models \neg A$ , then the claim holds trivially. Assume  $T_K \not\models \neg A$ .

Because  $(\bigvee_{i=0}^{n-1} T^i) \wedge A \equiv \bigvee_{i=0}^{n-1} (T^i \wedge A)$ , postulate (K2) is satisfied, and because  $T_K \wedge A \not\models \perp$ , the set  $S' = \{T^i \wedge A \mid 0 \leq i < n, T^i \wedge A \not\models \perp\}$  is nonempty. By definition the set  $S'$  is also finite and all elements in the set are satisfiable. Let  $s, s' \in S'$ . Then  $s = T^i \wedge A$  and  $s' = T^j \wedge A$  for some  $i, j, 0 \leq i, j < n$ . If  $s \neq s'$ , then  $i \neq j$ ,  $T^i \wedge T^j \models \perp$  and thus  $(T^i \wedge A) \wedge (T^j \wedge A) \models \perp$ . Equation  $i = j$  holds if and only if  $s = s'$ . Equation  $i < j$  holds if and only if  $(T^i \wedge A, T^j \wedge A) \in R'$ , equation  $j < i$  holds if and only if  $(T^j \wedge A, T^i \wedge A) \in R'$ . Thus either  $s = s'$ ,  $s < s'$  or  $s' < s$ . If  $(s, s'), (s', s'') \in R'$ , then  $s = T^i \wedge A$ ,  $s' = T^j \wedge A$ , and  $s'' = T^k \wedge A$  for some  $0 \leq i < j < k < n$ . Because  $i < k$ ,  $(s, s'') \in R'$ . Thus the relation  $R'$  is a linear ordering and the structure  $T \oplus A = \langle S', R' \rangle$  is an epistemic base and, as such, satisfies (K1) and (R3) by definition.

If  $T_B \not\models \neg A$ , then  $(T \oplus A)^0 = T^0 \wedge A$ . Postulate (R2) is satisfied.

To prove (RR1), assume  $B \models A$  and  $T_K \not\models \neg B$ . For the forward direction, assume  $w \models ((T \oplus A) \oplus B)_B$ . Then  $w \models ((T \oplus A) \oplus B)^0 \equiv (T^k \wedge A) \wedge B$  for some  $k, 0 \leq k < n$  with  $T^i \wedge A \models \neg B$  for all  $0 \leq i < k$ . Then  $T^i \models \neg B$  for all  $0 \leq i < k$  and  $w \models (T \oplus B)^0$ . For the converse direction, assume  $w \models (T \oplus B)_B$ . Then  $w \models (T^k \wedge B)$  for some  $0 \leq k < n$  with  $T^i \models \neg B$  for all  $0 \leq i < k$ . Then  $T^i \wedge A \models \neg B$  for all  $0 \leq i < k$  and  $w \models ((T \oplus A) \oplus B)^0$ . Thus postulate (RR1) is satisfied.  $\square$

## A Belief-revision operator

Even though ranking is not used in epistemic bases, it is used in the process of belief change. Our belief-revision operator  $\circ$  is a variant of the belief-revision operator (page 25) by Darwiche and Pearl [DaP94], applied on epistemic bases.

**Definition 6.3** (Operator  $\circ$ ). Given an epistemic base  $T = [T^0, T^1, \dots, T^{n-1}]$  and a propositional formula  $A$ , we define  $T \circ A$  as follows. If  $T_K \models \neg A$  or  $T_B \models A$ , then  $T \circ A =_{\text{def}} T$ . Otherwise, let  $m = \min(\{i \mid 0 \leq i < n, T^i \not\models \neg A\})$ . We define formulas  $U^0, U^1, \dots, U^n$ :

$$U^i = \begin{cases} T^m \wedge A & \text{if } i = 0, \\ (T^{m+i} \wedge A) \vee (T^{i-1} \wedge \neg A) & \text{if } 0 < i < n - m, \text{ and} \\ T^{i-1} \wedge \neg A & \text{if } n - m \leq i \leq n. \end{cases} \quad (6.7)$$

Finally, we define  $T \circ A =_{\text{def}} \langle S', R' \rangle$ , where  $S' = \{U^i \mid 0 \leq i \leq n, U^i \not\models \perp\}$ , and  $R' = \{(U^i, U^j) \mid 0 \leq i < j \leq n\} \cap (S' \times S')$ .



If the input is already believed or if it is unbelievable, the operator leaves the epistemic base unchanged. Otherwise, the operator splits each formula of the epistemic base into two formulas: one with a conjunction with the input and the other with a conjunction with the negation of the input. These formulas are then separately shifted in the ordering making the formulas with the negation of the new formula more disbelieved in the ordering compared to the others as shown in Table 6.1 (page 77).

**Example 6.3.** Let us consider revising an epistemic base  $T = \tau \oplus (a \vee c) \equiv \langle \{a \vee c\}, \emptyset \rangle$  first by  $a$  and then by  $a \wedge b$ . The base  $T$  and the new formulas are calculated as follows:

$$(T \circ a)^0 \equiv T^0 \wedge a \equiv (a \vee c) \wedge a \equiv a$$

$$(T \circ a)^1 \equiv T^0 \wedge \neg a \equiv (a \vee c) \wedge \neg a \equiv \neg a \wedge c$$

$$\begin{aligned} ((T \circ a) \circ (a \wedge b))^0 &\equiv (T \circ a)^0 \wedge (a \wedge b) \\ &\equiv a \wedge (a \wedge b) \\ &\equiv a \wedge b \end{aligned}$$

$$\begin{aligned} ((T \circ a) \circ (a \wedge b))^1 &\equiv ((T \circ a)^1 \wedge (a \wedge b)) \vee ((T \circ a)^0 \wedge \neg(a \wedge b)) \\ &\equiv ((\neg a \wedge c) \wedge (a \wedge b)) \vee (a \wedge \neg(a \wedge b)) \\ &\equiv a \wedge \neg b \end{aligned}$$

$$\begin{aligned} ((T \circ a) \circ (a \wedge b))^2 &\equiv (T \circ a)^1 \wedge \neg(a \wedge b) \\ &\equiv (\neg a \wedge c) \wedge \neg(a \wedge b) \\ &\equiv \neg a \wedge c. \end{aligned}$$

We get  $T \circ a \equiv [a, \neg a \wedge c]$  and  $(T \circ a) \circ (a \wedge b) \equiv [a \wedge b, a \wedge \neg b, \neg a \wedge c]$ . The result is the epistemic base introduced in Example 6.1.

**Example 6.4.** Let us consider revising an epistemic base  $T \equiv \tau \circ (a \wedge b) \equiv [a \wedge b, \neg a \vee \neg b]$  by a formula  $\neg a$ . The epistemic bases are calculated as follows:

$$\begin{aligned} (T \circ \neg a)^0 &\equiv T^1 \wedge \neg a \equiv (\neg a \vee \neg b) \wedge \neg a \equiv \neg a \\ (T \circ \neg a)^1 &\equiv T^0 \wedge \neg \neg a \equiv (a \wedge b) \wedge a \equiv a \wedge b \\ (T \circ \neg a)^2 &\equiv T^1 \wedge \neg \neg a \equiv (\neg a \vee \neg b) \wedge a \equiv a \wedge \neg b \end{aligned}$$

We then revise the base  $T \circ \neg a$  by a formula  $a$ :

$$\begin{aligned} U^0 &\equiv (T \circ \neg a)^1 \wedge a \equiv a \wedge b \\ U^1 &\equiv ((T \circ \neg a)^2 \wedge a) \vee ((T \circ \neg a)^0 \wedge \neg a) \equiv \neg a \vee \neg b \\ U^2 &\equiv (T \circ \neg a)^1 \wedge \neg a \equiv \perp \\ U^3 &\equiv (T \circ \neg a)^2 \wedge \neg a \equiv \perp. \end{aligned}$$

The result  $[a \wedge b, \neg a \vee \neg b]$  is equivalent to the result in Example 3.12. We can see that here  $T$  is recovered as when using semantically-oriented operators.

**Example 6.5.** Let us revise an epistemic base  $T \equiv \tau \circ (a \vee b) \equiv [a \vee b, \neg a \wedge \neg b]$  by a formula  $a$ . The new epistemic base is calculated as follows:

$$\begin{aligned} (T \circ a)^0 &\equiv T^0 \wedge a && \equiv a \\ (T \circ a)^1 &\equiv (T^1 \wedge a) \vee (T^0 \wedge \neg a) && \equiv \neg a \wedge b \\ (T \circ a)^2 &\equiv T^1 \wedge \neg a && \equiv \neg a \wedge \neg b \end{aligned}$$

We then revise the base  $(T \circ a)$  by a formula  $\neg a$ :

$$\begin{aligned} U^0 &\equiv (T \circ a)^1 \wedge \neg a && \equiv \neg a \wedge b \\ U^1 &\equiv ((T \circ a)^2 \wedge \neg a) \vee ((T \circ a)^0 \wedge \neg \neg a) && \equiv a \vee \neg b \\ U^2 &\equiv (T \circ a)^1 \wedge \neg \neg a && \equiv \perp \\ U^3 &\equiv (T \circ a)^2 \wedge \neg \neg a && \equiv \perp. \end{aligned}$$

The result  $[\neg a \wedge b, a \vee \neg b]$  is equivalent to the result in Example 3.13. As when using syntactically-oriented operators,  $b$  is believed in the resulting epistemic base.

By definition, the operator maps equivalent epistemic bases to equivalent epistemic bases.

**Lemma 6.1.** *The operator  $\circ$  is a function from epistemic bases and propositional formulas to epistemic bases.*

*Proof.* Let  $T = [T^0, T^1, \dots, T^{n-1}]$  denote an epistemic base,  $A$  a propositional formula, and  $\circ$  the belief-revision operator defined above. If  $T_K \models \neg A$  or  $T_B \models A$ , then by definition  $T \circ A = T$ . Thus  $T \circ A$  is an epistemic base.

Let us from now on assume that  $T_K \not\models \neg A$  and  $T_B \not\models A$ . Thus  $m = \min(\{i \mid 0 \leq i < n, T^i \not\models \neg A\})$  exists. Then  $U^0 \equiv T^m \wedge A \not\models \perp$ , thus  $S' = \{U^i \mid 0 \leq i \leq n, U^i \not\models \perp\}$  is nonempty. By definition it is also finite and all elements in  $S'$  are satisfiable. Let  $s, s' \in S'$ . Then  $s = U^i$  and  $s' = U^j$  for some  $i, j, 0 \leq i, j < n$ . Because  $T^k \wedge T^l \models \perp$  for all  $0 \leq k < l \leq n$ ,  $U^i \wedge U^j \models \perp$  for all  $i \neq j$ , and  $s = s'$  if and only if  $i = j$ . Moreover,  $(s, s') = (U^i, U^j) \in R'$ , if and only if  $i < j$ . Thus either  $s = s', s < s'$  or  $s' < s$ . If  $(s, s'), (s', s'') \in R'$ , then for some  $0 \leq i < j < k < n$ ,  $s = U^i, s' = U^j$  and  $s'' = U^k$ . But then  $i < k$  and  $(s, s'') \in R'$ . Thus the relation  $R'$  is a linear ordering and the structure  $T \circ A = \langle S', R' \rangle$  is an epistemic base.  $\square$

The operator also satisfies our postulates for belief revision.

**Theorem 6.2.** *The operator  $\circ$  is a function from epistemic bases and propositional formulas to epistemic bases, and it satisfies postulates (K0), (K1), (R0)–(R3), and (RR1)–(RR5).*

*Proof.* Let  $T = [T^0, T^1, \dots, T^{n-1}]$  denote an epistemic base,  $A$  a propositional formula, and  $\circ$  the belief-revision operator defined above. By Lemma 6.1,  $T \circ A$  is an epistemic base. If  $T_K \models \neg A$  or  $T_B \models A$ , then by definition  $T \circ A = T$ . Thus the claim holds trivially.

Let us from now on assume that  $T_K \not\models \neg A$  and  $T_B \not\models A$ . Thus  $m = \min(\{i \mid 0 \leq i < n, T^i \not\models \neg A\})$  exists.

Postulate (K0) is satisfied, because  $\bigvee_{i=0}^n U^i \equiv \bigvee_{i=m}^{n-1} (T^i \wedge A) \vee \bigvee_{i=1}^n (T^{i-1} \wedge \neg A) \equiv \bigvee_{i=0}^{n-1} (T^i \wedge A) \vee \bigvee_{i=0}^{n-1} (T^i \wedge \neg A) \equiv \bigvee_{i=0}^{n-1} T^i$ .

Postulate (R0) holds trivially. By definition  $U^0 = T^m \wedge A$ , thus postulates (R1) and (R3) hold. If  $T_B \not\models \neg A$ , then  $m = 0$  and  $(T \circ A)^0 = T^0 \wedge A$ . Postulate (R2) is satisfied.

To prove that (RR1) holds, assume a formula  $B$  with  $B \models A$  and  $T_K \not\models \neg B$ . Then  $k = \min(\{i \mid 0 \leq i < n, T^i \not\models \neg B\})$  exists, and  $m \leq k$ . Thus  $U^i \models \neg B$  for all  $i$ ,  $0 \leq i < k - m$ , and  $U^{k-m} \not\models \neg B$ .

Because  $B \models A$  and  $T_B \not\models A$ , we have  $T_B \not\models B$ , thus by definition  $(T \circ B)_B = T^k \wedge B$ . If  $(T \circ A)_B \not\models B$ , then  $((T \circ A) \circ B)^0 = (U^{k-m} \wedge B) \equiv (T^k \wedge A) \wedge B \equiv T^k \wedge B = (T \circ B)^0$ , thus postulate (RR1) holds.

Now assume  $(T \circ A)_B \models B$ , in which case we have  $m = k$ . By definition  $(T \circ A) \circ B = T \circ A$ , thus  $((T \circ A) \circ B)^0 = T^m \wedge A \equiv T^m \wedge A \wedge B \equiv T^k \wedge B = (T \circ B)^0$ . Postulate (RR1) is satisfied.

To prove that (RR2) holds, assume a formula  $B$  with  $B \models \neg A$  and  $T_K \not\models \neg B$ . Thus  $k = \min(\{i \mid 0 \leq i < n, T^i \not\models \neg B\})$  exists. Then by definition  $U^i \models \neg B$  for all  $i$ ,  $0 \leq i \leq k$ , and  $U^{k+1} \not\models \neg B$ . It follows  $((T \circ A) \circ B)^0 = U^{k+1} \wedge B \equiv T^k \wedge \neg A \wedge B \equiv T^k \wedge B$ .

If  $T_B \models B$ , by definition  $T \circ B = T$  and  $k = 0$ , thus  $(T \circ B)^0 = T^0 \equiv T^k \wedge B \equiv ((T \circ A) \circ B)^0$ . If  $T_B \not\models B$ , then  $(T \circ B)^0 = T^k \wedge B \equiv ((T \circ A) \circ B)^0$ . In both cases (RR2) holds.

To prove that postulates (RR3), (RR4), and (RR5) are satisfied, assume a formula  $B$  with  $(T \circ B)_B \not\models \neg A$ . If  $(T \circ A)_K \models \neg B$  or  $(T \circ A)_B \models B$ , then  $((T \circ A) \circ B)_B \models A$ , because by definition  $(T \circ A) \circ B = T \circ A$  and by (R1),  $(T \circ A)_B \models A$ .

Assume  $(T \circ A)_K \not\models \neg B$  and  $(T \circ A)_B \not\models B$ . Then by (K0),  $T_K \not\models \neg B$ , thus  $k = \min(\{i \mid 0 \leq i < n, T^i \not\models \neg B\})$  exists. By definition  $(T \circ B)^0 = T^k \wedge B$ . Because  $T^k \wedge B \not\models \neg A$ , we have  $m \leq k$ . Then  $U^i \models \neg B$  for all  $0 \leq i < k - m$ , and  $U^{k-m} \not\models \neg B$ . It follows that  $((T \circ A) \circ B)^0 = U^{k-m} \wedge B \equiv (T^k \wedge A) \wedge B \models A$ . Thus postulate (RR5) is satisfied. Because by (R3),  $(T \circ B)_B \models A$  implies  $(T \circ B)_B \not\models \neg A$ , and  $((T \circ A) \circ B)_B \models A$  implies  $((T \circ A) \circ B)_B \not\models \neg A$ , (RR3) and (RR4) are satisfied as well.  $\square$

Operator  $\circ$  is not equivalent to the operator by Darwiche and Pearl [DaP94], not even in the case where no knowledge is involved. This can

be seen in the following example.

**Example 6.6.** Let us revise an epistemic base  $T \equiv [\neg a, a \wedge b, a \wedge \neg b]$  by a formula  $a \wedge b$ :

$$\begin{aligned} U^0 &\equiv T^1 \wedge a \wedge b && \equiv a \wedge b \\ U^1 &\equiv (T^2 \wedge (a \wedge b)) \vee (T^0 \wedge \neg(a \wedge b)) && \equiv \neg a \\ U^2 &\equiv T^1 \wedge \neg(a \wedge b) && \equiv \perp \\ U^3 &\equiv T^2 \wedge \neg(a \wedge b) && \equiv a \wedge \neg b \end{aligned}$$

Because  $U^2$  is omitted,  $T \circ (a \wedge b) \equiv [a \wedge b, \neg a, a \wedge \neg b]$ . The absence of  $U^2$  can be seen in further revisions:

$$\begin{aligned} (T \circ (a \wedge b)) \circ \neg(a \wedge b) &\equiv [\neg a, a], \\ (T \circ (a \wedge b)) \circ \neg(a \wedge b) \circ a &\equiv [a, \neg a]. \end{aligned}$$

The result would have been different, if we had not omitted  $U^2$  (see Example 6.12).

### A belief-contraction operator

Our belief-contraction operator  $\bullet$  is used to contract beliefs from epistemic bases. Again, the operator splits the formulas and then shifts the new formulas in the ordering, in this case making the formula with the negation of the input less disbelieved as shown in Table 6.2 (page 77).

**Definition 6.4 (Operator  $\bullet$ ).** Given an epistemic base  $T = [T^0, T^1, \dots, T^{n-1}]$  and a propositional formula  $A$ , we define  $T \bullet A$  as follows. If  $T_K \models A$ , we define  $T \bullet A =_{\text{def}} T$ . Otherwise let  $m = \min(\{i \mid 0 \leq i < n, T^i \not\models A\})$ . We define formulas  $U^0, U^1, \dots, U^{n-1}$ :

$$U^i = \begin{cases} (T^i \wedge A) \vee (T^{m+i} \wedge \neg A) & \text{if } 0 \leq i < n - m, \text{ and} \\ T^i \wedge A & \text{if } n - m \leq i < n. \end{cases} \quad (6.8)$$

Finally, we define  $T \bullet A =_{\text{def}} \langle S', R' \rangle$ , where  $S' = \{U^i \mid 0 \leq i < n, U^i \not\models \perp\}$  and  $R' = \{(U^i, U^j) \mid 0 \leq i < j < n\} \cap (S' \times S')$ .

**Example 6.7.** Let us consider contracting a formula  $a \wedge b$  from the epistemic base  $T = ((\tau \oplus a \vee c) \circ a) \circ (a \wedge b) = [a \wedge b, a \wedge \neg b, \neg a \wedge c]$ :

$$\begin{aligned} U^0 &\equiv (T^0 \wedge a \wedge b) \vee (T^1 \wedge \neg(a \wedge b)) \equiv a \\ U^1 &\equiv (T^1 \wedge a \wedge b) \vee (T^2 \wedge \neg(a \wedge b)) \equiv \neg a \wedge c \\ U^2 &\equiv T^2 \wedge a \wedge b \equiv \perp. \end{aligned}$$

We get  $T \bullet (a \wedge b) \equiv [a, \neg a \wedge c]$ . We then contract  $a$  from the new base and calculate new formulas:

$$\begin{aligned} Z^0 &\equiv (T \bullet (a \wedge b))^0 \wedge a \vee (T \bullet (a \wedge b))^1 \wedge \neg a \equiv a \vee c \\ Z^1 &\equiv (T \bullet (a \wedge b))^1 \wedge a \equiv \perp. \end{aligned}$$

We get the result  $((((\tau \oplus a \vee c) \circ a) \circ (a \wedge b)) \bullet (a \wedge b)) \bullet a \equiv [a \vee c]$ .

By definition, the operator maps equivalent epistemic bases to equivalent epistemic bases. It also satisfies the postulates for belief contraction. In addition, it has the following property that corresponds to postulate (RR5):

$$(CR5): \text{ If } T_B \models A \text{ and } (T \circ B)_B \not\models A, \text{ then } ((T \bullet A) \circ B)_B \models \neg A.$$

If  $T_B \models A$ , then (CR5) implies (CR3) and (CR4).

**Theorem 6.3.** *The operator  $\bullet$  is a function from epistemic bases and propositional formulas to epistemic bases, and it satisfies postulates (K0), (K1), (C0)–(C3), and (CR1)–(CR5).*

*Proof.* Let  $T = [T^0, T^1, \dots, T^{n-1}]$  denote an epistemic base, let  $A$  denote a propositional formula,  $\circ$  denote the belief-revision operator defined in the previous section, and let  $\bullet$  denote the belief-contraction operator defined above. If  $T_K \models A$ , then by definition  $T \bullet A = T$ , and if  $T_B \not\models A$ , then by definition  $T \bullet A \equiv T$ . Thus in those cases,  $T \bullet A$  is an epistemic base and the claim holds trivially.

Let us from now on assume that  $T_K \not\models A$  and  $T_B \models A$ . Thus  $m = \min(\{i \mid 0 \leq i < n, T^i \not\models A\})$  exists, and  $m > 0$ . By definition  $T \bullet A$  is an epistemic base, and as such, postulate (K1) is satisfied.

Because by definition  $\bigvee_{i=0}^{n-1} U^i \equiv \bigvee_{i=0}^{n-1} (T^i \wedge A) \vee \bigvee_{i=0}^{n-m-1} (T^{m+i} \wedge \neg A) \equiv \bigvee_{i=0}^{n-1} T^i$ , (K0) is satisfied. By definition  $(T \bullet A)_B = U^0 = (T^0 \wedge A) \vee (T^m \wedge \neg A) \not\models A$ , thus postulate (C1) holds. Postulates (C0) and (C2) hold trivially. Because  $T_B \models A$ , by definition we have  $T_B \equiv T^0 \equiv T^0 \wedge A \models U^0 = (T \bullet A)^0 \equiv (T \bullet A)_B$ . Thus postulate (C3) is satisfied.

Assume a formula  $B$  with  $B \models A$  and  $T_K \not\models \neg B$ . Because  $B \models A$ , it follows  $T^i \wedge \neg A \models \neg B$  for all  $i$ ,  $0 \leq i < n$ . Let  $k = \min(\{i \mid 0 \leq i < n, T^i \not\models \neg B\})$ . Then  $\min(\{i \mid 0 \leq i < n, U^i \not\models \neg B\}) = k$ , and  $((T \bullet A) \circ B)^0 = U^k \wedge B \equiv T^k \wedge B = (T \circ B)^0$ . Thus postulate (CR1) holds.

Assume a formula  $B$  with  $B \models \neg A$  and  $T_K \not\models \neg B$ . Then  $T_B \not\models B$ , and  $(T \bullet A)_B \not\models B$ . Because  $B \models \neg A$ , we have  $T^i \wedge A \models \neg B$  for all  $i$ ,  $0 \leq i < n$ . Let  $k = \min(\{i \mid 0 \leq i < n, T^i \not\models \neg B\})$ . Then  $k \geq m$ , and  $((T \bullet A) \circ B)^0 = U^{k-m} \wedge B \equiv T^k \wedge B = (T \circ B)^0$ . Thus postulate (CR2) holds.

Assume a formula  $B$  with  $(T \circ B)_B \not\models A$ . If  $T_K \models \neg B$  or  $T_B \models B$ , then by definition  $T \circ B \equiv T$ , and thus  $T_B \not\models A$ , a contradiction. It follows  $T_K \not\models \neg B$  and  $T_B \not\models B$ . Then  $k = \min(\{i \mid 0 \leq i < n, T^i \not\models \neg B\})$  exists

and  $(T \circ B)^0 \equiv T^k \wedge B$ . Because  $T^k \wedge B \not\models A$ , we have  $0 < m \leq k$ . Then  $U^i \models \neg B$  for all  $i$ ,  $0 \leq i < k - m < k$  and  $U^{k-m} \not\models \neg B$ . Because  $T^k \wedge B \not\models A$ ,  $((T \bullet A) \circ B)^0 = U^{k-m} \wedge B = ((T^{k-m} \wedge A) \vee (T^k \wedge \neg A)) \wedge B \equiv (T^k \wedge \neg A) \wedge B \models \neg A$ . Thus postulate (CR5) holds and by (C3), postulates (CR3) and (CR4) hold as well.  $\square$

## Belief-update operators

Our belief-update operators  $\diamond$  and  $\diamond'$  are used to insert into epistemic states propositional formulas that record changes in the external world. The operators use the minimal-change update operator  $\diamond_W$  (page 21) by Winslett [Win86]. The idea is to use  $\diamond_W$  to update each one of the formulas in the list  $[T^0, \dots, T^{n-1}]$ . The operator  $\diamond$  gives the new piece of information only little plausibility while  $\diamond'$  gives the new formula maximal plausibility.

**Definition 6.5** (Operator  $\diamond$ ). Given a propositional formula  $A$  and an epistemic base  $T = [T^0, T^1, \dots, T^{n-1}]$ , let  $K$  denote a propositional formula such that  $K \equiv \bigvee_{i=0}^{n-1} T^i$ . If  $T_K \models \neg A$ , then  $T \diamond A =_{def} T$ . Otherwise we use the minimal-change update operator  $\diamond_W$  to define for all  $i$ ,  $0 \leq i < n$ , formulas

$$S^i \equiv T^i \diamond_W (A \wedge K), \quad (6.9)$$

and for all  $i$ ,  $0 < i < n$ , formulas

$$Z^i = \bigwedge_{j=0}^{i-1} \neg S^j. \quad (6.10)$$

We then define formulas  $U^0, U^1, \dots, U^n$  as follows:

$$U^i = \begin{cases} S^0 & \text{if } i = 0, \\ (S^i \wedge Z^i) \vee (T^{i-1} \wedge \neg A) & \text{if } 0 < i < n, \text{ and} \\ T^{n-1} \wedge \neg A & \text{if } i = n. \end{cases} \quad (6.11)$$

Finally, we define  $T \diamond A =_{def} \langle S', R' \rangle$ , where  $S' = \{U^i \mid 0 \leq i \leq n, U^i \not\models \perp\}$ , and  $R' = \{(U^i, U^j) \mid 0 \leq i < j \leq n\} \cap (S' \times S')$ .

In the definition above, formula  $A$  is used in conjunction with a formula representing the knowledge in the epistemic base. Formulas  $S^i$ ,  $0 \leq i < n$  denote the results of the updates of formulas  $T^i$  using the operator by Winslett. Then for each  $S^i$ ,  $S^i \models A$  and  $S^i \models T_K$  for all  $0 \leq i < n$ , but it is possible that the formulas are not pairwise inconsistent. Formulas  $U^i$  are then used to construct the resulting epistemic base by taking into account those possible situations in which the formula  $A$  is not true, and by imposing the formulas in the new epistemic base to be pairwise inconsistent by

using formulas  $Z^i$ . The new ranks of the subformulas are shown in Table 6.3 (page 77).

The operator of Winslett is, however, semantically oriented and we have no syntactic formulation for the part of update where  $\diamond_W$  is applied. Instead, we rely on converting the formulas into equivalent formulas in disjunctive normal form, interpreting the disjuncts as truth distributions and performing the update on these model sets as defined by Winslett. The result can then be interpreted as a formula in disjunctive normal form. If the formulas involved in the update have together  $n$  distinct propositional symbols, then we need to operate on  $n$ -ary disjuncts.

**Example 6.8.** Let us consider updating a formula  $T = \neg a \wedge \neg b$  by a formula  $p = (a \wedge b) \vee c$  using the operator  $\diamond_W$ . The formulas have together three propositional symbols,  $a$ ,  $b$  and  $c$ . We convert  $T$  into  $(\neg a \wedge \neg b \wedge c) \vee (\neg a \wedge \neg b \wedge \neg c)$  and  $p$  into  $(a \wedge b \wedge c) \vee (a \wedge b \wedge \neg c) \vee (a \wedge \neg b \wedge c) \vee (\neg a \wedge b \wedge c) \vee (\neg a \wedge \neg b \wedge c)$ . Then  $T \diamond_W p \equiv (\neg a \wedge \neg b \wedge c) \vee (\neg a \wedge \neg b \wedge \neg c) \vee (a \wedge b \wedge \neg c) \equiv (\neg a \wedge \neg b \wedge c) \vee (a \wedge b \wedge \neg c)$ .

**Example 6.9. a)** Let us consider updating an epistemic base  $T = [\neg a \wedge \neg b, a \vee b]$  by a formula  $(a \wedge b) \vee c$ . Then the new epistemic base is calculated as follows:

$$\begin{aligned} S^0 &\equiv T^0 \diamond_W (a \wedge b) \vee c \equiv (\neg a \wedge \neg b \wedge c) \vee (a \wedge b \wedge \neg c) \\ S^1 &\equiv T^1 \diamond_W (a \wedge b) \vee c \equiv (a \wedge b) \vee (a \wedge c) \vee (b \wedge c) \\ U^0 &\equiv S^0 && \equiv (\neg a \wedge \neg b \wedge c) \vee (a \wedge b \wedge \neg c) \\ U^1 &\equiv (S^1 \wedge \neg S^0) \vee \\ &\quad (T^0 \wedge \neg((a \wedge b) \vee c)) \equiv (a \wedge c) \vee (b \wedge c) \vee (\neg a \wedge \neg b \wedge \neg c) \\ U^2 &\equiv T^1 \wedge \neg((a \wedge b) \vee c) \equiv (a \wedge \neg b \wedge \neg c) \vee (\neg a \wedge b \wedge \neg c). \end{aligned}$$

We get  $T \diamond((a \wedge b) \vee c) \equiv [(\neg a \wedge \neg b \wedge c) \vee (a \wedge b \wedge \neg c), (a \wedge c) \vee (b \wedge c) \vee (\neg a \wedge \neg b \wedge \neg c), (a \wedge \neg b \wedge \neg c) \vee (\neg a \wedge b \wedge \neg c)]$ . The formula  $U^0$  is equivalent to the result when using operator  $\diamond_W$  in Example 3.6 (21).

**b)** Let us add to  $T$  a formula  $(a \wedge b) \vee (b \wedge c)$ . The formulas for the new epistemic base are calculated as follows:

$$\begin{aligned} S^0 &\equiv T^0 \diamond_W ((a \wedge b) \vee (b \wedge c)) \equiv (\neg a \wedge b \wedge c) \vee (a \wedge b \wedge \neg c) \\ S^1 &\equiv T^1 \diamond_W ((a \wedge b) \vee (b \wedge c)) \equiv (\neg a \wedge b \wedge c) \vee (a \wedge b \wedge \neg c) \vee (a \wedge b \wedge c) \\ U^0 &\equiv (\neg a \wedge b \wedge c) \vee (a \wedge b \wedge \neg c) \\ U^1 &\equiv (a \wedge b \wedge c) \vee (\neg a \wedge \neg b) \\ U^2 &\equiv (a \wedge \neg b) \vee (\neg a \wedge b \wedge \neg c). \end{aligned}$$

By definition, the operator maps equivalent epistemic bases to equivalent epistemic bases. It also satisfies the postulates for belief update.

**Theorem 6.4.** *The operator  $\diamond$  is a function from epistemic bases and propositional formulas to epistemic bases, and it satisfies postulates (K0), (K1), (U0)–(U8) and (RR4).*

*Proof.* Let  $T = [T^0, T^1, \dots, T^{n-1}]$  denote an epistemic base,  $A$  a propositional formula, and  $\diamond$  the belief-update operator defined above. The operator  $\diamond_W$  (page 35) used in the definition satisfies postulates (U1)–(U8) and (RR4).

If  $T_K \models \neg A$ , then by definition  $T \diamond A = T$ . Thus  $T \diamond A$  is an epistemic base and postulates (K0), (K1), (U0)–(U4), and (U8) hold trivially.

Let us for a while assume that  $T_K \not\models \neg A$ . Then by definition,  $U^0 = S^0 \equiv T^0 \diamond_W (A \wedge K)$  for some  $K \equiv T_K$ . Because  $\diamond_W$  satisfies postulate (U3) and  $A \wedge K$  is satisfiable,  $U^0$  is satisfiable, thus (U3) holds and the set  $S'$  is nonempty. By definition, relation  $R'$  is a well-ordering, and  $U^i \wedge U^j \models \perp$  for all  $0 \leq i < j \leq n$ . Thus  $T \diamond A$  is an epistemic base and by definition (K1) is satisfied.

Because  $T_K \not\models \neg A$ , postulate (U0) holds trivially. By postulate (U1),  $U^0 \models A \wedge K$ , thus (U1) holds. Postulate (U2) holds by (U2), postulate (U4) by (U4), and postulate (U8) by (U8).

By (U1),  $(T^i \diamond_W (A \wedge K)) \models A \wedge K$  for all  $i = 0, \dots, n$ , thus by definition  $(T \diamond A)_K \models T_K$ . By (U2),  $((T^i \wedge A) \diamond_W (A \wedge K)) \equiv T^i \wedge A$  for all  $i = 0, \dots, n$ , thus by (U8),  $T^i \wedge A \models (T \diamond A)_K$  for all  $i = 0, \dots, n$ . Then by definition,  $T_K \models (T \diamond A)_K$ . Thus (K0) is satisfied.

Now we are left to prove that postulates (U5)–(U7) and (RR4) hold.

If  $(T \diamond A)_B \models \neg B$ , (U5) holds trivially. If  $T_K \models \neg A$ , then  $(T \diamond A)_B \equiv T_B \equiv (T \diamond A \wedge B)_B$ , thus (U5) holds. Assume  $T_K \not\models \neg A$  and  $(T \diamond A)_B \not\models \neg B$ . Then by (U1),  $(T \diamond A)_B \not\models \neg(A \wedge B)$ , by (K1),  $(T \diamond A)_K \not\models \neg(A \wedge B)$ , and by (K0),  $T_K \not\models \neg(A \wedge B)$ . Then by (U5) and (U4),  $(T \diamond A)_B \wedge B \equiv (T^0 \diamond_W (A \wedge K)) \wedge B \models T^0 \diamond_W ((A \wedge K) \wedge B) \equiv T^0 \diamond_W ((A \wedge B) \wedge K) \equiv (T \diamond (A \wedge B))_B$ , thus (U5) holds.

To prove that (U6) holds, assume  $(T \diamond A)_B \models B$  and  $(T \diamond B)_B \models A$ . Then by (U3), (K1), and (K0),  $T_K \not\models \neg A$  and  $T_K \not\models \neg B$ . By (U6),  $(T \diamond A)_B \equiv T^0 \diamond_W (A \wedge K) \equiv T^0 \diamond_W (B \wedge K) \equiv (T \diamond B)_B$ , thus (U6) holds.

To prove that (U7) holds, assume that  $T_B$  is complete. If  $T_K \models \neg A$  and  $T_K \models \neg B$ , then (U7) is trivially satisfied.

If  $T_K \not\models \neg A$  and  $T_K \not\models \neg B$ , then  $T_K \not\models \neg(A \vee B)$ . By (U7) and (U4),  $(T \diamond A)_B \wedge (T \diamond B)_B \equiv T^0 \diamond_W (A \wedge K) \wedge T^0 \diamond_W (B \wedge K) \models T^0 \diamond_W ((A \wedge K) \vee (B \wedge K)) \equiv T^0 \diamond_W ((A \vee B) \wedge K) \equiv (T \diamond (A \vee B))_B$ .

If  $(T \diamond A)_B \wedge (T \diamond B)_B \models \perp$ , then (U7) holds trivially. Now assume  $T_K \not\models \neg A$ ,  $T_K \models \neg B$ , and  $(T \diamond A)_B \wedge (T \diamond B)_B \not\models \perp$ . Because  $T_B$  is complete and by (U1)  $(T \diamond A)_B \models A$ , it follows  $T_B \models A$ . Then by (U2),  $(T \diamond A)_B \equiv T_B \equiv T \diamond (A \vee B)$ . Thus (U7) holds.

To prove (RR4), assume  $(T \diamond B)_B \not\models \neg A$ . Thus by (K1) and (K0),  $T_K \not\models \neg A$ ,



by (U1),  $(T \diamond A)_B \models A$ , and by (U3),  $(T \diamond A)_B \not\models \neg A$ . If  $(T \diamond A)_K \models \neg B$ , then by definition  $(T \diamond A) \diamond B = T \diamond A$ , thus (RR4) is satisfied. Now assume  $(T \diamond A)_K \not\models \neg B$ . Then by (K0),  $T_K \not\models \neg B$ , and by definition  $(T \diamond B)_B \equiv T^0 \diamond_W (B \wedge K)$  and  $((T \diamond A) \diamond B)_B \equiv (T^0 \diamond_W (A \wedge K)) \diamond_W (B \wedge K)$ . Then  $T^0 \diamond_W (B \wedge K) \not\models \neg(A \wedge K)$ , and (RR4) is satisfied by (RR4).  $\square$

We also define an update operator that gives the new formula maximal plausibility.

**Definition 6.6** (Operator  $\diamond'$ ). Given a propositional formula  $A$  and an epistemic base  $T = [T^0, T^1, \dots, T^{n-1}]$ , let  $K$  denote a propositional formula such that  $K \equiv \bigvee_{i=0}^{n-1} T^i$ . If  $T_K \models \neg A$ , then  $T \diamond A =_{\text{def}} T$ . Otherwise we use the minimal-change update operator  $\diamond_W$  to define for all  $i$ ,  $0 \leq i < n$ , formulas  $S^i \equiv T^i \diamond_W (A \wedge K)$ , and for all  $i$ ,  $0 < i < n$ , formulas  $Z^i = \bigwedge_{j=0}^{i-1} \neg S^j$ .

We then define formulas  $U^0, U^1, \dots, U^{2n-1}$ :

$$U^i = \begin{cases} S^i & \text{if } i = 0, \\ S^i \wedge Z^i & \text{if } 0 < i < n, \text{ and} \\ T^{i-n} \wedge \neg A & \text{if } n \leq i < 2n. \end{cases} \quad (6.12)$$

Finally, we define  $T \diamond' A \equiv_{\text{def}} \langle S', R' \rangle$ , where  $S' = \{U^i \mid 0 \leq i < 2n, U^i \not\models \perp\}$ , and  $R' = \{(U^i, U^j) \mid 0 \leq i < j < 2n\} \cap (S' \times S')$ .

The new ranks of the subformulas are shown in Table 6.4 (page 77). By definition, the operator maps equivalent epistemic bases to equivalent epistemic bases. It also satisfies the postulates for belief update.

**Theorem 6.5.** *The operator  $\diamond'$  is a function from epistemic structures and propositional formulas to epistemic structures, and it satisfies postulates (K0), (K1), (U0)–(U8) and (RR4).*

*Proof.* The proof is analogous to the proof of Theorem 6.4.  $\square$

## 6.2 Operators on epistemic functions

We shall first give a formal definition of an epistemic function and then introduce a collection of operators that map epistemic functions and propositional formulas to epistemic functions. All but the new operator for inserting competitive evidence are analogous to the operators we introduced in the previous section. The only difference is that they work on epistemic functions instead of epistemic bases, that is, the gaps that may arise in the orderings will not be dispensed with.

## Epistemic functions

If we give up the restriction that all the formulas in the epistemic base must be satisfiable, we get a representation that resembles an ordinal conditional function. Unsatisfiable formulas mark gaps in the ordering of disbelief. We define epistemic functions as follows.

**Definition 6.7.** An epistemic function  $T_n$ ,  $n \in N$ , is a function  $\{0, \dots, n-1\} \rightarrow \mathcal{L}$  such that  $T_n(0) \not\models \perp$  and  $T_n(i) \wedge T_n(j) \models \perp$  for all  $0 \leq i < j < n$ .

Analogously to the definitions on epistemic bases, we may equivalently refer to  $T = T_n$  by using a list formulation  $T = [T^0, T^1, \dots, T^{n-1}]$  where  $T^i = T_n(i)$  for all  $i$ ,  $0 \leq i < n$ . Again we define  $T_B \equiv T_n(0)$  and  $T_K \equiv \bigvee_{i=0}^{n-1} T_n(i)$ . The state of ignorance  $\tau$  can be represented by the function  $\tau = \{(0, \top)\}$ .

**Example 6.10.** Let us consider an epistemic function  $T = \{(0, a \wedge b), (1, a \wedge \neg b), (2, \neg a \wedge c)\}$ , or equivalently, a list of formulas  $T = [T(0), T(1), T(2)]$ , where  $T(0) = a \wedge b$ ,  $T(1) = a \wedge \neg b$ , and  $T(2) = \neg a \wedge c$ . Then  $T_B \equiv a \wedge b$  and  $T_K \equiv (a \wedge b) \vee (a \wedge \neg b) \vee (\neg a \wedge c) \equiv a \vee c$ .

The models of the formula  $T_K \equiv \bigvee_{i=0}^{n-1} T(i)$  are the possible models in the epistemic state represented by the knowledge base. The models of the formula  $T(0)$  are the most plausible models in the state. The models of the formula  $T(i)$  are less disbelieved than the models of the formula  $T(j)$ , for all  $0 \leq i < j < n$ .

Epistemic functions  $T_n$  and  $T'_m$  are defined to be *equivalent*, if either

1.  $n \leq m$ , and for all  $i, j$ ,  $0 \leq i < n \leq j < m$ ,  
 $T_n(i) \equiv T'_m(i)$  and  $T'_m(j) \equiv \perp$ , or
2.  $m \leq n$ , and for all  $i, j$ ,  $0 \leq i < m \leq j < n$ ,  
 $T_n(i) \equiv T'_m(i)$  and  $T'_m(j) \equiv \perp$ .

The equivalence of epistemic functions is an equivalence relation, also denoted by  $\equiv$ .

Assume  $T = T_n$  is an epistemic function. Then again for all  $w, w' \in \llbracket T_K \rrbracket$ ,  $w \in \llbracket T(i) \rrbracket$  holds for exactly one  $0 \leq i < n$  and  $w' \in \llbracket T(j) \rrbracket$  holds for exactly one  $0 \leq j < n$ . We define  $w \leq_{T_n} w'$  if and only if  $i \leq j$ . Then  $\leq_{T_n}$  is an ordering of disbelief on possible models.

Epistemic functions resemble prioritized theories, which were reviewed in Chapter 3. However, in epistemic functions prioritization is developed dynamically by the system.

### A knowledge-expansion operator

The knowledge-expansion operator  $\oplus$  is defined only in the cases in which the new formula is consistent with the knowledge set of the epistemic state.

**Definition 6.8** (Operator  $\oplus$ ). Assume an epistemic function  $T = T_n$  and a propositional formula  $A$  that is consistent with  $T_K$ . Let  $m = \min(\{i \mid 0 \leq i < n, T(i) \not\models \neg A\})$ . We then define a function  $(T \oplus A)$  from  $\{0, \dots, n - m - 1\}$  into  $\mathcal{L}$  as follows:

$$(T \oplus A)(i) = T(i + m) \wedge A, \text{ for all } 0 \leq i < n - m. \quad (6.13)$$

The operator is analogous to the corresponding operator on epistemic bases. The new ranks of the subformulas are shown in Table 6.5 (page 77).

A formula  $A$  is consistent with the knowledge set  $T_K \equiv \bigvee_{i=0}^{n-1} T_n(i)$ , if and only if  $T(i) \not\models \neg A$  for some  $i$ ,  $0 \leq i < n$ . Thus the minimum  $m$  is well defined.

**Example 6.11.** Let us consider expanding the epistemic base  $T = [a \wedge b, a \wedge \neg b, \neg a \wedge c]$  by a formula  $b$ :

$$\begin{aligned} T(0) &\equiv a \wedge b \\ T(1) &\equiv a \wedge \neg b \\ T(2) &\equiv \neg a \wedge c \end{aligned}$$

$$\begin{aligned} (T \oplus a)(0) &\equiv a \wedge b \\ (T \oplus a)(1) &\equiv a \wedge \neg b \wedge b \equiv \perp \\ (T \oplus a)(2) &\equiv \neg a \wedge c \wedge b. \end{aligned}$$

The result  $T \oplus b \equiv [a \wedge b, \perp, \neg a \wedge b \wedge c]$  is not equivalent to  $[a \wedge b, \neg a \wedge b \wedge c]$ . The knowledge set  $(T \oplus b)_K \equiv (a \wedge b) \vee \perp \vee (\neg a \wedge b \wedge c) \equiv (a \vee c) \wedge b$  is not affected by the unsatisfiable formula  $(T \oplus a)(1)$ , but in further revisions the difference can become visible in the belief set as can be seen in Example 6.12 (page 74).

**Theorem 6.6.** *The operator  $\oplus$  is a function from epistemic functions and propositional formulas to epistemic functions, and it satisfies postulates (K2), (K1), (R2), (R3), and (RR1).*

*Proof.* The proof is analogous to the proof of Theorem 6.1. □

## A belief-revision operator

Our belief revision operator differs from the operator by Darwiche and Pearl [DaP94] only in the case the new information is inconsistent with the knowledge in the state.

**Definition 6.9** (Operator  $\circ$ ). Given an epistemic function  $T = T_n$  and a propositional formula  $A$ , we define  $T \circ A$  as follows. If  $T_K \models \neg A$  or  $T_B \models A$ , we define  $T \circ A =_{\text{def}} T$ . Otherwise let  $m = \min(\{i \mid 0 \leq i < n, T(i) \not\models \neg A\})$ . We then define

$$(T \circ A)(i) = \begin{cases} T(m) \wedge A & \text{if } i = 0, \\ (T(m+i) \wedge A) \vee (T(i-1) \wedge \neg A) & \text{if } 0 < i < n - m, \text{ and} \\ T(i-1) \wedge \neg A & \text{if } n - m \leq i \leq n. \end{cases} \quad (6.14)$$

The operator is analogous to the corresponding operator on epistemic bases. Examples 6.3–6.5 are also valid for this operator. However, Example 6.6 is not valid, because when using this operator, unsatisfiable formulas will not be excluded from the resulting list of formulas. In the long run, this will affect the result of belief change.

**Example 6.12.** Let us revise an epistemic function  $T \equiv [-a, a \wedge b, a \wedge \neg b]$  by a formula  $a \wedge b$ :

$$\begin{aligned} T(0) &\equiv \neg a \\ T(1) &\equiv a \wedge b \\ T(2) &\equiv a \wedge \neg b \end{aligned}$$

$$\begin{aligned} (T \circ (a \wedge b))(0) &\equiv T(1) \wedge a \wedge b && \equiv a \wedge b \\ (T \circ (a \wedge b))(1) &\equiv (T(2) \wedge (a \wedge b)) \vee (T(0) \wedge \neg(a \wedge b)) && \equiv \neg a \\ (T \circ (a \wedge b))(2) &\equiv T(1) \wedge \neg(a \wedge b) && \equiv \perp \\ (T \circ (a \wedge b))(3) &\equiv T(2) \wedge \neg(a \wedge b) && \equiv a \wedge \neg b \end{aligned}$$

Thus  $T \circ (a \wedge b) \equiv [a \wedge b, \neg a, \perp, a \wedge \neg b]$ . The effect of having the formula  $\perp$  in the list can be seen in further revisions:

$$\begin{aligned} (T \circ (a \wedge b)) \circ \neg(a \wedge b) &\equiv [-a, a \wedge b, a \wedge \neg b], \\ (T \circ (a \wedge b)) \circ \neg(a \wedge b) \circ a &\equiv [a \wedge b, \neg a \vee \neg b]. \end{aligned}$$

The result is different from the result  $[a, \neg a]$  in Example 6.6 (page 66), in which epistemic bases were used.

**Theorem 6.7.** *The operator  $\circ$  is a function from epistemic functions and propositional formulas to epistemic functions, and it satisfies postulates (K0), (K1), (R0)–(R3), and (RR1)–(RR5).*

*Proof.* The proof is analogous to the proof of Theorem 6.2. □

### A belief-contraction operator

**Definition 6.10** (Operator  $\bullet$ ). Given an epistemic function  $T = T_n$  and a propositional formula  $A$ , if  $T_K \models A$  or  $T_B \not\models A$ , we define  $T \bullet A =_{def} T$ . Otherwise let  $m = \min(\{i \mid 0 \leq i < n, T(i) \not\models A\})$ . We then define

$$(T \bullet A)(i) = \begin{cases} (T(i) \wedge A) \vee (T(m+i) \wedge \neg A) & \text{if } 0 \leq i < n - m, \text{ and} \\ T(i) \wedge A & \text{if } n - m \leq i < n. \end{cases} \quad (6.15)$$

The operator is analogous to the corresponding operator on epistemic bases. Example 6.7 is also valid for this operator.

**Theorem 6.8.** *The operator  $\bullet$  is a function from epistemic functions and propositional formulas to epistemic functions, and it satisfies postulates (K0), (K1), (C0)–(C3), and (CR1)–(CR5).*

*Proof.* The proof is analogous to the proof of Theorem 6.3.  $\square$

### A belief-update operator

**Definition 6.11** (Operator  $\diamond$ ). Given a propositional formula  $A$  and an epistemic function  $T = T_n$ , let  $K$  denote a propositional formula such that  $K \equiv \bigvee_{i=0}^{n-1} T(i)$ . If  $T_K \models \neg A$ , then  $T \diamond A =_{def} T$ . Otherwise we use the minimal-change update operator  $\diamond_W$  (page 21) by Winslet [Win86] to define for all  $i, 0 \leq i < n$ , formulas  $S^i \equiv T(i) \diamond_W (A \wedge K)$ , and for all  $i, 0 < i < n$ , formulas  $Z^i = \bigwedge_{j=0}^{i-1} \neg S^j$ . We then define

$$(T \diamond A)(i) = \begin{cases} S^0 & \text{if } i = 0, \\ (S^i \wedge Z^i) \vee (T(i-1) \wedge \neg A) & \text{if } 0 < i < n, \text{ and} \\ T(n-1) \wedge \neg A & \text{if } i = n. \end{cases} \quad (6.16)$$

The operator is analogous to the corresponding operator on epistemic bases. Example 6.9 is also valid for this operator.

**Theorem 6.9.** *The operator  $\diamond$  is a function from epistemic functions and propositional formulas to epistemic functions, and it satisfies postulates (K0), (K1), (U0)–(U8) and (RR4).*

*Proof.* The proof is analogous to the proof of Theorem 6.4.  $\square$

## A competing-evidence operator

We introduce an operator for entering competitive information into the knowledge base. The competing evidence operator  $*$  also splits the formulas and performs shiftings. If the new formula is considered plausible in the epistemic state, then its negation is made more disbelieved, otherwise the new formula is made less disbelieved as shown in tables 6.6 and 6.7 (page 77).

**Definition 6.12** (Operator  $*$ ). Given a propositional formula  $A$  and an epistemic function  $T = T_n$ , if  $T_K \models \neg A$ , we define  $T * A =_{def} T$ . Otherwise, if  $T_B \not\models \neg A$ , then we define a function  $T * A : \{0, \dots, n\} \rightarrow \mathcal{L}$  as follows:

$$(T * A)(i) = \begin{cases} T(i) \wedge A & \text{if } i = 0, \\ (T(i) \wedge A) \vee (T(i-1) \wedge \neg A) & \text{if } 0 < i < n, \text{ and} \\ T(i-1) \wedge \neg A & \text{if } i = n. \end{cases} \quad (6.17)$$

If  $T_B \models \neg A$ , then we define a function  $T * A : \{0, \dots, n-1\} \rightarrow \mathcal{L}$  as follows:

$$(T * A)(i) = \begin{cases} (T(i) \wedge \neg A) \vee (T(i+1) \wedge A) & \text{if } 0 \leq i < n-1, \text{ and} \\ T(i) \wedge \neg A & \text{if } i = n-1. \end{cases} \quad (6.18)$$

**Example 6.13.** Let us change an epistemic function  $T \equiv \tau * a$  by formulas  $a \vee b$  and  $\neg a$ .  $T$  and the new formulas are calculated as follows:

$$\begin{aligned} T(0) &\equiv a \\ T(1) &\equiv \neg a \end{aligned}$$

$$\begin{aligned} (T * a \vee b)(0) &\equiv a \\ (T * a \vee b)(1) &\equiv \neg a \wedge b \\ (T * a \vee b)(2) &\equiv \neg a \wedge \neg b \end{aligned}$$

$$\begin{aligned} ((T * a \vee b) * \neg a)(0) &\equiv a \vee b \\ ((T * a \vee b) * \neg a)(1) &\equiv \neg a \wedge \neg b. \end{aligned}$$

At the beginning, the result of  $\tau * a$  is the same as when revising  $\tau$  by  $a$ .

The previous example shows that in the absence of competition the result of inserting competing evidence is the same as that of revision. It also shows that competing evidence has effects even if it was already entailed by the epistemic state.

Table 6.1: New ranks of subformulas in belief revision  $T \circ A$  (6.7, 6.14).

	$T^0$	$T^1$	...	$T^m$	$T^{m+1}$	...	$T^{n-1}$
$T^i \wedge A$	-	-	-	0	1	...	$n - m - 1$
$T^i \wedge \neg A$	1	2	...	$m + 1$	$m + 2$	...	$n$

Table 6.2: New ranks of subformulas in belief contraction  $T \bullet A$  (6.8, 6.15).

	$T^0$	$T^1$	...	$T^m$	$T^{m+1}$	...	$T^{n-1}$
$T^i \wedge A$	0	1	...	$m$	$m + 1$	...	$n - 1$
$T^i \wedge \neg A$	-	-	-	0	1	...	$n - m - 1$

Table 6.3: New ranks of subformulas in belief update  $T \diamond A$  (6.11, 6.16).

	$T^0$	$T^1$	...	$T^{n-1}$
$S^i \wedge Z^i$	0	1	...	$n - 1$
$T^i \wedge \neg A$	1	2	...	$n$

Table 6.4: New ranks of subformulas in belief update  $T \diamond' A$  (6.12).

	$T^0$	$T^1$	...	$T^{n-1}$
$S^i \wedge Z^i$	0	1	...	$n - 1$
$T^i \wedge \neg A$	$n$	$n + 1$	...	$2n - 1$

Table 6.5: New ranks of subformulas in knowledge expansion  $T \oplus A$  (6.13).

	$T^0$	$T^1$	...	$T^m$	$T^{m+1}$	...	$T^{n-1}$
$T^i \wedge A$	-	-	-	0	1	...	$n - m - 1$
$T^i \wedge \neg A$	-	-	-	-	-	-	-

Table 6.6: New ranks of subformulas in  $T * A$  when  $T_B \not\models \neg A$  (6.17).

	$T^0$	$T^1$	$T^2$	...	$T^{n-1}$
$T^i \wedge A$	0	1	2	...	$n - 1$
$T^i \wedge \neg A$	1	2	3	...	$n$

Table 6.7: New ranks of subformulas in  $T * A$  when  $T_B \models \neg A$  (6.18).

	$T^0$	$T^1$	$T^2$	...	$T^{n-1}$
$T^i \wedge A$	-	0	1	...	$n - 2$
$T^i \wedge \neg A$	0	1	2	...	$n - 1$

**Theorem 6.10.** *The operator  $*$  is a function from epistemic functions and propositional formulas to epistemic functions, and it satisfies the postulates (K0), (K1), (R0), (R2), (R3), (NP1), and (NR1)–(NR3).*

*Proof.* Let  $T = [T(0), T(1), \dots, T(n-1)]$  denote an epistemic function, let  $A$  and  $B$  denote propositional formulas, and let  $*$  denote the competing-evidence operator defined above. It is easy to see that by definition,  $(T * A)$  is an epistemic function that satisfies postulates (K0), (K1), (R0), (R2), and (R3).

To prove (NR1), assume propositional formulas  $A$  and  $B$  such that  $B \models A$  and  $T_K \not\models \neg B$ . Then also  $T_K \not\models \neg A$ , and  $(T \circ B)(0) = T(m) \wedge B$ , where  $m = \min(\{i \mid 0 \leq i < n, T^i \not\models \neg B\})$ . Then  $((T * A) \circ B)_B \equiv T(m) \wedge A \wedge B \equiv T(m) \wedge B \equiv (T \circ B)_B$ .

To prove (NR2), assume propositional formulas  $A$  and  $B$  such that  $B \models \neg A$  and  $T_K \not\models \neg B$ . Then  $(T \circ B)(0) = T(m) \wedge B$ , where  $m = \min(\{i \mid 0 \leq i < n, T^i \not\models \neg B\})$ . We get  $((T * A) \circ B)_B \equiv T(m) \wedge \neg A \wedge B \equiv T(m) \wedge B \equiv (T \circ B)_B$ .

To prove (NR3), assume  $(T \circ B)_B \not\models \neg A$ . Then by (K0) and (K1),  $T_K \not\models \neg A$ . If  $T_K \models \neg B$ , then by (R0),  $(T \circ B)_B \equiv T_B$ . Thus  $((T * A) \circ B)_B \equiv T(0) \wedge A \models A$ . If  $T_K \not\models \neg B$ , then  $(T \circ B)(0) = T(m) \wedge B$ , where  $m = \min(\{i \mid 0 \leq i < n, T^i \not\models \neg B\})$ . Then  $((T * A) \circ B)_B \equiv T(m) \wedge A \wedge B \models A$ . Thus (NR3) holds.

We are left to prove that postulate (PP1) is satisfied.

If  $T_K \models \neg A$  or  $T_K \models \neg B$ , then by definition  $T * A = T$  or  $T * B = T$ , thus the claim holds trivially. Let us from now on assume that  $T_K \not\models \neg A$  and  $T_K \not\models \neg B$ .

Let us name the rules used in the process. When inserting  $A$ , let  $A+$  denote the rule 6.17 applied when  $A$  is compatible with the belief set of the epistemic state, and let  $A-$  denote the rule 6.18 applied when  $A$  is not compatible with the belief set of the epistemic state. Rules  $B+$  and  $B-$  are defined analogously. When calculating both  $(T * A) * B$  or  $(T * B) * A$ , four different combinations of rules are possible, and combined we have the following six disjoint cases:

1. Assume  $T(0) \not\models \neg(A \wedge B)$ . Then  $(T * A) * B$  is calculated by rules  $A+/B+$  and  $(T * B) * A$  by rules  $B+/A+$ . It is easy to see that  $((T * A) * B)(0) \equiv T(0) \wedge A \wedge B \equiv ((T * B) * A)(0)$ .
2. Assume  $T(0) \models \neg(A \wedge B)$ ,  $T(0) \not\models \neg A$  and  $T(0) \not\models \neg B$ . Then  $(T * A) * B$  is calculated by rules  $A+/B-$ , whereas  $(T * B) * A$  by rules  $B+/A-$ . If  $n = 1$ , then  $((T * A) * B)(0) \equiv (T(0) \wedge A \wedge \neg B) \vee (T(0) \wedge \neg A \wedge B) \equiv ((T * B) * A)(0)$ . If  $n > 1$ , then  $((T * A) * B)(0) \equiv (T(0) \wedge A \wedge \neg B) \vee (T(1) \wedge A \wedge B) \vee (T(0) \wedge \neg A \wedge B) \equiv ((T * B) * A)(0)$ .



3. Assume  $T(0) \not\models \neg A$  and  $T(0) \models \neg B$ . Then rules  $A+/B-$  and  $B-/A+$  are used, and  $n > 1$ . Because  $T(0) \models \neg B$ , we get  $((T * A) * B)(0) \equiv (T(0) \wedge A \wedge \neg B) \vee (T(1) \wedge A \wedge B) \equiv ((T * B) * A)(0)$ .
4. Assume  $T(0) \models \neg A$  and  $T(0) \not\models \neg B$ . Then rules  $A-/B+$  and  $B+/A-$  are used, and  $n > 1$ . Because  $T(0) \models \neg A$ , we get  $((T * A) * B)(0) \equiv (T(0) \wedge \neg A \wedge B) \vee (T(1) \wedge A \wedge B) \equiv ((T * B) * A)(0)$ .
5. Assume  $T(0) \models \neg A \wedge \neg B$  and  $T(1) \not\models \neg(A \wedge B)$ . Then rules  $A-/B+$  and  $B-/A+$  are used and  $n > 1$ . Because  $T(0) \models \neg A \wedge \neg B$ , we get  $((T * A) * B)(0) \equiv (T(1) \wedge A \wedge B) \equiv ((T * B) * A)(0)$ .
6. Assume  $T(0) \models \neg A \wedge \neg B$  and  $T(1) \models \neg(A \wedge B)$ . Then rules  $A-/B-$  and  $B-/A-$  are used and  $n > 1$ . If  $n = 2$ , then  $((T * A) * B)(0) \equiv (T(0) \wedge \neg A \wedge \neg B) \vee (T(1) \wedge A) \wedge \neg B \vee (T(1) \wedge \neg A \wedge B) \equiv ((T * B) * A)(0)$ . If  $n > 2$ , then  $((T * A) * B)(0) \equiv (T(0) \wedge \neg A \wedge \neg B) \vee (T(1) \wedge A \wedge \neg B) \vee (T(1) \wedge \neg A \wedge B) \vee (T(2) \wedge A \wedge B) \equiv ((T * B) * A)(0)$ .

Thus in all cases,  $((T * A) * B)(0) \equiv ((T * B) * A)(0)$  and postulate (PP1) is satisfied.  $\square$

### 6.3 Discussion

We have given two finite, concrete representations of knowledge bases along with collections of change operators. Our operator collections include basic operations with some variation. We believe that the agent can use these operators along with the accessories to build more elaborate behaviour when necessary.

Even though our operators reject unbelievable input, the agent need not do so. For example, the agent could start a conversation with the source of the input in order to solve the inconsistencies, as suggested by Nykänen et al. [Nyk11]. Otherwise, the agent might wish to manipulate the input in order to make it believable, as is done in accommodative belief revision [Elo08]. In *accommodative belief revision* the input is first revised by the knowledge of the agent to make it acceptable. The idea is to try to guess what the source of the input might have believed, had it known what the agent does.

We have not studied the computational complexity of our proposals. In the end the proposals have been an attempt to sketch an overall picture of the knowledge base. In this chapter, two concrete structures have been given, and the corresponding abstract structures have been proved to have the desired properties.

The space requirements of our proposals are severe: the lengths of the formulas double at each step. In order to reduce the space requirements, the epistemic state might be represented by a collection of lists of formulas, "local epistemic states", instead of a single list of formulas. The idea would then be to have separate domains in separate lists, that is, if there are groups of atomic formulas with no dependencies between the groups, we might have an epistemic base or an epistemic function separately for each of them. Queries might then be answered by using subqueries to these local epistemic states.

Changing the knowledge base with a formula that involves several local states and creates dependencies between the groups would then require combining those local states before the change. Epistemic functions  $T_n$  and  $U_m$  with separate sets of atomic formulas could be combined into a function  $S = T_n \times U_m : 0, \dots, n + m - 2 \rightarrow \mathcal{L}$  by defining for each  $i = 0, \dots, n + m - 2$  formulas

$$(T_n \times U_m)(i) = \bigvee \{T_j \wedge U_k \mid 0 \leq j < n, 0 \leq k < m, j + k = i\}. \quad (6.19)$$

Then  $S(0) = T(0) \wedge U(0)$  would be satisfiable, and we would have  $S_B \equiv T_B \cup U_B$  and  $S_K \equiv T_K \cup U_K$ .

# Chapter 7

## Modelling knowledge expansion and belief revision

In this chapter we see how the postulates of belief revision imply the existence of ordering of disbelief. We show this by constructing such an ordering by applying Grove's construction [Gro88] to create a system of spheres. We see how the ordering of disbelief is modified in belief revision and knowledge expansion.

### 7.1 Knowledge change

We consider all knowledge-change and belief-change operators as functions from epistemic states and propositional formulas to epistemic states such that the epistemic states satisfy constraints (S1) and (S2). As before, let  $T$  denote an epistemic state, and let  $T_K$  denote the set of propositional formulas known in the state. We assume that  $T_K$  is deductively closed.

We assume that knowledge-expansion operators satisfy postulate (K2) and that all other operators satisfy postulate (K0), thus in all circumstances, changing the epistemic state always satisfies either one of those postulates. We can express postulate (K0) by the following equivalent condition (KN) and postulate (K2) by the following equivalent condition (KE):

$$\begin{aligned} \text{(KN):} \quad & \llbracket (T \circ A)_K \rrbracket = \llbracket T_K \rrbracket, \\ \text{(KE):} \quad & \llbracket (T \oplus A)_K \rrbracket = \llbracket T_K \rrbracket \cap \llbracket A \rrbracket. \end{aligned}$$

Both conditions (KN) and (KE) imply the following condition (KX):

$$\text{(KX):} \quad \llbracket T_K \rrbracket \cap \llbracket A \rrbracket \subseteq \llbracket (T \circ A)_K \rrbracket \subseteq \llbracket T_K \rrbracket.$$

Condition (KX) expresses a condition shared by both knowledge expansion and belief change.

If we assume that in the beginning  $T_K \equiv \top$  and that the knowledge base has evolved through a finite series of change, then  $T_K$  is actually finitely axiomatizable by the conjunction of the formulas that have appeared in the knowledge expansion steps in the series. We will discuss finitariness issues later in this chapter.

## 7.2 Orderings of disbelief

To see how it is necessary to have an ordering among the elements of  $\llbracket T_K \rrbracket$  whenever the belief-change operator is to satisfy our postulates for belief revision, we first define an ordering of disbelief, and then we construct such an ordering.

Let  $T_B$  denote the set of propositional formulas believed in the state. We assume that  $T_B$  is deductively closed. To enhance readability, we in this chapter usually write  $\llbracket T \rrbracket$  short for  $\llbracket T_B \rrbracket$ . Thus  $\llbracket T \rrbracket$  is the set of the doxastic alternatives in state  $T$ , whereas  $\llbracket T_K \rrbracket$  is the set of the epistemic alternatives in the state.

**Definition 7.1.** For a given epistemic state  $T$ , a total pre-order  $\leq_T$  on  $\llbracket T_K \rrbracket$  is an *ordering of disbelief*, if it satisfies the following conditions:

- (D1):  $\min(\llbracket A \rrbracket \cap \llbracket T_K \rrbracket, \leq_T) \neq \emptyset$  for all  $A \in \mathcal{L}$  s.th.  $\llbracket T_K \rrbracket \cap \llbracket A \rrbracket \neq \emptyset$ ,  
 (D2):  $\min(\llbracket T_K \rrbracket, \leq_T) = \llbracket T \rrbracket \neq \emptyset$ .

Thus by definition, having an ordering of disbelief in an epistemic state guarantees that the static constraints (S1) and (S2) are satisfied in the state.

## 7.3 Grove's system of spheres

We now construct an ordering of disbelief by applying Grove's construction [Gro88] to create a system of spheres. We first give a definition of the construction, then we prove that it has the properties that make it a system of spheres.

**Definition 7.2.** Given a belief-revision operator  $\circ$  and an epistemic state  $T$  satisfying constraints (S1) and (S2), we define for all  $A \in \mathcal{L}$  such that  $T_K \not\models \neg A$ ,

$$z(T, \circ, A) =_{\text{def}} \bigcup \{ \llbracket T \circ B \rrbracket \mid B \in \mathcal{L}, \llbracket A \rrbracket \subseteq \llbracket B \rrbracket \}. \quad (7.1)$$

We then define

$$Z(T, \circ) =_{\text{def}} \{ z(T, \circ, A) \mid A \in \mathcal{L}, T_K \not\models \neg A \}, \quad (7.2)$$

and finally, we define a system

$$\mathbf{S}(T, \circ) =_{\text{def}} Z(T, \circ) \cup \{\llbracket T_K \rrbracket\}. \quad (7.3)$$

The elements of the construction  $\mathbf{S}(T, \circ)$  are called *spheres*.

As we shall see in the next lemma (Lemma 7.1), the system  $\mathbf{S}(T, \circ)$  is a set of subsets of  $\llbracket T_K \rrbracket$ . The elements of the set  $\mathbf{S}(T, \circ)$  are totally ordered by set inclusion. The smallest element is the set  $\llbracket T \rrbracket$ , the largest element is the set  $\llbracket T_K \rrbracket$ . For each possible model  $w \in \llbracket T_K \rrbracket$  we define the sphere of the model as

$$z_w =_{\text{def}} \bigcap \{S \in \mathbf{S}(T, \circ) \mid w \in S\}. \quad (7.4)$$

Thus after proving the lemma, we are able to define a total ordering upon the possible models by using set inclusion on their spheres. Before proving these properties, let us look at some examples.

**Example 7.1.** Let us consider an epistemic base  $T = [a \wedge b, a \wedge \neg b, \neg a \wedge b]$  and our revision operator  $\circ$  on epistemic bases. The system of spheres then contains the sets  $\llbracket a \wedge b \rrbracket$ ,  $\llbracket a \wedge b \rrbracket \cup \llbracket a \wedge \neg b \rrbracket = \llbracket a \rrbracket$ , and  $\llbracket a \wedge b \rrbracket \cup \llbracket a \wedge \neg b \rrbracket \cup \llbracket \neg a \wedge b \rrbracket = \llbracket a \vee b \rrbracket$ . We have  $\mathbf{S}(T, \circ) = Z(T, \circ)$  and  $\llbracket T \rrbracket = \llbracket a \wedge b \rrbracket \subset \llbracket a \rrbracket \subset \llbracket a \vee b \rrbracket = \llbracket T_K \rrbracket$ .

The second example is more elaborate.

**Example 7.2.** Let us consider an epistemic base  $T = [T^0, T^1, \dots, T^{n-1}]$  with the revision operator  $\circ$  defined on epistemic bases. Recall that we then have  $T_B \equiv T^0$  and  $T_K \equiv \bigvee_{i=0}^{n-1} T^i$ . Let  $A$  denote any propositional formula such that  $T_K \not\models \neg A$ . What can we say about  $z(T, \circ, A)$ ?

Let  $m$  denote the smallest index such that  $T^m \not\models \neg A$ . Because  $T_K \not\models \neg A$ , such a formula  $T^m$  exists for some  $0 \leq m < n$ . We show that  $z(T, \circ, A) = \bigcup_{i=0}^m \llbracket T^i \rrbracket$ .

Assume  $k$  such that  $0 \leq k \leq m$ . Because  $\llbracket A \rrbracket \subseteq \llbracket T^k \vee A \rrbracket$ , by definition  $\llbracket T \circ (T^k \vee A) \rrbracket \subseteq z(T, \circ, A)$ . Because  $T^k \not\models \neg(T^k \vee A)$  and  $T^i \models \neg(T^k \vee A)$  for all  $i$ ,  $0 \leq i < k$ , by the definition of our revision operator we have  $(T \circ (T^k \vee A))_B \equiv T^k \wedge (T^k \vee A) \equiv T^k$ . Thus  $\llbracket T^k \rrbracket \subseteq z(T, \circ, A)$  for all  $k$ ,  $0 \leq k \leq m$ , and therefore  $\bigcup_{i=0}^m \llbracket T^i \rrbracket \subseteq z(T, \circ, A)$ .

To prove that actually  $\bigcup_{i=0}^m \llbracket T^i \rrbracket = z(T, \circ, A)$ , let  $B$  denote a formula such that  $\llbracket A \rrbracket \subseteq \llbracket B \rrbracket$ . Then by definition of the revision operator,  $(T \circ B)_B \equiv T^j \wedge B$  for some  $j$ ,  $0 \leq j \leq m$ . Thus  $z(T, \circ, A) \subseteq \bigcup_{i=0}^m \llbracket T^i \rrbracket$ .

We have now seen, that for all the formulas  $A \in \mathcal{L}$  such that  $T_K \not\models \neg A$ , only  $n$  different elements  $z(T, \circ, A)$  exist, namely

$$Z(T, \circ) = \left\{ \bigcup_{i=0}^m \llbracket T^i \rrbracket \mid m = 0, \dots, n-1 \right\}.$$

This formula shows how the elements are contained in each other. The innermost element is  $\llbracket T^0 \rrbracket$ , the next one is  $\llbracket T^0 \rrbracket \cup \llbracket T^1 \rrbracket$ , and so on.

Because  $T_K \equiv \bigvee_{i=0}^{n-1} T^i$ , we have  $\llbracket T_K \rrbracket = \bigcup_{i=0}^{n-1} \llbracket T^i \rrbracket$ . Our system is therefore  $\mathbf{S}(T, \circ) = Z(T, \circ)$ .

In the next lemma we prove nine properties of the construction  $\mathbf{S}(T, \circ)$ , most of which are based on properties defined by Grove [Gro88]. Some of the properties constitute that the construction  $\mathbf{S}(T, \circ)$  is a system of spheres centered on  $\llbracket T \rrbracket$ , that is:

- the elements of the construction are totally ordered by set inclusion (property 6),
- the set  $\llbracket T \rrbracket$  is the smallest element of the construction (properties 1+2),
- the set  $\llbracket T_K \rrbracket$  is the largest element of the construction (property 4).

Property 3 of the lemma characterizes all possible models in  $\bigcup Z(T, \circ)$  saying that they can be reached in single revisions. By property 5 of the lemma, if a sphere contains any model of  $A$ , it contains all the models in  $\llbracket T \circ A \rrbracket$ . By property 7, the set  $\llbracket T \circ A \rrbracket$  actually is the set of all the models of  $A$  that are in the sphere  $z(T, \circ, A)$ . By property 8, the sphere  $z(T, \circ, A)$  is the smallest sphere containing any model of  $A$ . By property 9 of the lemma each  $z_w$  is an element of the construction  $\mathbf{S}(T, \circ)$ . We can therefore refer to  $z_w$  as the sphere of the possible model  $w$ . Together these properties help us later to characterize the set  $\llbracket T \circ A \rrbracket$  as the set of those possible models that are minimal models of  $A$  in the ordering based on their spheres.

**Lemma 7.1.** *For any given epistemic state  $T$  that satisfies constraints (S1) and (S2), if a belief-revision operator  $\circ$  satisfies condition (KX) and postulates (K1) and (R1)–(R6), then the system of spheres  $\mathbf{S}(T, \circ)$  has the following properties:*

1.  $\emptyset \neq \llbracket T \rrbracket \in Z(T, \circ)$ .
2. If  $S \in \mathbf{S}(T, \circ)$ , then  $\llbracket T \rrbracket \subseteq S$ .
3. For each  $w \in S \in Z(T, \circ)$ , a formula  $A \in \mathcal{L}$  exists such that  $T_K \not\models \neg A$  and  $w \in \llbracket T \circ A \rrbracket \subseteq \llbracket A \rrbracket$ .
4.  $\bigcup Z(T, \circ) \subseteq \llbracket T_K \rrbracket$ .
5. If  $S \in \mathbf{S}(T, \circ)$  and  $S \cap \llbracket A \rrbracket \neq \emptyset$ , then  $\llbracket T \circ A \rrbracket \subseteq S$ .
6. If  $S$  and  $S'$  are in  $\mathbf{S}(T, \circ)$ , then  $S \subseteq S'$  or  $S' \subseteq S$ .

7. If  $T_K \not\models \neg A$ , then  $z(T, \circ, A) \cap \llbracket A \rrbracket = \llbracket T \circ A \rrbracket$ .
8. If  $S \in \mathbf{S}(T, \circ)$  and  $S \subset z(T, \circ, A)$ , then  $S \cap \llbracket A \rrbracket = \emptyset$ .
9. For each  $w \in \llbracket T_K \rrbracket$ ,  $z_w \in \mathbf{S}(T, \circ)$ .

*Proof.* Assume an epistemic state  $T$  satisfying constraints (S1) and (S2), and a belief-change operator  $\circ$  satisfying condition (KX) and postulates (K1) and (R1)–(R6).

1. By constraint (S2),  $\llbracket T \rrbracket \neq \emptyset$ . Then by constraint (S1),  $\llbracket T_K \rrbracket \neq \emptyset$ , thus  $T_K \not\models \perp$  and by definition  $z(T, \circ, \top) = \bigcup \{ \llbracket T \circ B \rrbracket \mid B \in \mathcal{L} \text{ and } \llbracket \top \rrbracket \subseteq \llbracket B \rrbracket \}$ . Then by postulate (R4),  $z(T, \circ, \top) = \llbracket T \circ \top \rrbracket$ . Postulate (R2) says that  $\llbracket T \circ \top \rrbracket = \llbracket T \rrbracket \cap \llbracket \top \rrbracket = \llbracket T \rrbracket$ . Thus  $\llbracket T \rrbracket = z(T, \circ, \top) \in Z(T, \circ)$ .
2. If  $S = \llbracket T_K \rrbracket$ , then  $\llbracket T \rrbracket \subseteq S$  by constraint (S1).  
In case  $S \in Z(T, \circ)$ , by definition  $S = z(T, \circ, A)$  for some  $A \in \mathcal{L}$  such that  $T_K \not\models \neg A$ . Because  $\llbracket A \rrbracket \subseteq \llbracket \top \rrbracket$ , by definition  $\llbracket T \circ \top \rrbracket \subseteq z(T, \circ, A)$ . Thus  $\llbracket T \rrbracket = \llbracket T \circ \top \rrbracket \subseteq z(T, \circ, A) = S$ .
3. Assume  $w \in S \in Z(T, \circ)$ . Then by definition,  $w \in z(T, \circ, B)$  for some  $B \in \mathcal{L}$  such that  $T_K \not\models \neg B$ . By definition, for some  $A \in \mathcal{L}$  such that  $B \models A$ ,  $w \in \llbracket T \circ A \rrbracket$ . Because  $T_K \not\models \neg B$  and  $B \models A$ , we have  $T_K \not\models \neg A$ , and hence postulate (R1) gives  $\llbracket T \circ A \rrbracket \subseteq \llbracket A \rrbracket$ .
4. If  $w \in \bigcup Z(T, \circ)$ , then by property 3,  $A \in \mathcal{L}$  exists such that  $T_K \not\models \neg A$  and  $w \in \llbracket T \circ A \rrbracket$ . Postulate (K1) and condition (KX) say that  $\llbracket T \circ A \rrbracket \subseteq \llbracket (T \circ A)_K \rrbracket \subseteq \llbracket T_K \rrbracket$ . Thus  $w \in \llbracket T_K \rrbracket$ .
5. Assume  $S \in \mathbf{S}(T, \circ)$  and  $A \in \mathcal{L}$  such that  $S \cap \llbracket A \rrbracket \neq \emptyset$ . If  $S = \llbracket T_K \rrbracket$ , then postulate (K1) and condition (KX) give us  $\llbracket T \circ A \rrbracket \subseteq \llbracket (T \circ A)_K \rrbracket \subseteq \llbracket T_K \rrbracket$ .  
If  $S \in Z(T, \circ)$ , then  $S = z(T, \circ, C)$  for some  $C \in \mathcal{L}$ . Because  $S \cap \llbracket A \rrbracket \neq \emptyset$ , then by definition, there exists  $B \in \mathcal{L}$  such that  $T_K \not\models \neg B$ ,  $\llbracket T \circ B \rrbracket \subseteq S$ ,  $\llbracket C \rrbracket \subseteq \llbracket B \rrbracket$ , and  $\llbracket T \circ B \rrbracket \cap \llbracket A \rrbracket \neq \emptyset$ .  
Because  $\llbracket B \rrbracket \subseteq \llbracket A \vee B \rrbracket$ , then by definition  $\llbracket T \circ (A \vee B) \rrbracket \subseteq S$ . If  $\llbracket A \rrbracket \cap \llbracket T \circ (A \vee B) \rrbracket = \emptyset$ , then postulates (R1) and (R3) give us  $\emptyset \neq \llbracket T \circ (A \vee B) \rrbracket \subseteq \llbracket B \rrbracket$ . Thus by postulates (R4) and (R6),  $\llbracket T \circ B \rrbracket = \llbracket T \circ ((A \vee B) \wedge B) \rrbracket \subseteq \llbracket B \rrbracket \cap \llbracket T \circ (A \vee B) \rrbracket = \llbracket T \circ (A \vee B) \rrbracket$ . Because we assumed  $\llbracket T \circ B \rrbracket \cap \llbracket A \rrbracket \neq \emptyset$  and  $\llbracket A \rrbracket \cap \llbracket T \circ (A \vee B) \rrbracket = \emptyset$ , equation  $\llbracket T \circ B \rrbracket = \llbracket T \circ (A \vee B) \rrbracket$  gives us a contradiction. Thus  $\llbracket A \rrbracket \cap \llbracket T \circ (A \vee B) \rrbracket \neq \emptyset$ . Then by postulates (R4) and (R6),  $\llbracket T \circ A \rrbracket = \llbracket T \circ ((A \vee B) \wedge A) \rrbracket \subseteq \llbracket A \rrbracket \cap \llbracket T \circ (A \vee B) \rrbracket \subseteq S$ .

6. If either  $S = \llbracket T_K \rrbracket$  or  $S' = \llbracket T_K \rrbracket$ , then property 4 gives the answer. Otherwise both  $S$  and  $S'$  are in  $Z(T, \circ)$ , and for showing the converse, assume that  $w \in S \setminus S'$  and  $w' \in S' \setminus S$  exist.

By property 3,  $w \in \llbracket T \circ A \rrbracket \subseteq \llbracket A \rrbracket$  and  $w' \in \llbracket T \circ B \rrbracket \subseteq \llbracket B \rrbracket$  for some  $A, B \in \mathcal{L}$  such that  $T_K \not\models \neg A$  and  $T_K \not\models \neg B$ . Because  $\llbracket A \vee B \rrbracket \cap S \neq \emptyset$  and  $\llbracket A \vee B \rrbracket \cap S' \neq \emptyset$ , property 5 gives us  $\llbracket T \circ (A \vee B) \rrbracket \subseteq S$  and  $\llbracket T \circ (A \vee B) \rrbracket \subseteq S'$ .

On one hand, if  $\llbracket T \circ (A \vee B) \rrbracket \cap \llbracket A \rrbracket \neq \emptyset$ , then by postulates (R4) and (R6),  $\llbracket T \circ (A \vee B) \rrbracket \cap \llbracket A \rrbracket \supseteq \llbracket T \circ ((A \vee B) \wedge A) \rrbracket = \llbracket T \circ A \rrbracket$ . Thus  $w \in \llbracket T \circ (A \vee B) \rrbracket \subseteq S'$ , a contradiction.

On the other hand, if  $\llbracket T \circ (A \vee B) \rrbracket \cap \llbracket A \rrbracket = \emptyset$ , then by postulates (R3) and (R1),  $\emptyset \neq \llbracket T \circ (A \vee B) \rrbracket \subseteq \llbracket B \rrbracket$ . Then  $\llbracket B \rrbracket \cap S \neq \emptyset$ , and by property 5,  $\llbracket T \circ B \rrbracket \subseteq S$ , a contradiction. Thus either  $S \subseteq S'$  or  $S' \subseteq S$ .

7. Assume  $A \in \mathcal{L}$  such that  $T_K \not\models \neg A$ . By definition  $z(T, \circ, A) \cap \llbracket A \rrbracket = \bigcup \{ \llbracket T \circ B \rrbracket \mid B \in \mathcal{L} \text{ and } \llbracket A \rrbracket \subseteq \llbracket B \rrbracket \} \cap \llbracket A \rrbracket = \bigcup \{ \llbracket T \circ B \rrbracket \cap \llbracket A \rrbracket \mid \llbracket A \rrbracket \subseteq \llbracket B \rrbracket \}$ . By postulates (R4), (R5) and (R6), if  $\llbracket T \circ B \rrbracket \cap \llbracket A \rrbracket \neq \emptyset$ , then  $\llbracket T \circ B \rrbracket \cap \llbracket A \rrbracket = \llbracket T \circ (A \wedge B) \rrbracket = \llbracket T \circ A \rrbracket$  for each  $B$  such that  $\llbracket A \rrbracket \subseteq \llbracket B \rrbracket$ . Thus  $z(T, \circ, A) \cap \llbracket A \rrbracket = \llbracket T \circ A \rrbracket$ .

8. Assume  $S \in \mathbf{S}(T, \circ)$  and  $S \subset z(T, \circ, A)$ , thus  $w \in z(T, \circ, A) \setminus S$  exists. Assume for the converse that  $S \cap \llbracket A \rrbracket \neq \emptyset$ . By definition  $w \in \llbracket T \circ B \rrbracket$  for some  $B \in \mathcal{L}$  such that  $\llbracket A \rrbracket \subseteq \llbracket B \rrbracket$ . But then  $S \cap \llbracket B \rrbracket \neq \emptyset$ , in which case property 5 gives us  $\llbracket T \circ B \rrbracket \subseteq S$ , a contradiction.

9. If  $w \in \bigcup Z(T, \circ)$ , then by property 3, there exists  $A \in \mathcal{L}$  such that  $T_K \not\models \neg A$  and  $w \in \llbracket T \circ A \rrbracket \subseteq \llbracket A \rrbracket$ . Property 7 then says that  $\llbracket T \circ A \rrbracket = z(T, \circ, A) \cap \llbracket A \rrbracket$ . By property 8,  $S \subset z(T, \circ, A)$  implies  $w \notin S$ . Property 6 then gives us  $z_w = \bigcap \{ S \in \mathbf{S}(T, \circ) \mid w \in S \} = z(T, \circ, A)$ , and by definition,  $z(T, \circ, A) \in Z(T, \circ) \subseteq \mathbf{S}(T, \circ)$ .

If  $w \in \llbracket T_K \rrbracket \setminus \bigcup Z(T, \circ)$ , then  $z_w = \bigcap \{ S \in \mathbf{S}(T, \circ) \mid w \in S \} = \llbracket T_K \rrbracket$ , and by definition,  $\llbracket T_K \rrbracket \in \mathbf{S}(T, \circ)$ .

□

Let us now construct an ordering based on a system of spheres  $\mathbf{S}(T, \circ)$  on  $\llbracket T_K \rrbracket$ . By property 9 of Lemma 7.1, each  $z_w$  is an element in  $\mathbf{S}(T, \circ)$ . Then by property 6 of Lemma 7.1, we have  $z_w \subseteq z_{w'}$  or  $z_{w'} \subseteq z_w$  for all  $w, w' \in \llbracket T_K \rrbracket$ . An ordering on the possible models is defined by using the set inclusion relation of the spheres as follows.



**Definition 7.3.** Given a system of spheres  $\mathbf{S}(T, \circ)$  on  $\llbracket T_K \rrbracket$ ,

$$\leq_{\mathbf{S}(T, \circ)} =_{\text{def}} \{(w, w') \in \llbracket T_K \rrbracket \times \llbracket T_K \rrbracket \mid z_w \subseteq z_{w'}\}. \quad (7.5)$$

The following proposition tells us a property that is shared by both knowledge expansion and belief revision.

**Theorem 7.1.** *Given a belief-revision operator  $\circ$  and an epistemic state  $T$  satisfying constraints (S1) and (S2), then if the operator satisfies condition (KX), postulates (K1), and (R1)–(R6), the relation  $\leq_{\mathbf{S}(T, \circ)}$  is an ordering of disbelief and for each  $A \in \mathcal{L}$ , if  $T_K \not\models \neg A$ , then  $\llbracket T \circ A \rrbracket = \min(\llbracket A \rrbracket \cap \llbracket T_K \rrbracket, \leq_{\mathbf{S}(T, \circ)})$ .*

*Proof.* Let  $\leq$  denote the relation  $\leq_{\mathbf{S}(T, \circ)}$ . Because the relation  $\leq$  is based on set inclusion, the relation is reflexive and transitive. Because by property 9 of Lemma 7.1,  $z_w$  is an element in  $\mathbf{S}(T, \circ)$  for each possible model  $w$ , property 6 of Lemma 7.1 then says that for all possible models  $w$  and  $w'$ ,  $z_w \subseteq z_{w'}$  or  $z_{w'} \subseteq z_w$ , thus the preorder is total.

Properties 1 and 2 of Lemma 7.1 give us  $\min(\llbracket T_K \rrbracket, \leq) = \llbracket T \rrbracket \neq \emptyset$ , condition (D2). For proving condition (D1), let  $A$  be a propositional formula such that  $\llbracket A \rrbracket \cap \llbracket T_K \rrbracket \neq \emptyset$ . By postulate (R3), a model  $w \in \llbracket T \circ A \rrbracket$  exists. By definition,  $w \in z(T, \circ, A)$ . By postulate (R1),  $w \in \llbracket A \rrbracket$  and by postulate (K1) and condition (KX),  $w \in \llbracket (T \circ A)_K \rrbracket \subseteq \llbracket T_K \rrbracket$ . For any  $w' \in \llbracket T_K \rrbracket$ , property 8 of Lemma 7.1 implies  $w' < w$ , only if  $w' \notin \llbracket A \rrbracket$ , therefore  $w \in \min(\llbracket A \rrbracket \cap \llbracket T_K \rrbracket, \leq)$ . Thus condition (D1) holds and  $\llbracket T \circ A \rrbracket \subseteq \min(\llbracket A \rrbracket \cap \llbracket T_K \rrbracket, \leq)$ .

For proving condition  $\min(\llbracket A \rrbracket \cap \llbracket T_K \rrbracket, \leq) \subseteq \llbracket T \circ A \rrbracket$ , assume  $w \in \min(\llbracket A \rrbracket \cap \llbracket T_K \rrbracket, \leq)$ . Then properties 5, 7, and 8 of Lemma 7.1 give us  $z_w = z(T, \circ, A)$ , and  $w \in \llbracket T \circ A \rrbracket$ .  $\square$

We can construct a system  $\mathbf{S}(T \circ A, \circ)$  and an ordering  $\leq_{\mathbf{S}(T \circ A, \circ)}$  correspondingly. For further use, we shall next prove a property relating these constructions  $\mathbf{S}(T, \circ)$  and  $\mathbf{S}(T \circ A, \circ)$  in cases when the operator also satisfies some postulates for iterated belief revision.

**Lemma 7.2.** *If an epistemic state  $T$  satisfies constraints (S1) and (S2), and a belief-revision operator  $\circ$  satisfies condition (KX) and postulates (K1), (R1)–(R6), and (RR1), then  $\bigcup Z(T \circ A, \circ) \cap \llbracket A \rrbracket = \bigcup Z(T, \circ) \cap \llbracket A \rrbracket$ .*

*Proof.* Assume an epistemic state  $T$  satisfying constraints (S1) and (S2), and a belief-revision operator  $\circ$  that satisfies condition (KX) and postulates (K1), (R1)–(R6), and (RR1). Assume  $A \in \mathcal{L}$ .

To prove  $\bigcup Z(T, \circ) \cap \llbracket A \rrbracket \subseteq \bigcup Z(T \circ A, \circ)$ , assume  $w \in \bigcup Z(T, \circ) \cap \llbracket A \rrbracket$ . By property 4 of Lemma 7.1,  $w \in \llbracket T_K \rrbracket \cap \llbracket A \rrbracket$ . Property 3 of Lemma 7.1 says

that some  $B \in \mathcal{L}$  exists such that  $w \in \llbracket T \circ B \rrbracket \subseteq \llbracket B \rrbracket$ . Thus  $T_K \not\models \neg(A \wedge B)$ . By postulate (R5),  $w \in \llbracket T \circ (A \wedge B) \rrbracket$ , and postulate (RR1) gives us  $w \in \llbracket (T \circ A) \circ (A \wedge B) \rrbracket$ . Thus by definition,  $w \in \bigcup Z(T \circ A, \circ)$ .

To prove  $\bigcup Z(T \circ A, \circ) \cap \llbracket A \rrbracket \subseteq \bigcup Z(T, \circ)$ , assume  $w \in \bigcup Z(T \circ A, \circ) \cap \llbracket A \rrbracket$ . Then, by property 4 of Lemma 7.1,  $w \in \llbracket (T \circ A)_K \rrbracket$ , thus condition (KX) gives us  $w \in \llbracket T_K \rrbracket$ . Property 3 of Lemma 7.1 says that  $w \in \llbracket (T \circ A) \circ B \rrbracket \subseteq \llbracket B \rrbracket$  for some  $B \in \mathcal{L}$ . Thus  $T_K \not\models \neg(A \wedge B)$  and  $(T \circ A)_K \not\models \neg(A \wedge B)$ . By postulate (R6),  $w \in \llbracket (T \circ A) \circ (A \wedge B) \rrbracket$ , and postulate (RR1) gives us  $w \in \llbracket T \circ (A \wedge B) \rrbracket$ . Thus  $w \in \bigcup Z(T, \circ)$ .  $\square$

**Lemma 7.3.** *If an epistemic state  $T$  satisfies constraints (S1) and (S2), and a belief-revision operator  $\circ$  satisfies postulates (K0), (K1), (R1)–(R6), and (RR2), then  $\bigcup Z(T \circ A, \circ) \cap \llbracket \neg A \rrbracket = \bigcup Z(T, \circ) \cap \llbracket \neg A \rrbracket$ .*

*Proof.* The proof is analogous to the proof of Lemma 7.2.  $\square$

The Following lemma is a corollary of lemmas 7.2 and 7.3: among the possible models of the new state, those models reachable before belief revision remain reachable after the revision and those unreachable remain so after the revision.

**Lemma 7.4.** *If an epistemic state  $T$  satisfies constraints (S1) and (S2), and a belief-revision operator  $\circ$  satisfies postulates (K0), (K1), (R1)–(R6), (RR1), and (RR2), then  $\bigcup Z(T \circ A, \circ) = \bigcup Z(T, \circ)$ .*

Because knowledge expansion also carries out belief revision, we can prove a property corresponding to the previous lemma for knowledge-expansion operators.

**Lemma 7.5.** *If an epistemic state  $T$  satisfies constraints (S1) and (S2), and a knowledge expansion-operator  $\oplus$  satisfies postulates (K1), (K2), (R1)–(R6), and (RR1), then  $\bigcup Z(T \oplus A, \oplus) = \bigcup Z(T, \oplus) \cap \llbracket A \rrbracket$ .*

*Proof.* Postulate (K2) implies condition (KX), thus by Lemma 7.2,  $\bigcup Z(T \oplus A, \oplus) \cap \llbracket A \rrbracket = \bigcup Z(T, \oplus) \cap \llbracket A \rrbracket$ . By property 4 of Lemma 7.1,  $\bigcup Z(T \oplus A, \oplus) \subseteq \llbracket (T \oplus A)_K \rrbracket$ , by postulate (K2),  $\llbracket (T \oplus A)_K \rrbracket \subseteq \llbracket A \rrbracket$ . Thus  $\bigcup Z(T \oplus A, \oplus) = \bigcup Z(T \oplus A, \oplus) \cap \llbracket A \rrbracket = \bigcup Z(T, \oplus) \cap \llbracket A \rrbracket$ .  $\square$

Notice that for any belief-revision operator  $\circ$  satisfying the postulates in question, these lemmas imply that  $\llbracket (T \circ A)_K \rrbracket \in Z(T \circ A, \circ)$  if and only if  $\llbracket T_K \rrbracket \in Z(T, \circ)$ , and for any  $A \in \mathcal{L}$  such that  $T_K \not\models \neg A$ ,  $\llbracket (T \oplus A)_K \rrbracket \in Z(T \oplus A, \oplus)$  if and only if  $\llbracket T_K \rrbracket \in Z(T, \oplus)$ .

The following theorems show how the orderings of disbelief relate in iterative revisions.

**Theorem 7.2.** *Given a propositional formula  $A$ , an epistemic state  $T$  satisfying constraints (S1) and (S2), and a belief-revision operator  $\circ$  that satisfies condition (KX) and postulates (K1), (R0)–(R6), and (RR1), then for each  $w, w' \in \llbracket (T \circ A)_K \rrbracket \cap \llbracket A \rrbracket$ ,  $w \leq_{\mathcal{S}(T, \circ)} w'$  if and only if  $w \leq_{\mathcal{S}(T \circ A, \circ)} w'$ .*

*Proof.* Assume an epistemic state  $T$  satisfying constraints (S1) and (S2), and a belief-revision operator  $\circ$  that satisfies condition (KX) and postulates (K1), (R0)–(R6), and (RR1).

Let  $\leq_T$  and  $\leq_{T \circ A}$  denote relations  $\leq_{\mathcal{S}(T, \circ)}$  and  $\leq_{\mathcal{S}(T \circ A, \circ)}$  based on respective systems of spheres on  $\llbracket T_K \rrbracket$ . By Theorem 7.1, relations  $\leq_T$  and  $\leq_{T \circ A}$  are orderings of disbelief.

By Lemma 7.2,  $\bigcup Z(T, \circ) \cap \llbracket A \rrbracket = \bigcup Z(T \circ A, \circ) \cap \llbracket A \rrbracket$ . If  $w \in \llbracket (T \circ A)_K \rrbracket \cap \llbracket A \rrbracket \setminus \bigcup Z(T \circ A, \circ)$ , then  $w' \leq_T w$  and  $w' \leq_{T \circ A} w$  for all  $w' \in \llbracket (T \circ A)_K \rrbracket$ .

Thus we only need to prove the condition in the case  $w, w' \in \bigcup Z(T \circ A, \circ) \cap \llbracket (T \circ A)_K \rrbracket \cap \llbracket A \rrbracket$ . By condition (KX),  $w, w' \in \bigcup Z(T, \circ) \cap \llbracket T_K \rrbracket \cap \llbracket A \rrbracket$ . Then by property 3 of Lemma 7.1, there exist  $B, C \in \mathcal{L}$  such that  $w \in \llbracket T \circ B \rrbracket \subseteq \llbracket B \rrbracket$  and  $w' \in \llbracket T \circ C \rrbracket \subseteq \llbracket C \rrbracket$ . Thus  $w, w' \in \llbracket A \wedge (B \vee C) \rrbracket \cap \llbracket T_K \rrbracket$  and  $w, w' \in \llbracket A \wedge (B \vee C) \rrbracket \cap \llbracket (T \circ A)_K \rrbracket$ . Then by Theorem 7.1,  $w \in \min(\llbracket B \rrbracket \cap \llbracket T_K \rrbracket, \leq_T)$ ,  $w' \in \min(\llbracket C \rrbracket \cap \llbracket T_K \rrbracket, \leq_T)$ ,  $w \in \llbracket T \circ (A \wedge B) \rrbracket$ , and  $w' \in \llbracket T \circ (A \wedge C) \rrbracket$ .

Because  $\llbracket A \wedge (B \vee C) \rrbracket \subseteq \llbracket A \rrbracket$ , postulate (RR1) implies that  $\llbracket (T \circ A) \circ (A \wedge (B \vee C)) \rrbracket = \llbracket T \circ (A \wedge (B \vee C)) \rrbracket$ . Because  $T_K \not\models \neg(A \wedge (B \vee C))$  and  $(T \circ A)_K \not\models \neg(A \wedge (B \vee C))$  and Theorem 7.1 then gives us  $\min(\llbracket A \wedge (B \vee C) \rrbracket \cap \llbracket T_K \rrbracket, \leq_T) = \llbracket T \circ (A \wedge (B \vee C)) \rrbracket = \llbracket (T \circ A) \circ (A \wedge (B \vee C)) \rrbracket = \min(\llbracket A \wedge (B \vee C) \rrbracket \cap \llbracket (T \circ A)_K \rrbracket, \leq_{T \circ A})$ .

If  $w \leq_T w'$ , then  $w \in \min(\llbracket A \wedge (B \vee C) \rrbracket \cap \llbracket T_K \rrbracket, \leq_T) = \min(\llbracket A \wedge (B \vee C) \rrbracket \cap \llbracket (T \circ A)_K \rrbracket, \leq_{T \circ A})$ . Thus  $w \leq_{T \circ A} w'$ .

If  $w \not\leq_T w'$ , then  $w' \in \min(\llbracket A \wedge (B \vee C) \rrbracket \cap \llbracket T_K \rrbracket, \leq_T) = \min(\llbracket A \wedge (B \vee C) \rrbracket \cap \llbracket (T \circ A)_K \rrbracket, \leq_{T \circ A})$  but  $w \notin \min(\llbracket A \wedge (B \vee C) \rrbracket \cap \llbracket T_K \rrbracket, \leq_T) = \min(\llbracket A \wedge (B \vee C) \rrbracket \cap \llbracket (T \circ A)_K \rrbracket, \leq_{T \circ A})$ . Thus  $w \not\leq_{T \circ A} w'$ .  $\square$

**Theorem 7.3.** *Given a propositional formula  $A$ , an epistemic state  $T$  satisfying constraints (S1) and (S2), and a belief-revision operator  $\circ$  that satisfies condition (KX) and postulates (K1), (R0)–(R6), and (RR2), then for each  $w, w' \in \llbracket (T \circ A)_K \rrbracket \setminus \llbracket A \rrbracket$ ,  $w \leq_{\mathcal{S}(T, \circ)} w'$  if and only if  $w \leq_{\mathcal{S}(T \circ A, \circ)} w'$ .*

*Proof.* The proof is analogous to that of Theorem 7.2. Because  $\llbracket \neg A \wedge (B \vee C) \rrbracket \cap \llbracket A \rrbracket = \emptyset$ , postulate (RR2) implies that  $\llbracket (T \circ A) \circ (\neg A \wedge (B \vee C)) \rrbracket = \llbracket T \circ (\neg A \wedge (B \vee C)) \rrbracket$ .  $\square$

**Theorem 7.4.** *Given a propositional formula  $A$ , an epistemic state  $T$  satisfying constraints (S1) and (S2), and a belief-revision operator  $\circ$  that satisfies postulates (K0), (K1), (R0)–(R6), and (RR1), then*

1. if the operator  $\circ$  satisfies postulate (RR3), then  
for each  $w \in \llbracket (T \circ A)_K \rrbracket \cap \llbracket A \rrbracket$  and  $w' \in \llbracket (T \circ A)_K \rrbracket \setminus \llbracket A \rrbracket$ ,  
 $w <_{S(T, \circ)} w'$  implies  $w <_{S(T \circ A, \circ)} w'$ .
2. if the operator  $\circ$  satisfies postulate (RR4), then  
for each  $w \in \llbracket (T \circ A)_K \rrbracket \cap \llbracket A \rrbracket$  and  $w' \in \llbracket (T \circ A)_K \rrbracket \setminus \llbracket A \rrbracket$ ,  
 $w \leq_{S(T, \circ)} w'$  implies  $w \leq_{S(T \circ A, \circ)} w'$ .

*Proof.* Assume a belief-revision operator  $\circ$  that satisfies postulates (K0), (K1), (R0)–(R6), and (RR1). Postulate (K0) implies condition (KX).

Given a propositional formula  $A$  and an epistemic state  $T$  that satisfies constraints (S1) and (S2), let  $\leq_T$  and  $\leq_{T \circ A}$  denote relations  $\leq_{S(T, \circ)}$  and  $\leq_{S(T \circ A, \circ)}$  based on respective systems of spheres on  $\llbracket T_K \rrbracket$ . By Theorem 7.1,  $\leq_T$  and  $\leq_{T \circ A}$  are orderings of disbelief.

By Lemma 7.4,  $\bigcup Z(T, \circ) = \bigcup Z(T \circ A, \circ)$ . If  $w \in \llbracket (T \circ A)_K \rrbracket \setminus \bigcup Z(T \circ A, \circ)$ , then  $w' \leq_T w$  and  $w' \leq_{T \circ A} w$  for all  $w' \in \llbracket (T \circ A)_K \rrbracket$ . Thus we only need to prove the claims in the case  $w, w' \in \bigcup Z(T \circ A, \circ) \cap \llbracket (T \circ A)_K \rrbracket$ . By postulate (K0),  $\bigcup Z(T \circ A, \circ) \cap \llbracket (T \circ A)_K \rrbracket = \bigcup Z(T, \circ) \cap \llbracket T_K \rrbracket$ .

Assume  $w \in \llbracket A \rrbracket \cap \bigcup Z(T \circ A, \circ) \cap \llbracket (T \circ A)_K \rrbracket = \llbracket A \rrbracket \cap \bigcup Z(T, \circ) \cap \llbracket T_K \rrbracket$  and  $w' \in \llbracket \neg A \rrbracket \cap \bigcup Z(T \circ A, \circ) \cap \llbracket (T \circ A)_K \rrbracket = \llbracket \neg A \rrbracket \cap \bigcup Z(T, \circ) \cap \llbracket T_K \rrbracket$ . By property 3 of Lemma 7.1, there exist  $B, C \in \mathcal{L}$  such that  $w \in \llbracket T \circ B \rrbracket \subseteq \llbracket B \rrbracket$  and  $w' \in \llbracket T \circ C \rrbracket \subseteq \llbracket C \rrbracket$ . Theorem 7.1 gives us  $w \in \min(\llbracket B \rrbracket \cap \llbracket T_K \rrbracket, \leq_T)$  and  $w' \in \min(\llbracket C \rrbracket \cap \llbracket T_K \rrbracket, \leq_T)$ , then  $w \in \llbracket T \circ (A \wedge B) \rrbracket$  and  $w' \in \llbracket T \circ (\neg A \wedge C) \rrbracket$ .

Thus  $w, w' \in \llbracket (A \wedge B) \vee (\neg A \wedge C) \rrbracket \cap \llbracket T_K \rrbracket = \llbracket (A \wedge B) \vee (\neg A \wedge C) \rrbracket \cap \llbracket (T \circ A)_K \rrbracket$ . Because  $T_K \not\models \neg((A \wedge B) \vee (\neg A \wedge C))$  and  $(T \circ A)_K \not\models \neg((A \wedge B) \vee (\neg A \wedge C))$ , Theorem 7.1 then gives us  $\min(\llbracket (A \wedge B) \vee (\neg A \wedge C) \rrbracket \cap \llbracket (T \circ A)_K \rrbracket, \leq_{T \circ A}) = \llbracket (T \circ A) \circ ((A \wedge B) \vee (\neg A \wedge C)) \rrbracket$  and  $\min(\llbracket (A \wedge B) \vee (\neg A \wedge C) \rrbracket \cap \llbracket T_K \rrbracket, \leq_T) = \llbracket T \circ ((A \wedge B) \vee (\neg A \wedge C)) \rrbracket$ .

If  $\llbracket (T \circ A) \circ ((A \wedge B) \vee (\neg A \wedge C)) \rrbracket \cap \llbracket A \rrbracket \neq \emptyset$  were the case, then we would have by postulate (R1),  $\llbracket (T \circ A) \circ ((A \wedge B) \vee (\neg A \wedge C)) \rrbracket \cap \llbracket A \wedge B \rrbracket \neq \emptyset$ , by postulates (R4), (R5), and (R6),  $\llbracket (T \circ A) \circ ((A \wedge B) \vee (\neg A \wedge C)) \rrbracket \cap \llbracket A \wedge B \rrbracket = \llbracket (T \circ A) \circ (A \wedge B) \rrbracket$ , and by postulate (RR1),  $\llbracket (T \circ A) \circ (A \wedge B) \rrbracket = \llbracket (T \circ (A \wedge B)) \rrbracket$ .

Assume that  $w <_T w'$  and the operator  $\circ$  satisfies postulate (RR3). Then  $\min(\llbracket (A \wedge B) \vee (\neg A \wedge C) \rrbracket \cap \llbracket T_K \rrbracket, \leq_T) = \min(\llbracket A \wedge B \rrbracket \cap \llbracket T_K \rrbracket, \leq_T) \subseteq \llbracket A \rrbracket$ . By postulate (RR3),  $\llbracket (T \circ A) \circ ((A \wedge B) \vee (\neg A \wedge C)) \rrbracket \subseteq \llbracket A \rrbracket$ , thus  $w' \notin \min(\llbracket (A \wedge B) \vee (\neg A \wedge C) \rrbracket \cap \llbracket (T \circ A)_K \rrbracket, \leq_{T \circ A})$ . Because  $w \leq_T w'$ , it follows  $w \in \min(\llbracket (A \wedge B) \vee (\neg A \wedge C) \rrbracket \cap \llbracket T_K \rrbracket, \leq_T)$ . By postulate (R3),  $\llbracket (T \circ A) \circ ((A \wedge B) \vee (\neg A \wedge C)) \rrbracket \cap \llbracket A \rrbracket \neq \emptyset$ , thus  $w \in \min(\llbracket (A \wedge B) \vee (\neg A \wedge C) \rrbracket \cap \llbracket (T \circ A)_K \rrbracket, \leq_{T \circ A})$  and  $w <_{T \circ A} w'$  holds.

Assume that  $w \leq_T w'$  and the operator  $\circ$  satisfies postulate (RR4). Because  $w \leq_T w'$ , it follows  $w \in \min(\llbracket (A \wedge B) \vee (\neg A \wedge C) \rrbracket \cap \llbracket T_K \rrbracket, \leq_T)$ .

By postulate (RR4),  $\llbracket (T \circ A) \circ ((A \wedge B) \vee (\neg A \wedge C)) \rrbracket \cap \llbracket A \rrbracket \neq \emptyset$ , thus  $w \in \min(\llbracket (A \wedge B) \vee (\neg A \wedge C) \rrbracket \cap \llbracket (T \circ A)_K \rrbracket, \leq_{T \circ A})$  and  $w \leq_{T \circ A} w'$  holds.  $\square$

## 7.4 Modelling belief revision

When characterizing the effect of belief revision on the set of the most plausible models of an epistemic state, we can use the following conditions:

- (BN): If  $\llbracket A \rrbracket \cap \llbracket T_K \rrbracket = \emptyset$ , then  $\llbracket T \circ A \rrbracket = \llbracket T \rrbracket$ .  
 (BR): If  $\llbracket A \rrbracket \cap \llbracket T_K \rrbracket \neq \emptyset$ , then  $\llbracket T \circ A \rrbracket = \min(\llbracket A \rrbracket \cap \llbracket T_K \rrbracket, \leq_T)$ .

The following theorem for single belief revisions says how the set of propositional formulas known in the state and how the set of propositional formulas believed in the state change in belief revision.

**Theorem 7.5.** *If a belief-revision operator  $\circ$  satisfies postulates (K0), (K1), and (R0)–(R6), then there is a function that maps each epistemic state  $T$  that satisfies constraints (S1) and (S2) to an ordering of disbelief  $\leq_T$  such that conditions (KN), (BN), and (BR) are satisfied.*

*Proof.* Assume that a belief-revision operator  $\circ$  satisfies postulates (K0), (K1), and (R0)–(R6). Given an epistemic state  $T$  satisfying constraints (S1) and (S2), let  $\leq_{S(T, \circ)}$  denote the ordering based on the system of spheres  $S(T, \circ)$  on  $\llbracket T_K \rrbracket$ .

Condition (KN) holds by postulate (K0), condition (BN) by postulate (R0). Because postulate (K0) implies condition (KX), Theorem 7.1 says that  $\leq_{S(T, \circ)}$  is an ordering of disbelief and that condition (BR) holds.  $\square$

We phrase the conditions characterizing the effect of belief revision on orderings of disbelief as follows:

- (O1): If  $w, w' \in \llbracket (T \circ A)_K \rrbracket \cap \llbracket A \rrbracket$ ,  
 then  $w \leq_T w'$  if and only if  $w \leq_{T \circ A} w'$ .  
 (O2): If  $w, w' \in \llbracket (T \circ A)_K \rrbracket \setminus \llbracket A \rrbracket$ ,  
 then  $w \leq_T w'$  if and only if  $w \leq_{T \circ A} w'$ .  
 (O3): If  $w \in \llbracket (T \circ A)_K \rrbracket \cap \llbracket A \rrbracket$  and  $w' \in \llbracket (T \circ A)_K \rrbracket \setminus \llbracket A \rrbracket$ ,  
 then  $w <_T w'$  implies  $w <_{T \circ A} w'$ .  
 (O4): If  $w \in \llbracket (T \circ A)_K \rrbracket \cap \llbracket A \rrbracket$  and  $w' \in \llbracket (T \circ A)_K \rrbracket \setminus \llbracket A \rrbracket$ ,  
 then  $w \leq_T w'$  implies  $w \leq_{T \circ A} w'$ .

The next theorem characterizes (prioritized) belief revision, and gives us the intuition on which our belief-revision operators are based: the ordering among the possible models of the input formula remains unchanged,

the ordering among the possible models not modelling the input formula remains the same, but the two sets of possible models may get shifted in the ordering compared to each other so that the models of the input formula get less disbelieved.

**Theorem 7.6.** *A belief-revision operator  $\circ$  satisfies postulates (K0), (K1), (R0)–(R3) and (RR1)–(RR4), if and only if there is a function that maps each epistemic state that satisfies constraints (S1) and (S2) to an ordering of disbelief such that conditions (KN), (BN), (BR), and (O1)–(O4) hold.*

*Proof.* ( $\Rightarrow$ ) Assume a belief-revision operator  $\circ$  that satisfies postulates (K0), (K1), (R0)–(R3), and (RR1)–(RR4). By theorems 5.1 and 5.2, postulates (R4)–(R6) are also satisfied. Given a propositional formula  $A$  and an epistemic state  $T$  that satisfies constraints (S1) and (S2), let  $\leq_T$  and  $\leq_{T \circ A}$  denote relations  $\leq_{\mathcal{S}(T, \circ)}$  and  $\leq_{\mathcal{S}(T \circ A, \circ)}$  based on respective systems of spheres on  $\llbracket T_K \rrbracket$ . Condition (KN) holds by postulate (K0), condition (BN) by postulate (R0). Because postulate (K0) implies condition (KX), Theorem 7.1 says that  $\leq_{\mathcal{S}(T, \circ)}$  and  $\leq_{\mathcal{S}(T \circ A, \circ)}$  are orderings of disbelief and that condition (BR) holds. By theorems 7.2, 7.3, and 7.4, orderings  $\leq_{\mathcal{S}(T, \circ)}$  and  $\leq_{\mathcal{S}(T \circ A, \circ)}$  satisfy conditions (O1)–(O4).

( $\Leftarrow$ ) Assume a belief-revision operator  $\circ$ , propositional formulas  $A$  and  $B$ , and assume epistemic states  $T$  and  $T \circ A$  with orderings of disbelief satisfying conditions (KN), (BN), (BR), and (O1)–(O4). By definition, condition (KN) implies postulate (K0).

If  $T_K \models \neg A$ , condition (BN) implies postulate (R0). If  $T_K \not\models \neg A$ , then condition (BR) gives  $\llbracket T \circ A \rrbracket = \min(\llbracket A \rrbracket \cap \llbracket T_K \rrbracket, \leq_T) \subseteq \llbracket A \rrbracket$ . Postulate (R1) is satisfied. If  $T_B \not\models \neg A$ , then conditions (D2) and (BR) give us  $\llbracket T \rrbracket \cap \llbracket A \rrbracket = \min(\llbracket T_K \rrbracket, \leq_T) \cap \llbracket A \rrbracket = \min(\llbracket A \rrbracket \cap \llbracket T_K \rrbracket, \leq_T) = \llbracket T \circ A \rrbracket$ . Postulate (R2) is satisfied.

To prove postulate (RR1), assume  $T_K \not\models \neg B$  and  $B \models A$ . Then conditions (BR), (KN), and (O1) give us  $\llbracket (T \circ A) \circ B \rrbracket = \min(\llbracket B \rrbracket \cap \llbracket (T \circ A)_K \rrbracket, \leq_{T \circ A}) = \min(\llbracket B \rrbracket \cap \llbracket T_K \rrbracket, \leq_T) = \llbracket T \circ B \rrbracket$ .

To prove postulate (RR2), assume propositional formulas  $A$  and  $B$  such that  $B \models \neg A$  and  $T_K \not\models \neg B$ . Then conditions (BR), (KN), and (O2) give us  $\llbracket (T \circ A) \circ B \rrbracket = \min(\llbracket B \rrbracket \cap \llbracket (T \circ A)_K \rrbracket, \leq_{T \circ A}) = \min(\llbracket B \rrbracket \cap \llbracket T_K \rrbracket, \leq_T) = \llbracket T \circ B \rrbracket$ .

To prove postulate (RR3), assume  $(T \circ B)_B \models A$ . Then by condition (D2),  $T_K \not\models \neg A$ . If  $T_K \not\models \neg B$ , then by conditions (BR) and (D1), for every  $w \in \llbracket \neg A \wedge B \rrbracket$  there is  $w' \in \llbracket A \wedge B \rrbracket$  such that  $w' <_T w$ . Condition (O3) then gives us  $w' <_{T \circ A} w$ . Thus by conditions (BR) and (KN),  $\llbracket (T \circ A) \circ B \rrbracket = \min(\llbracket B \rrbracket \cap \llbracket (T \circ A)_K \rrbracket, \leq_{T \circ A}) = \min(\llbracket B \rrbracket \cap \llbracket T_K \rrbracket, \leq_{T \circ A}) \subseteq \llbracket A \rrbracket$ . If  $\llbracket B \rrbracket \cap \llbracket T_K \rrbracket = \emptyset \neq \llbracket A \rrbracket \cap \llbracket T_K \rrbracket$ , then by condition (BN),  $\llbracket T \circ B \rrbracket \subseteq \llbracket A \rrbracket$  implies  $\llbracket T \rrbracket \subseteq \llbracket A \rrbracket$ .

Thus by conditions (BN) and (BR),  $\llbracket (T \circ A) \circ B \rrbracket = \llbracket T \circ A \rrbracket = \min(\llbracket A \rrbracket \cap \llbracket T_K \rrbracket, \leq_T) \subseteq \llbracket A \rrbracket$ .

To prove postulate (RR4), assume  $(T \circ B)_B \not\models \neg A$ . Then by condition (D2),  $T_K \not\models \neg A$ . If  $\llbracket B \rrbracket \cap \llbracket T_K \rrbracket \neq \emptyset$ , then a model  $w \in \llbracket A \wedge B \rrbracket$  exists such that for all  $w' \in \llbracket B \rrbracket$ ,  $w \leq_T w'$ . Condition (O4) then says that  $w \leq_{T \circ A} w'$ . Thus by condition (BR),  $\llbracket (T \circ A) \circ B \rrbracket = \min(\llbracket B \rrbracket \cap \llbracket (T \circ A)_K \rrbracket, \leq_{T \circ A}) = \min(\llbracket B \rrbracket \cap \llbracket T_K \rrbracket, \leq_{T \circ A}) \not\subseteq \llbracket \neg A \rrbracket$ . If  $\llbracket B \rrbracket \cap \llbracket T_K \rrbracket = \emptyset$ , then condition (BN) gives us  $\llbracket T \circ B \rrbracket = \llbracket T \rrbracket \not\subseteq \llbracket \neg A \rrbracket$ . Then by postulate (R2),  $\llbracket T \circ A \rrbracket = \llbracket T \rrbracket \cap \llbracket A \rrbracket \neq \emptyset$ . By condition (KN),  $\llbracket B \rrbracket \cap \llbracket (T \circ A)_K \rrbracket = \emptyset$ , thus by condition (BN),  $\llbracket (T \circ A) \circ B \rrbracket = \llbracket T \circ A \rrbracket \not\subseteq \llbracket \neg A \rrbracket$ .  $\square$

## 7.5 Modelling knowledge expansion

Expansion of knowledge means deleting possible models. The possible models that do not model the new formula become impossible. In addition to this, knowledge expansion carries out belief revision, which means that the ordering of disbelief among the possible models does not change. These properties are formally characterized by conditions (KE), (BR), and (O1).

**Example 7.3.** Let us consider our knowledge-expanding operator  $\oplus$  on epistemic functions. Assume that  $T_n$  is an epistemic function, let  $A$  denote a propositional formula with  $T_K \not\models \neg A$ . Then by definition,  $m = \min(\{i \mid 0 \leq i < n, T(i) \not\models \neg A\})$  exists and  $T \oplus A$  is an epistemic function from  $0, \dots, n - m - 1$  into  $\mathcal{L}$ .

We have  $(T \oplus A)_K \equiv \bigvee_{i=m}^{n-1} (T(i) \wedge A) \equiv \left( \bigvee_{i=0}^{n-1} T(i) \right) \wedge A \equiv T_K \cup \{A\}$ . It is easy to see that for all  $w, w' \in \llbracket (T \oplus A)_K \rrbracket$ , relation  $w \leq_{T \oplus A} w'$  holds if and only if  $w \leq_T w'$ . Conditions (KE) and (O1) hold.

Our knowledge-expanding operators are based on the intuition given by the following theorem.

**Theorem 7.7.** *A knowledge-expansion operator  $\oplus$  satisfies postulates (K1), (K2), (R2), (R3), and (RR1), if and only if there is a function that maps each epistemic state to an ordering of disbelief such that conditions (KE), (BR), and (O1) hold.*

*Proof.* ( $\Rightarrow$ ) Given an epistemic state  $T$  that satisfies constraints (S1) and (S2) and a propositional formula  $A$  such that  $T_K \not\models \neg A$ , let us assume we have a knowledge-expansion operator  $\oplus$  that satisfies postulates (K1), (K2), (R2), (R3), and (RR1).

Condition (KE) is satisfied by postulate (K2). Postulate (K2) implies (KX), and because the state  $T \oplus A$  satisfies postulate (K1), postulate (K2)

also implies (R1). Postulate (RR2) holds trivially. Then by theorems 5.1 and 5.2, postulates (R4)–(R6) are also satisfied.

Let  $\leq_T$  and  $\leq_{T \oplus A}$  denote relations  $\leq_{\mathcal{S}(T, \circ)}$  and  $\leq_{\mathcal{S}(T \oplus A, \circ)}$  based on corresponding systems of spheres on  $\llbracket T_K \rrbracket$  and  $\llbracket (T \oplus A)_K \rrbracket$ . By Theorem 7.1,  $\leq_T$  and  $\leq_{T \oplus A}$  are orderings of disbelief and condition (BR) is satisfied. By Theorem 7.2, condition (O1) holds.

( $\Leftarrow$ ) Assume there is a function that maps each epistemic state to an ordering of disbelief such that conditions (KE), (BR), and (O1) hold. Assume epistemic states  $T$  and  $T \oplus A$ . By definition, condition (KE) implies postulate (K2). To prove postulate (R2), assume  $T_B \not\models \neg A$ . By condition (D1),  $\min(\llbracket (T \oplus A)_K \rrbracket, \leq_{T \oplus A})$  exists, and by conditions (D2), (KE), and (O1),  $\llbracket T \oplus A \rrbracket = \min(\llbracket (T \oplus A)_K \rrbracket, \leq_{T \oplus A}) = \min(\llbracket T_K \rrbracket \cap \llbracket A \rrbracket, \leq_T) = \min(\llbracket T_K \rrbracket, \leq_T) \cap \llbracket A \rrbracket = \llbracket T \rrbracket \cap A$ . Thus postulate (R2) holds.

To prove postulate (RR1), let  $B$  denote a formula such that  $B \models A$  and  $T_K \not\models \neg B$ . Then also  $T_K \not\models \neg A$ . By condition (KE),  $\llbracket ((T \oplus A) \oplus B)_K \rrbracket = \llbracket T_K \rrbracket \cap \llbracket A \rrbracket \cap \llbracket B \rrbracket = \llbracket T_K \rrbracket \cap \llbracket B \rrbracket = \llbracket (T \oplus B)_K \rrbracket$ . By condition (D1),  $\min(\llbracket ((T \oplus A) \oplus B)_K \rrbracket, \leq_{(T \oplus A) \oplus B}) \neq \emptyset$ , and thus by conditions (D2) and (O1),  $\llbracket (T \oplus A) \oplus B \rrbracket = \min(\llbracket ((T \oplus A) \oplus B)_K \rrbracket, \leq_{(T \oplus A) \oplus B}) = \min(\llbracket T_K \rrbracket \cap \llbracket B \rrbracket, \leq_T) = \llbracket T \oplus B \rrbracket$ . Thus postulate (RR1) holds.  $\square$

## 7.6 Elementary epistemic states

Because of the principles we have adopted in our assumptions about the knowledge base, we actually do not have arbitrary sets of possible models nor arbitrary orderings of disbelief.

Let us, however, consider arbitrary sets of possible models. Given a set  $X$  of arbitrary logically possible models, let us define

$$th(X) =_{def} \{A \in \mathcal{L} \mid X \subseteq \llbracket A \rrbracket\}. \quad (7.6)$$

Then we have  $th(\llbracket T \rrbracket) = T$  for any logically closed set  $T$  of formulas. However,  $\llbracket th(X) \rrbracket \subseteq X$  does not hold as a rule.

**Example 7.4.** [PST96] Assume that the set of atomic formulas in  $\mathcal{L}$  is infinite, and assume some logically possible model  $w \in W = \llbracket \top \rrbracket$ . Let  $X = W \setminus \{w\}$ . Then  $\llbracket th(X) \rrbracket = W \neq X$ .

Now let us take into account our assumption that in the beginning nothing except tautologies is known nor believed. Let  $T_0$  denote the initial state. We thus assume that as a collection of possible models in state  $T_0$  we have  $W$  and as the ordering of disbelief in the state we have  $W \times W$ . Thus in the case  $T = T_0$ , the set  $T_K$  is finitely axiomatizable.



**Theorem 7.8.** If an epistemic state results from the initial state  $T_0$  through a finite chain of change such that at each step, condition (KN) or condition (KE) holds, then the knowledge set of the state is finitely axiomatizable.

*Proof.* By induction. □

Let us introduce a concept of elementary ordering of disbelief to characterize those epistemic states that result from a finite chain of change such that it starts at the initial state  $T_0$  and at each step, conditions (KX), (O1), and (O2) hold.

**Definition 7.4.** An ordering of disbelief  $\leq_T$  is *elementary*, if it satisfies the following conditions:

- (D3): each equivalence class imposed by  $\leq_T$  is finitely axiomatizable,
- (D4): the number of the equivalence classes imposed by  $\leq_T$  is finite.

We call an epistemic state *elementary*, if it has an elementary ordering of disbelief.

**Theorem 7.9.** *If an epistemic state results from the initial state  $T_0$  through a finite chain of change such that at each step, conditions (O1), (O2), and (KN) or (KE) hold, then the state is elementary.*

*Proof.* By Theorem 7.8, the knowledge set of the initial state  $T_0$  is finitely axiomatized. In the initial state, the number of equivalence classes imposed by  $\leq_{T_0}$  is 1, and the equivalence class is finitely axiomatized.

Now assume an elementary epistemic state  $T$ . Let us consider epistemic state  $T \circ A$  such that conditions (O1), (O2), and (KN) or (KE) hold.

By Theorem 7.8, the knowledge set  $(T \circ A)_K$  is finitely axiomatized.

Let  $T_1, T_2, \dots, T_n$  denote the formulas representing the equivalence classes imposed by the ordering of disbelief  $\leq_T$ . Then for all  $i, 1 \leq i \leq n$ , all the models of  $T_i \wedge A$  are in the same equivalence class imposed by  $\leq_{T \circ A}$ , as well as the all models of  $T_i \wedge \neg A$  are in the same class. Thus each equivalence class in the state  $T \circ A$  is axiomatizable, and the number of equivalence classes can at most double. □

Thus to prove that our epistemic state is elementary, it is enough to prove that all the change operators used in the series of change satisfy the conditions in question.

Because of our assumptions, even though the language may be infinite, the set of formulas implied by the epistemic state is infinite, and the set of logically possible models may be infinite, in belief revision the size of the well-ordered partition representing the ordering of disbelief is finite.

## 7.7 Discussion

### New motivation for the postulates for belief revision

We wish to give new motivation for the postulates for belief revision. According to Niiniluoto [Nii99], the concept of truth should not be neglected in belief revision. He argues that from Plato to Hintikka, knowledge is not just well justified belief. We therefore bear in mind that one of the possible states of affairs is the true one.

In belief revision, it has been the postulates that have been given motivation for, and orderings on possible models have been used just to characterize their effects. But instead of characterizing the effect of belief revision on the ordering of disbelief with conditions (O1), (O3), and (O4), we might have used the following, equivalent condition:

$$(O1'): \text{ For all } w, w' \in \llbracket (T \circ A)_K \rrbracket, \text{ if } w \in \llbracket A \rrbracket, \text{ then} \\ w <_T w' \text{ only if } w <_{T \circ A} w' \text{ and } w \leq_T w' \text{ only if } w \leq_{T \circ A} w'.$$

Condition (O1') carries out the following principle:

*When learning a true formula, the epistemic alternative representing the true state of affairs should never become less plausible compared to any other epistemic alternative.*

We believe that this principle could have been used as fundamental motivation when deriving postulates for belief revision. Then again, we do not necessarily know whether a formula is true or not. That is why we must treat the two cases symmetrically: condition (O1') is used whether the input is true or not.

### Modelling belief contraction and competing evidence

As well as our belief-revision operators, our belief contraction operators and our competing-evidence operator are based on keeping the ordering of disbelief unchanged within the two subsets of possible models: those modelling the input formula, and those not. The only difference between various types of belief revision is how the two sets of possible models may get shifted in the ordering compared to each other. In belief revision, the models of the input formula get less disbelieved up to the point that all the least disbelieved models are models of the input formula. In belief contraction, the models of the input formula get more disbelieved up to the point that not all of the least disbelieved models are models of the input formula.

Let  $\bullet$  denote a belief-contraction operator. When characterizing the effect of belief contraction, we could use the following condition (BC):

$$(BC): \quad \llbracket T \bullet A \rrbracket = \llbracket T \rrbracket \cup \min(\llbracket T_K \rrbracket \setminus \llbracket A \rrbracket, \leq_T).$$

The intuition behind our competing-evidence operator is the following: if possible, the epistemic alternatives modelling the input are shifted up one step, otherwise the epistemic alternatives not modelling the input formula are shifted down one step. Here shifting up means becoming less disbelieved, and shifting down means becoming more disbelieved.

Let us exemplify the differences by using our epistemic functions. Figure 7.1 shows various cases before receiving input  $A$ . In each case, within the outermost circle are the possible models of the epistemic state, within the innermost circle are the least disbelieved possible models of the epistemic state. In the first case,  $A$  is believed in the state, in the third case  $A$  is disbelieved in the state, and in the second case  $A$  is neither believed nor disbelieved.

Figures 7.2, 7.4, and 7.3 illustrate shifting the orderings of disbelief in various types of belief change:

- in belief revision, possible models in  $\llbracket A \rrbracket$  are shifted upwards, if possible (case 3), and models in  $\llbracket T_K \rrbracket \setminus \llbracket A \rrbracket$  are shifted downwards, when necessary (cases 2 and 3),
- in competing evidence, possible models in  $\llbracket A \rrbracket$  are shifted upwards, if possible (case 3), or else models in  $\llbracket T_K \rrbracket \setminus \llbracket A \rrbracket$  are shifted downwards (cases 1 and 2),
- in contraction, possible models in  $\llbracket T_K \rrbracket \setminus \llbracket A \rrbracket$  are shifted upwards, if possible (case 1).

### Avoiding triviality of logic

Even when the belief set and the knowledge set of an epistemic state are finitely axiomatizable, the epistemic state cannot be represented by the pair of them, because in that case, by Theorem 4.1, the logic would be trivial.

By Theorem 7.6, belief revision changes the ordering of disbelief of a state, and the ordering of disbelief and the input formula completely determine the beliefs in the revised state.

Ordering  $\leq_T$  on possible models corresponds to epistemic entrenchment on the formulas of the language. An epistemic entrenchment  $\leq_e$  can be constructed from an ordering  $\leq_T$  on a set of possible models  $\llbracket T_K \rrbracket$  by

defining  $A \leq_e B$  if and only if  $\llbracket T_K \rrbracket \subseteq \llbracket B \rrbracket$  or a model  $w \in \llbracket T_K \rrbracket \setminus \llbracket A \rrbracket$  exists such that  $w \leq_T w'$  for all  $w' \in \llbracket T_K \rrbracket \setminus \llbracket B \rrbracket$ .

Should we assume that an ordering of disbelief, or an epistemic entrenchment, is equivalent to a revision operator, we would have to consider that we have a different operator at each revision. But even in that interpretation, by Theorem 7.6, any revision affects the choice of the operator in the next revision; the epistemic state therefore needs to contain the information for the choice. As Spohn [Spo88] has argued, the ordering of disbelief should be part of the epistemic state, because it is acted upon in revision.

Having an ordering of disbelief in epistemic states is sufficient for iterated belief revision to satisfy the rationality criteria. Spohn [Spo88] introduced ordinal conditional functions to facilitate revocability of revisions, but revocability of revisions is not demanded by the postulates; some [Wil93] have even considered such a property undesirable.

### Elementary epistemic states

An error in Grove's proof of sphere semantics satisfying the AGM-postulates [Gro88] has been reported [PST96]. The error involves assuming in the proof of Theorem 1 (page 161) that  $\llbracket th(\llbracket A \rrbracket \cap z(T, \circ, A)) \rrbracket = \llbracket (T \circ A)_B \rrbracket$  based on  $th(\llbracket A \rrbracket \cap z(T, \circ, A)) = (T \circ A)_B$ . However, Grove later (page 163) says that actually the system he constructed satisfies the former statement. So in his Theorem 1, the former statement could have been used in place of the latter, and no false deduction would have been needed. In our theorems, we have used condition (BR) thus avoiding the error.

The authors of the report [PST96] suggest as one solution to the error the assumption that the spheres are elementary, that is, finitely axiomatisable. We have shown, how conditions (KX), (O1), and (O2) together with our assumption concerning the initial state in fact imply elementarity.

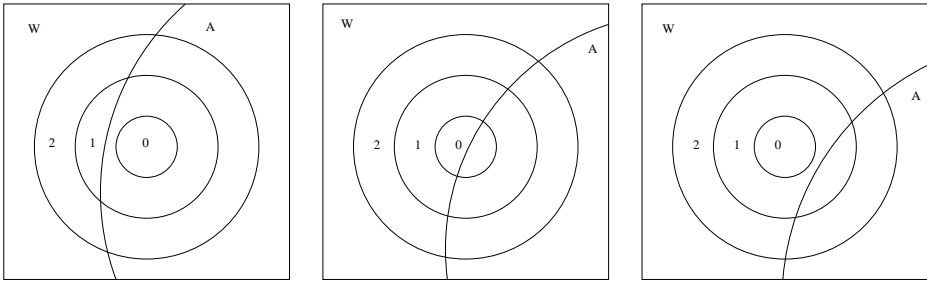


Figure 7.1: Various cases before input A.

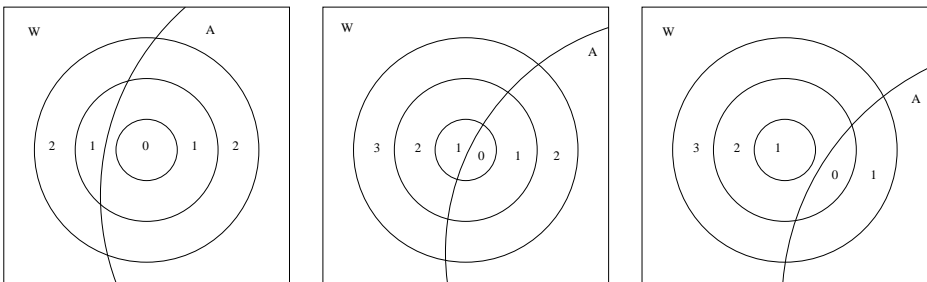


Figure 7.2: Various cases when revising by A.

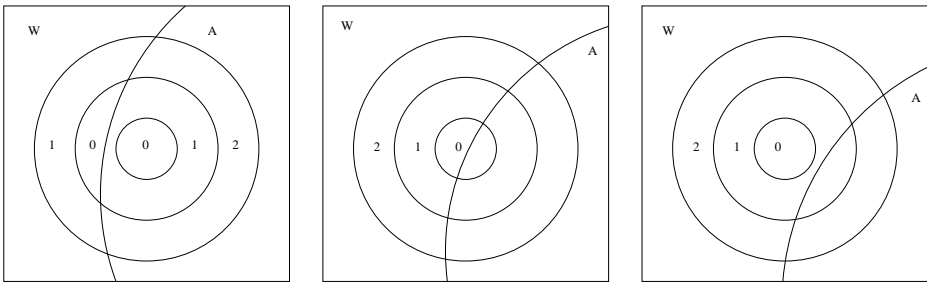


Figure 7.3: Various cases when contracting A.

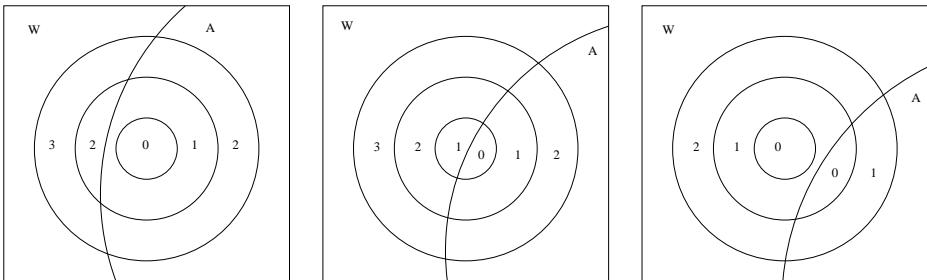


Figure 7.4: Various cases when receiving competing evidence A.



# Chapter 8

## Modelling belief update

Proving a refined representation theorem for belief update is not straightforward. Some of the postulates for belief update involve complete theories, but if the language of the input has an infinite number of atomic formulas, it is not possible to address single valuations within the language.

We will, however, adhere to belief update for elementary epistemic states. That is why we will give one extra postulate for belief update.

### 8.1 Finite proposition fields

In this chapter we assume that the knowledge base has finitely axiomatizable knowledge and belief sets. Then the number of atomic formulas in each of those formulas is finite. We shall change the representation theorem so that it deals with atoms of finite proposition fields instead of single valuations or models. Recall that a field of propositions is a non-empty set of subsets of all logically possible truth distributions  $W$  and it is closed under complementation and arbitrary union and intersection [Spo88]. An atom is a minimal nonempty element of the field.

For a given finite set of atomic formulas  $P$  we define

$$\mathcal{F}(P) =_{def} \bigcap \{F \subseteq \mathcal{P}(W) \mid F \text{ is a field of propositions such that } \llbracket p \rrbracket \in F \text{ for all } p \in P\}.$$

The intersection  $\mathcal{F}(P)$  is itself a proposition field and it is the minimal field of propositions satisfying the conditions. Each element in  $\mathcal{F}(P)$  can be addressed by the language by using formulas containing only those atomic formulas that appear in the set  $P$ .

Given a propositional formula  $A$  and a field of propositions  $\mathcal{F}(P)$  such that  $Voc(A) \subseteq P$ , we define

$$M_P(A) =_{def} \{X \in \mathcal{F}(P) \mid X \text{ is an atom in } \mathcal{F}(P) \text{ and } X \subseteq \llbracket A \rrbracket\}.$$

If the proposition field is fixed in the context, we write just  $M(A)$  as a short form of  $M_P(A)$ .

## 8.2 Adjusting the postulates for belief update

In order to revise the representation theorem for belief update in the case of an infinite language, we need to introduce an extra postulate (U9):

$$(U9): \quad \text{If } (T \diamond (A \vee B))_B \not\models \neg A \text{ and } A \text{ is complete,} \\ \text{then } A \models (T \diamond (A \vee B))_B.$$

Because the language may be infinite, we have to apply contextual completeness instead of absolute completeness. A formula is complete in a given proposition field, if and only if its model set is an atom in the field, that is, a formula  $A$  is complete in a field, if and only if the set  $M(A)$  is a singleton.

## 8.3 A representation theorem for belief update

Given a knowledge base  $T$  with finitely axiomatizable knowledge and belief sets let  $D$  denote a formula equivalent to the belief set  $T_B$  and let  $K$  denote a formula equivalent to the knowledge set  $T_K$ . The representation theorem for update  $T \diamond A$  involves orderings upon atoms of proposition fields  $\mathcal{F}(P)$  such that all the atomic formulas in  $D$ ,  $K$ , and  $A$  are in  $P$ , that is,  $Voc(D) \cup Voc(K) \cup Voc(A) \subseteq P$ .

The orderings need to satisfy the faithfulness conditions (page 36). Given the set  $M(K)$ , we say that a function that maps each element  $X \in M(K)$  to a partial preorder  $\leq_X$  on  $M(K)$  is *faithful*, if and only if for each  $\leq_X$ , atom  $X$  is the minimum in the ordering, that is, for all  $Y \in M(K)$  the following conditions hold:

$$(F1): \quad X \leq_X Y, \\ (F2): \quad \text{If } Y \leq_X X, \text{ then } X = Y.$$

In the representation theorem, conditions (KN) and (BN) are applied to belief update also:



- (KN):  $\llbracket (T \diamond A)_K \rrbracket = \llbracket T_K \rrbracket$ ,  
 (BN): If  $\llbracket A \rrbracket \cap \llbracket T_K \rrbracket = \emptyset$ , then  $\llbracket T \diamond A \rrbracket = \llbracket T \rrbracket$ .

The representation theorem for belief update is refined as follows.

**Theorem 8.1.** *Let  $\diamond$  denote a belief-update operator that, given a propositional formula  $A$  and a knowledge base  $T$  that satisfies the static rationality criteria and has finitely axiomatizable knowledge and belief sets, maps them to the knowledge base  $T \diamond A$ . Then the following conditions are equivalent:*

1. *The operator  $\diamond$  satisfies postulates (K0), (K1) and (U0)–(U9).*
2. *Conditions (KN) and (BN) hold, and for each formula  $K$  equivalent to the knowledge set of  $T$ , for each formula  $D$  equivalent to the belief set of  $T$ , and for each finite set of atomic formulas  $P$  such that it contains all the atomic formulas in  $K$ ,  $D$ , and  $A$ , a faithful function exists that maps each atom  $X$  in  $M_P(K)$  to a partial preorder  $\leq_X$  on  $M_P(K)$  such that the following condition holds:*

$$(BU): \text{ If } T_K \not\models \neg A, \text{ then } \llbracket T \diamond A \rrbracket = \bigcup \bigcup_{X \in M_P(D)} \min(M_P(A \wedge K), \leq_X).$$

3. *Conditions (KN) and (BN) hold, and for each formula  $K$  equivalent to the knowledge set of  $T$  for each formula  $D$  equivalent to the belief set of  $T$ , and for each finite set of atomic formulas  $P$  such that it contains all the atomic formulas in  $K$ ,  $D$ , and  $A$ , a faithful function exists that maps each  $X \in M_P(K)$  to a partial order  $\leq_X$  on  $M_P(K)$  such that condition (BU) holds.*

*Proof.* Assume a belief-update operator  $\diamond$ , a propositional formula  $A$ , and a knowledge base  $T$  that satisfies constraints (S1) and (S2) and has finitely axiomatizable sets  $T_K$  and  $T_B$ .

Because condition (3) trivially implies condition (2), it is sufficient to prove that condition (1) implies condition (3) and condition (2) implies condition (1).

[1  $\Rightarrow$  3]

Assume that the operator  $\diamond$  satisfies postulates (K0), (K1), and (U0)–(U9).

By postulate (K0), condition (KN) holds. If  $T_K \models \neg A$ , then by (U0), condition (BN) holds.

To prove that (BU) holds, assume  $T_K \not\models \neg A$ . Because the sets  $T_K$  and  $T_B$  are finitely axiomatizable, formulas  $K$  and  $D$  equivalent to  $T_K$  and  $T_B$  exist. Then assume a proposition field  $\mathcal{F}(P)$  with some arbitrary finite  $P$  such that  $Voc(K) \cup Voc(D) \cup Voc(A) \subseteq P$ . Because  $T$  satisfies (S1) and (S2),  $\emptyset \neq M(D) \subseteq M(K)$ .

For each  $X \in M(K)$ , relation  $\leq_X \subseteq M(K) \times M(K)$  is constructed as follows. Let  $T_X$  denote a knowledge base such that  $(T_X)_K \equiv T_K$  and  $\llbracket T_X \rrbracket = X$ . Thus constraints (S1) and (S2) hold for  $T_X$ , and  $(T_X)_B$  and  $(T_X)_K$  are finitely axiomatizable. Then we can define for all atoms  $Y, Z \in M(K)$  and formulas  $x, y$  and  $z$  such that  $\llbracket x \rrbracket = X$ ,  $\llbracket y \rrbracket = Y$  and  $\llbracket z \rrbracket = Z$ :

$$Y \leq_X Z, \text{ if and only if } \llbracket T_X \diamond (y \vee z) \rrbracket \subseteq Y.$$

In order to prove that the relations are partial preorders, we have to prove that they are reflexive and transitive. First we prove that  $\leq_X$  is reflexive. Postulates (U4) and (U1) say that  $\llbracket T_X \diamond (y \vee y) \rrbracket = \llbracket T_X \diamond y \rrbracket \subseteq Y$ . Thus  $Y \leq_X Y$ . In order to prove that the relation is transitive, assume atoms  $Y, Z$ , and  $V$  in  $M(K)$  such that  $Y \leq_X Z$  and  $Z \leq_X V$ . Let  $S$  denote the theory  $(T_X \diamond (y \vee z \vee v))_B$ . Postulates (U5) and (U4) say that  $\llbracket S \rrbracket \cap (Y \cup Z) \subseteq Y$  and  $\llbracket S \rrbracket \cap (Z \cup V) \subseteq Z$ . Thus  $\llbracket S \rrbracket \cap Z = \emptyset$  and  $\llbracket S \rrbracket \cap V = \emptyset$ , and  $(T_X \diamond (y \vee z \vee v))_B \models y \vee v$ . Because by (U1),  $(T_X \diamond (y \vee v))_B \models y \vee z \vee v$ , (U6) gives  $(T_X \diamond (y \vee z \vee v))_B \equiv (T_X \diamond (y \vee v))_B$ . Then  $\llbracket T_X \diamond (y \vee v) \rrbracket \cap V = \emptyset$  and by (U1),  $\llbracket T_X \diamond (y \vee v) \rrbracket \subseteq Y$ . Thus  $Y \leq_X V$ . The relation is a partial preorder.

By definition and postulate (U3), if  $Y \leq_X Z$  and  $Z \leq_X Y$ , then  $Y = Z$ . Thus the relation is a partial order. Postulate (U2) says that  $\llbracket T_X \diamond (x \vee y) \rrbracket = X$ . Thus  $X \leq_X Y$  and  $X \neq Y$  implies  $Y \not\leq_X X$ . Conditions (F1) and (F2) hold.

To prove that  $\llbracket T \diamond A \rrbracket \subseteq \bigcup \bigcup_{X \in M(D)} \min(M(A \wedge K), \leq_X)$ , assume  $w \in \llbracket T \diamond A \rrbracket$ . Because  $\llbracket T_B \rrbracket = \bigcup M(D)$ , by (U8) an atom  $X \in M(D)$  exists such that  $w \in \llbracket T_x \diamond A \rrbracket$ . By (K0), (K1), and (U1),  $w \in Y$  for some  $Y \in M(A \wedge K)$ . Assume for the converse that  $Y \notin \min(M(A \wedge K), \leq_X)$ . Thus an atom  $Z$ ,  $Z \neq Y$ , in  $M(A \wedge K)$  exists such that  $\llbracket T_x \diamond (y \vee z) \rrbracket \subseteq Z$ . Because by (U5)  $\llbracket T_x \diamond A \rrbracket \cap \llbracket y \vee z \rrbracket \subseteq \llbracket T_x \diamond (y \vee z) \rrbracket$ , we have a contradiction with  $w \in \llbracket T_x \diamond A \rrbracket \cap Y$ . Thus  $w \in Y \in \bigcup \bigcup_{X \in M(D)} \min(M(A \wedge K), \leq_X)$ .

To prove that  $\llbracket T \diamond A \rrbracket \supseteq \bigcup \bigcup_{X \in M(D)} \min(M(A \wedge K), \leq_X)$ , assume  $w \in \bigcup \bigcup_{X \in M(D)} \min(M(A \wedge K), \leq_X)$ . Then  $w \in Y$  for one atom  $Y \in \min(M(A \wedge K), \leq_X)$  for some atom  $X \in M(D)$ . Thus by (U1) and (U3),  $\llbracket T_x \diamond (y \vee z) \rrbracket \cap Y \neq \emptyset$  for all atoms  $Z \in M(A \wedge K)$ . Because  $Y$  is an atom in  $F$ , it is complete, and by (U9)  $Y \subseteq \llbracket T_x \diamond (y \vee z) \rrbracket$  for all atoms  $Z \in M(A \wedge K)$ . Because  $T_X$  is complete, postulate (U7) then gives us  $Y \subseteq \llbracket T_x \diamond A \rrbracket$  and (U8) that  $Y \subseteq \llbracket T \diamond A \rrbracket$ . Condition (BU) holds.

[2  $\Rightarrow$  1]

Assume that conditions (KN) and (BN) hold and for each formula  $K$  equivalent to  $T_K$ ,  $D$  equivalent to  $T_B$ , and for each finite set of atomic formulas  $P$  such that  $Voc(K) \cup Voc(D) \cup Voc(A) \subseteq P$ , a faithful function exists that maps each atom  $X$  in  $M_P(K)$  to a partial preorder  $\leq_X$  on  $M_P(K)$  such that condition (BU) holds.

By condition (KN), postulate (K0) holds.

Assume for a while the case that  $T_K \models \neg A$ . Then by condition (BN) postulate (U0) holds. Conditions (BN), (S1), and (KN) give us  $\llbracket T \diamond A \rrbracket = \llbracket T \rrbracket \subseteq \llbracket T_K \rrbracket = \llbracket (T \diamond A)_K \rrbracket$ . Postulate (K1) holds. By (BN)  $\llbracket T \diamond A \rrbracket = \llbracket T \rrbracket \neq \emptyset$ , because  $T$  satisfies (S2). Postulate (U3) holds.

Because the sets  $T_K$  and  $T_B$  are finitely axiomatizable, formulas  $K$  and  $D$  equivalent to  $T_K$  and  $T_B$  exist. To prove that postulates (K1), (U1), (U2), and (U3) are satisfied in case  $T_K \not\models \neg A$ , let us consider proposition field  $\mathcal{F}(Voc(K) \cup Voc(D) \cup Voc(A))$ .

Assume  $T_K \not\models \neg A$ . Then conditions (BU) and (KN) give us  $\llbracket T \diamond A \rrbracket = \bigcup \bigcup_{X \in M(D)} \min(M(A \wedge K), \leq_X) \subseteq \llbracket T_K \rrbracket = \llbracket (T \diamond A)_K \rrbracket$ . Thus (K1) holds. Condition (BU) gives us  $\llbracket T \diamond A \rrbracket = \bigcup \bigcup_{X \in M(D)} \min(M(A \wedge K), \leq_X) \subseteq \llbracket A \rrbracket$ . Thus  $(T \diamond A)_B \models A$  and postulate (U1) holds.

If  $\llbracket T \rrbracket \subseteq \llbracket A \rrbracket$ , then by (S1) and (S2)  $\llbracket A \rrbracket \cap \llbracket T_K \rrbracket \neq \emptyset$ , and conditions (BU), (F1), and (F2) give us  $\llbracket T \diamond A \rrbracket = \bigcup \bigcup_{X \in M(D)} \min(M(A \wedge K), \leq_X) = \bigcup \bigcup_{X \in M(D)} X = \llbracket T \rrbracket$ . Thus postulate (U2) holds.

If  $\llbracket A \rrbracket \cap \llbracket T_K \rrbracket \neq \emptyset$ , then by (BU),  $\llbracket T \diamond A \rrbracket = \bigcup \bigcup_{X \in M(D)} \min(M(A \wedge K), \leq_X)$  and  $M(A \wedge K) \neq \emptyset$ . Because  $F$  is finite,  $\min(M(A \wedge K), \leq_X) \neq \emptyset$  and  $(T \diamond A)_B \not\models \perp$ . Postulate (U3) holds.

To prove postulates (U4), (U5), (U6), (U7), and (U9), assume a propositional formula  $B$  and proposition field  $\mathcal{F}(Voc(K) \cup Voc(D) \cup Voc(A) \cup Voc(B))$ .

To prove (U4), assume a knowledge base  $T'$  such that  $\llbracket T \rrbracket = \llbracket T' \rrbracket$ ,  $\llbracket T_K \rrbracket = \llbracket T'_K \rrbracket$  and assume  $\llbracket A \rrbracket = \llbracket B \rrbracket$ . If  $\llbracket A \rrbracket \cap \llbracket T_K \rrbracket = \emptyset$ , then  $\llbracket B \rrbracket \cap \llbracket T'_K \rrbracket = \emptyset$ , and by (BN)  $\llbracket T \diamond A \rrbracket = \llbracket T \rrbracket = \llbracket T' \rrbracket = \llbracket T' \diamond B \rrbracket$ . If  $\llbracket A \rrbracket \cap \llbracket T_K \rrbracket \neq \emptyset$ , then by (BU)  $\llbracket T \diamond A \rrbracket = \bigcup \bigcup_{X \in M(D)} \min(M(A \wedge K), \leq_X) = \bigcup \bigcup_{X \in M(D)} \min(M(B \wedge K), \leq_X) = \llbracket T' \diamond B \rrbracket$ . Postulate (U4) holds.

If  $\llbracket T \diamond A \rrbracket \cap \llbracket B \rrbracket = \emptyset$ , then (U5) trivially holds. Assume  $\llbracket T \diamond A \rrbracket \cap \llbracket B \rrbracket \neq \emptyset$ . If  $\llbracket A \rrbracket \cap \llbracket T_K \rrbracket = \emptyset$ , then  $\llbracket A \wedge B \rrbracket \cap \llbracket T_K \rrbracket = \emptyset$  and by (BN)  $\llbracket T \diamond A \rrbracket \cap \llbracket B \rrbracket = \llbracket T \rrbracket \cap \llbracket B \rrbracket \subseteq \llbracket T \rrbracket = \llbracket T \diamond (A \wedge B) \rrbracket$ . If  $\llbracket A \rrbracket \cap \llbracket T_K \rrbracket \neq \emptyset$ , then by (BU)  $\llbracket T \diamond A \rrbracket \cap \llbracket B \rrbracket = \bigcup \bigcup_{X \in M(D)} \min(M(A \wedge K), \leq_X) \cap \llbracket B \rrbracket \subseteq \bigcup \bigcup_{X \in M(D)} \min(M(A \wedge B \wedge K), \leq_X) = \llbracket T \diamond (A \wedge B) \rrbracket$ . Thus (U5) holds.

To prove (U6), assume  $(T \diamond A)_B \models B$  and  $(T \diamond B)_B \models A$ . Then by (K0), (K1), and (U3)  $\llbracket A \rrbracket \cap \llbracket T_K \rrbracket \neq \emptyset$  and  $\llbracket B \rrbracket \cap \llbracket T_K \rrbracket \neq \emptyset$ , thus by (BU)  $\llbracket T \diamond A \rrbracket = \bigcup \bigcup_{X \in M(D)} \min(M(A \wedge K), \leq_X)$  and  $\llbracket T \diamond B \rrbracket = \bigcup \bigcup_{X \in M(D)} \min(M(B \wedge K), \leq_X)$ .

Let  $Y \in \min(M(A \wedge K), \leq_X)$  for some  $X \in M(D)$ . Assume for the converse that  $Y \notin \min(M(B \wedge K), \leq_X)$ . Then  $Z \in \min(M(B \wedge K), \leq_X)$  exists such that  $Z \leq Y$  but  $Y \not\leq Z$ . Because  $(T \diamond B)_B \models A$ ,  $Z \in M(A \wedge K)$ , a contradiction with  $Y \in \min(M(A \wedge K), \leq_X)$ . Thus  $\llbracket T \diamond A \rrbracket \subseteq \llbracket T \diamond B \rrbracket$ , and analogously,  $\llbracket T \diamond B \rrbracket \subseteq \llbracket T \diamond A \rrbracket$ .

To prove (U7), assume that  $D$  is complete, that is,  $M(D)$  is a singleton.

If  $\llbracket A \rrbracket \cap \llbracket T_K \rrbracket = \emptyset$  and  $\llbracket B \rrbracket \cap \llbracket T_K \rrbracket = \emptyset$ , then  $\llbracket T \diamond A \rrbracket = \llbracket T \diamond B \rrbracket = \llbracket T \diamond (A \vee B) \rrbracket = \llbracket T \rrbracket$  and the case is trivial.

Assume  $\llbracket A \rrbracket \cap \llbracket T_K \rrbracket = \emptyset$  and  $\llbracket B \rrbracket \cap \llbracket T_K \rrbracket \neq \emptyset$ . Then by (BN) and (BU)  $\llbracket T \diamond A \rrbracket \cap \llbracket T \diamond B \rrbracket = \llbracket T \rrbracket \cap \bigcup \bigcup_{X \in M(D)} \min(M(B \wedge K), \leq_X) \subseteq \bigcup \bigcup_{X \in M(D)} \min(M(B \wedge K), \leq_X) = \bigcup \bigcup_{X \in M(D)} \min(M((A \vee B) \wedge K), \leq_X) = \llbracket T \diamond (A \vee B) \rrbracket$ .

If  $\llbracket A \rrbracket \cap \llbracket T_K \rrbracket \neq \emptyset$  and  $\llbracket B \rrbracket \cap \llbracket T_K \rrbracket \neq \emptyset$ , then by (BU)  $\llbracket T \diamond A \rrbracket \cap \llbracket T \diamond B \rrbracket = \bigcup \bigcup_{X \in M(D)} \min(M(A \wedge K), \leq_X) \cap \bigcup \bigcup_{X \in M(D)} \min(M(B \wedge K), \leq_X)$ . If the intersection is empty, then the claim trivially holds. Otherwise, assume  $Y \in \bigcup \bigcup_{X \in M(D)} \min(M(A \wedge K), \leq_X) \cap \bigcup \bigcup_{X \in M(D)} \min(M(B \wedge K), \leq_X)$ . Then because  $D$  is complete,  $Y \in \bigcup \bigcup_{X \in M(D)} \min(M(A \wedge K) \cup M(B \wedge K), \leq_X) = \llbracket T \diamond (A \vee B) \rrbracket$ .

To prove (U9), assume that  $A$  is complete and  $(T \diamond (A \vee B))_B \not\models \neg A$ . Then by (K0) and (K1)  $T_K \not\models \neg A$  and by (BU)  $\llbracket T \diamond (A \vee B) \rrbracket = \bigcup \bigcup_{X \in M(D)} \min(M((A \vee B) \wedge K), \leq_X)$ . Because of the assumptions,  $\llbracket A \rrbracket$  is an atom in the field  $F'$ . Then  $\bigcup \bigcup_{X \in M(D)} \min(M(A \wedge K) \cup M(B \wedge K), \leq_X) \cap \llbracket A \rrbracket \neq \emptyset$  implies  $\llbracket A \rrbracket \in \bigcup \bigcup_{X \in M(D)} \min(M(A \wedge K) \cup M(B \wedge K), \leq_X)$ .

To prove (U8), assume knowledge bases  $T'$  and  $T''$  such that  $(T'_B$  and  $T''_B)$  are finitely axiomatizable,  $T_K \equiv T'_K \equiv T''_K$ , and  $T_B \equiv (T'_B \vee T''_B)$ . Let  $D'$  and  $D''$  denote formulas equivalent to  $T'_B$  and  $T''_B$  correspondingly. Let  $P = \text{Voc}(K) \cup \text{Voc}(D) \cup \text{Voc}(D') \cup \text{Voc}(D'') \cup \text{Voc}(A)$  and let  $\mathcal{F}(P)$  denote the proposition field under consideration.

If  $\llbracket A \rrbracket \cap \llbracket T_K \rrbracket = \emptyset$ , then by (BN)  $\llbracket T \diamond A \rrbracket = \llbracket T \rrbracket = \llbracket T' \rrbracket \cup \llbracket T'' \rrbracket = \llbracket T' \diamond A \rrbracket \cup \llbracket T'' \diamond A \rrbracket$ .

If  $\llbracket A \rrbracket \cap \llbracket T_K \rrbracket \neq \emptyset$ , then by (BU)  $\llbracket T \diamond A \rrbracket = \bigcup \bigcup_{X \in M(D)} \min(M(A \wedge K), \leq_X) = \bigcup \bigcup_{X \in M(D')} \min(M(A \wedge K), \leq_X) \cup \bigcup \bigcup_{X \in M(D'')} \min(M(A \wedge K), \leq_X) = \llbracket T' \diamond A \rrbracket \cup \llbracket T'' \diamond A \rrbracket$ .  $\square$

The representation theorem describes how the set of the most plausible models is changed in belief update, but it gives no hint for changing the ordering of disbelief otherwise. In that sense the rationality criteria are incomplete compared to those of belief revision. We cannot say that due to the rationality criteria, belief update maps elementary knowledge bases to elementary knowledge bases. Instead, we can say that due to the rationality criteria, belief update maps knowledge bases that have finitely axiomatizable belief and knowledge sets and satisfy the static rationality criteria, to knowledge bases that have finitely axiomatizable belief and knowledge sets and satisfy the static rationality criteria.

## 8.4 Modifying update operators

If the ordering of disbelief is contained in the knowledge base, updates have to deal with it, as done by the update operators proposed in Chapter 6.

Those operators were defined using Winslett's update operator  $\diamond_W$  (page 21) as a component. However, Winslett's operator cannot be used, if the language is infinite. Therefore we will now propose an operator  $\diamond_i$  that can be used instead of operator  $\diamond_W$ . Operator  $\diamond_i$  is a modification of  $\diamond_W$  such that it uses atoms of proposition fields instead of possible models. In fact, given an atom  $X$  of a proposition field  $\mathcal{F}(P)$ , the intersection  $\cap X$  can be taken as an interpretation in the language whose set of atomic formulas is  $P$ .

**Definition 8.1** (Operator  $\diamond_i$ ). *Given a propositional formula  $A$  and a knowledge base  $T$  that satisfies the static rationality criteria and has finitely axiomatizable knowledge and belief sets, let  $K$  denote a formula equivalent to the knowledge set of  $T$ , let  $D$  denote a formula equivalent to the belief set of  $T$ , and let  $P$  denote a set of atomic formulas such that it contains all the atomic formulas in  $K$ ,  $D$ , and  $A$ . Then  $\llbracket (T \diamond_i A)_K \rrbracket =_{\text{def}} \llbracket T_K \rrbracket$  and if  $A$  is inconsistent with  $K$ , the input is rejected, that is,  $\llbracket T \diamond_i A \rrbracket =_{\text{def}} \llbracket T \rrbracket$ . If  $A$  is consistent with  $K$ , then*

$$\llbracket T \diamond_i A \rrbracket =_{\text{def}} \bigcup \bigcup_{X \in M_P(D)} \min(M_P(A \wedge K), \leq_X^e),$$

where for all atoms  $X$ ,  $Y$ , and  $Z$  in  $M_P(K)$ ,

$$Y \leq_X^e Z, \text{ if and only if } (\cap X \triangle \cap Y) \subseteq (\cap X \triangle \cap Z).$$

This definition recognizes only the belief and knowledge sets of the knowledge base, not the epistemic entrenchment, but it can be used to build operators that do recognize the ordering.

We then need to prove that the choice of  $P$  does not affect the result of the operation.

**Lemma 8.1.** *Given a knowledge base  $T$  that satisfies the static rationality criteria and has finitely axiomatizable knowledge and belief sets, let  $K$  denote a formula equivalent to the knowledge set of  $T$ , let  $D$  denote a formula equivalent to the belief set of  $T$ . Let  $A$  denote a propositional formula that is consistent with  $K$ , and let  $P$  denote a set of atomic formulas such that it contains all the atomic formulas in  $K$ ,  $D$ , and  $A$ . Then given an atomic formula  $q$  such that  $q \notin P$  and  $P' = P \cup \{q\}$ , the following holds:*

$$\bigcup \bigcup_{X \in M_P(D)} \min(M_P(A \wedge K), \leq_X^e) = \bigcup \bigcup_{X \in M_{P'}(D)} \min(M_{P'}(A \wedge K), \leq_X^e).$$

*Proof.* In the proposition field  $\mathcal{F}(P')$  each atom of the field  $\mathcal{F}(P)$  is split into two.

If  $Y \in \bigcup_{X \in M_P(D)} \min(M_P(A \wedge K), \leq_X^e)$ , then  $Y \in \min(M_P(A \wedge K), \leq_X^e)$  for some  $X \in M_P(D)$ . Let  $X_1, X_2, Y_1$ , and  $Y_2$  denote the atoms in  $\mathcal{F}(P')$  such that  $X = X_1 \cup X_2$  and  $Y = Y_1 \cup Y_2$  with  $X_1$  and  $Y_1$  containing possible

models that model  $q$  and  $X_2$  and  $Y_2$  containing possible models that do not model  $q$ . Then  $\bigcap X \triangle \bigcap Y = \bigcap X_1 \triangle \bigcap Y_1 = \bigcap X_2 \triangle \bigcap Y_2$ .

Because  $q \notin P$ , then for all atoms in  $\mathcal{F}(P)$ , both of its halves are in  $M_{P'}(A \wedge K)$  or  $M_{P'}(D)$  or neither of the halves is. Therefore  $Y \in \min(M_{P'}(A \wedge K), \leq_X^e)$  for some  $X \in M_{P'}(D)$  if and only if  $Y_1 \in \min(M_{P'}(A \wedge K), \leq_{X_1}^e)$  and  $Y_2 \in \min(M_{P'}(A \wedge K), \leq_{X_2}^e)$ .  $\square$

**Theorem 8.2.** *Operator  $\diamond_i$  satisfies postulates (K0), (K1), and (U0)–(U9).*

*Proof.* Given a propositional formula  $A$  and a knowledge base  $T$  that satisfies conditions (S1) and (S2) and has finitely axiomatizable sets  $T_K$  and  $T_B$ , let formulas  $K_1$  and  $D_1$  as well as  $K_2$  and  $D_2$  denote formulas equivalent to  $T_K$  and  $T_B$  correspondingly, and let  $P_1$  and  $P_2$  denote sets of atomic formulas such that  $\text{Voc}(D_1) \cup \text{Voc}(K_1) \cup \text{Voc}(A) \subseteq P_1$  and  $\text{Voc}(D_2) \cup \text{Voc}(K_2) \cup \text{Voc}(A) \subseteq P_2$ .

Let us consider proposition field  $\mathcal{F}(P)$  such that  $P = P_1 \cup P_2$ . Then by induction, Lemma 8.1 gives us  $\bigcup \bigcup_{X \in M_{P_1}(D_1)} \min(M_{P_1}(A \wedge K_1), \leq_X^e) = \bigcup \bigcup_{X \in M_P(D_1)} \min(M_P(A \wedge K_1), \leq_X^e) = \bigcup \bigcup_{X \in M_P(D_2)} \min(M_P(A \wedge K_2), \leq_X^e) = \bigcup \bigcup_{X \in M_{P_2}(D_2)} \min(M_{P_2}(A \wedge K_2), \leq_X^e)$ . Thus by Theorem 8.1, operator  $\diamond_i$  satisfies postulates (K0), (K1), and (U0)–(U9).  $\square$

# Chapter 9

## Conclusion and future work

### Knowledge base as an abstract data type

In this thesis an abstract data type called knowledge base has been described. The knowledge base has been considered as an object that can carry the epistemic state of an independent agent. The agent can also use further instantiations of the knowledge base to carry its estimates of the epistemic states of other agents.

When describing the knowledge base, the emphasis has been on the change operators while the explicit assumptions on the knowledge base and its use implicitly define a set of access functions and a constructor for the knowledge base.

What is essential with the knowledge base is that it contains both knowledge and beliefs. Various change-operator types are characterized by sets of rationality criteria for each of them. The previous rationality criteria have been modified so that beliefs known to be false will not be accepted. Due to the refined rationality criteria, knowledge acts as integrity constraints actively taking part in belief change. Also a new, commutative type of change is proposed for entering competing evidence into the knowledge base.

### Representing the knowledge base using finite structures

Two finite representations of knowledge bases were introduced to demonstrate that even if the language is infinite, the knowledge base can be implemented using finite structures.

Both representations of knowledge bases contain a dynamic epistemic entrenchment. It is created and modified by the change operations, not given from outside the system. The refined representation theorems for knowledge expansion, belief revision and iterated belief revision are consistent with these results: In order to satisfy the postulates for iterated belief revision, knowledge bases must contain orderings of disbelief on possible models. These orderings correspond to epistemic entrenchments on formulas.

In this thesis, elementarity of knowledge bases was defined. If the knowledge base is elementary, it remains so in belief revision and in knowledge expansion. In that sense the rationality criteria for belief revision and knowledge expansion might be considered as complete. The representation theorem for belief update guarantees that the belief set remains finitely axiomatizable, but it says nothing about the other elementarity aspects. In that respect the rationality criteria for belief update (and erasure) can be considered as incomplete. Nevertheless, the representation theorems confirm that the knowledge base need not know the language in advance.

Representation theorems for belief contraction, belief erasure, and competing evidence as well as further elementarity proofs are left for future work.

## Further work

In this thesis the classification of input has been left to the responsibility of the agent using the knowledge base, because it is the agent that can take the necessary actions needed to solve the problems involved in the task.

Even though the revision operators proposed in this thesis reject beliefs known to be false, the agents using the knowledge base are not forced to do so. What should an agent do when it hears something it knows to be false? Accepting some information might still be reasonable, as shown by the following example.

**Example 9.1.** Imagine you hear that Sibelius violin contest winner Jaakko Kuusisto is to play in a forthcoming concert, and yet you know for sure that it was Jaakko's younger brother Pekka, not Jaakko, who won the contest. However, instead of rejecting the information you might be willing to accept that either of the two brothers is playing in the concert.

One proposal for solving the problem of unbelievable input is *accommodative belief revision* [Elo08], in which the input is first revised by the knowledge of the agent, and the resulting formula is then taken as the new



input. Accommodative belief revision has an implementation written by M. Nykänen using the Haskell programming language. The implementation uses ranking functions and extends the vocabulary dynamically as needed.

The intuition behind accommodative belief revision is to guess what the source of the information would have said, had it had the knowledge that the agent has. Of course, if communication is possible, the agent needs not guess it, instead the agent can ask for it. Another proposal [Nyk11] to solve the problem of unbelievable input is to enter into a dialog with the source of the information to find out what the source actually would have said had it known better.

## **Future work**

The scope of this thesis has been mainly confined to demonstrating the finiteness of the knowledge base. A natural question to be explored in the extension of this study is the computational complexity analysis for implementing the knowledge base.

Another interesting direction for future work is the study of independence aspects. The knowledge base could be implemented by a dynamic set of ranking functions, each of which being independent of the others. This implementation would in a way resemble that of relational databases, in which data is stored in a collection of relations, not in a universal relation.



## References

- [AbG85] Abiteboul, S., Grahne, G., Update Semantics for Incomplete Databases. Pages 1–12 in Proceedings of the Eleventh International Conference on Very Large Data Bases. Eds. A. Pirotte and Y. Vassiliou. Morgan Kaufmann Publishers, 1985.
- [AGM85] Alchourrón, C.E., Gärdenfors, P., Makinson, D., On the Logic of Theory Change: Partial Meet Contraction and Revision Functions. *The Journal of Symbolic Logic* 50 (1985), 2, 510–530.
- [BaS81] Bancilhon, F., Spyrtatos, N., Update Semantics of Relational Views. *ACM Transactions on Database Systems* 6 (1981), 4, 557–575.
- [Bor85] Borgida, A., Language Features for Flexible Handling of Exceptions in Information Systems. *ACM Transactions on Database Systems* 10 (1985), 565–603.
- [Bre91] Brewka, G., Belief Revision in a Framework for Default Reasoning. Pages 206–222 in *The Logic of Theory Change. Lecture Notes in Artificial Intelligence* 465. Eds. A. Fuhrmann and M. Morreau. Springer-Verlag 1991.
- [Dal88] Dalal, M., Investigations into a Theory of Knowledge Base Revision: Preliminary Report. Pages 475–479 in Proceedings AAAI -88 Seventh National Conference on Artificial Intelligence. Morgan Kaufmann Publishers, 1988.
- [DaP94] Darwiche, A., Pearl, J., On the Logic of Iterated Belief Revision. Pages 5–23 in *proc. of the Fifth Conference on Theoretical Aspects of Reasoning About Knowledge*. Ed. R. Fagin. Morgan Kaufmann, 1994.
- [DaP97] Darwiche, A., Pearl, J., On the Logic of Iterated Belief Revision. *Artificial Intelligence* 89 (1997), 1–29.

- [Doy79] Doyle, J., A Truth Maintenance System. *Artificial Intelligence* 12 (1979), 3, 231–272.
- [DVa92] Del Val, A., Computing Knowledge Base Updates. Pages 740–750 in *Proceedings of the Third International Conference on Principles of Knowledge Representation and Reasoning*. Eds. B. Nebel, C. Rich and W. Swartout. Morgan Kaufmann Publishers, 1992.
- [DVS92] Del Val, A., Shoham, Y., Deriving Properties of Belief Update from Theories of Action. Pages 584–589 in *Proceedings AAAI - 92 Tenth National Conference on Artificial Intelligence*, The MIT Press, 1992.
- [DVS94] Del Val, A., Shoham, Y., A Unified View of Belief Revision and Update. *Journal of Logic and Computation* 4 (1994), 5, 797–810.
- [Eit93] Eiter, T., Complexity of Nonmonotonic Reasoning. A course handout, University of Helsinki, August 1993.
- [Elo95] Eloranta, S., Tietämyskannan päivittäminen. Report C-1995-5, Helsinki University, 1995.
- [Elo97] Eloranta, S., On Iterated Belief Change. Pages 83–94 in *SCAI'97 - Proceedings of the Sixth Scandinavian Conference on Artificial Intelligence*. Ed. G. Grahne. IOS Press, Amsterdam, 1997.
- [Elo04] Eloranta, S., Dynamic features of the knowledge base. Report C-2004-090, Helsinki University, 2004.
- [Elo08] Eloranta, S., Hakli, R., Niinivaara, O., Nykänen, M., Accommodative Belief Revision. Pages 180–191 in *Proc. of the 11th European conference on Logics in Artificial Intelligence (JELIA'08)*. Springer-Verlag, 2008.
- [End77] Enderton, H. B., *Elements of Set Theory*. Academic Press, 1977.
- [Fag86] Fagin, R., Kuper, G. M., Ullman, J. D., Vardi, M. Y., Updating Logical Databases. Pages 1–18 in *Advances in Computing Research* 3. Eds. P. Kanellakis and F. Preparata. JAI Press, 1986.
- [FrH94] Friedman, N., Halpern, J., A Knowledge-Based Framework for Belief Change, Part I: Foundations. Pages 44–64 in *Proceedings of the Fifth Conference on Theoretical Aspects of Reasoning About Knowledge*. Ed. R. Fagin. Morgan Kaufmann Publishers, 1994.

- [FrH96] Friedman, N., and Halpern, J. Y., Belief Revision: A Critique. Pages 421–431 in KR'96: Principles of Knowledge Representation and Reasoning. Eds. L. C. Aiello, J. Doyle and S. Shapiro. Morgan Kaufmann, 1996.
- [FrL94] Freund, M., Lehmann, D., Belief Revision and Rational Inference. Technical Report TR 94-16, Hebrew University, 1994.
- [FUV83] Fagin, R., Ullman, J. D., Vardi, M. Y., On the Semantics of Updates in Databases. Pages 352–365 in Proceedings of the Second ACM SIGACT-SIGMOD-SIGART Symposium on Principles of Database Systems. ACM Press, 1983.
- [For89] Forbus, K.D., Introducing Actions into Qualitative Simulation. Pages 1273–1278 in Proceedings of the Eleventh International Joint Conference on Artificial Intelligence. Eds. S. Sridharan. Morgan Kaufmann Publishers, 1989.
- [Gin86] Ginsberg, M. L., Counterfactuals. *Artificial Intelligence* 30 (1986), 35–79.
- [GiS87] Ginsberg, M. L., Smith, D. E., Reasoning About Action I: A Possible Worlds Approach. Pages 434–463 in Readings in Nonmonotonic Reasoning. Eds. M.L. Ginsberg. Morgan Kaufmann Publishers, 1987.
- [GMR92] Grahne, G., Mendelzon, A. O., Reiter, R., On the Semantics of Belief Revision Systems. Pages 132–142 in Proceedings of the Fourth Conference on Theoretical Aspects of Reasoning About Knowledge. Eds. Y. Moses. Morgan Kaufmann Publishers, 1992.
- [Gra91a] Grahne, G., Updates and Counterfactuals. Pages 269–276 in Proceedings of the Second International Conference of Principles of Knowledge Representation and Reasoning. Eds. J. Allen, R. Fikes and E. Sandewall. Morgan Kaufmann Publishers, 1991.
- [Gra91b] Grahne, G., The Problem of Incomplete Information in Relational Databases. *Lecture Notes in Computer Science* 554. Springer-Verlag 1991.
- [Gro88] Grove, G., Two Modellings for Theory Change. *Journal of Philosophical Logic* 17 (1988), 157–170.
- [Gär88] Gärdenfors, P., Knowledge in Flux, Modeling the Dynamics of Epistemic States. MIT Press, 1988.

- [GäM88] Gärdenfors, P., Makinson, D., Revisions of Knowledge Systems Using Epistemic Entrenchment. Pages 83–95 in Proceedings of the Second Conference on Theoretical Aspects of Reasoning About Knowledge. Eds. M.Y. Vardi. Morgan Kaufmann Publishers, 1988.
- [HaV91] Halpern, J. Y., Vardi, M. Y., Model Checking vs. Theorem Proving: A Manifesto. Pages 325–334 in Proceedings of the Second International Conference of Principles of Knowledge Representation and Reasoning. Eds. J. Allen, R. Fikes ja E. Sandewall. Morgan Kaufmann Publishers, 1991.
- [Han99] Hansson, S. O., A survey of non-prioritized belief revision. *Erkenntnis* 50 (1999), 413–427.
- [HeR98] Herzig, A., Rifi, O., Update Operations: A Review. Pages 13–17 in ECAI98, Thirteenth European Conference on Artificial Intelligence. Ed. H. Prade. John Wiley and Sons, Chichester, 1998.
- [HeR99] Herzig, A., Rifi, O., Propositional Belief Base Update and Minimal Change. *Artificial Intelligence* 115, 1 (1999), pages 107–138.
- [Hin62] Hintikka, J., *Knowledge and Belief*. Cornell University Press, 1962.
- [KaM91a] Katsuno, H., Mendelzon, A. O., On the Difference between Updating a Knowledge Base and Revising it. Pages 387–394 in Proceedings of the Second International Conference on Principles of Knowledge Representation and Reasoning. Eds. J. Allen, R. Fikes and E. Sandewall. Morgan Kaufmann Publishers, 1991.
- [KaM91b] Katsuno, H., Mendelzon, A.O., Propositional Knowledgebase Revision and Minimal Change. *Artificial Intelligence* 52 (1991), 3, 263–294.
- [KeW85] Keller, A.M., Winslett Wilkins, M., On the Use of an Extended Relational Model to Handle Changing Incomplete Information. *IEEE Transactions on Software Engineering* 11 (1985), 7, 620–633.
- [KoP01] Konieczny, S., Prez, R., Some Operators for Iterated Revision. Pages 498–509 in Proceedings of the 6th European Conference on Symbolic and Quantitative Approaches to Reasoning with Uncertainty. Eds. S. Benferhat and P. Besnard. Springer-Verlag 2001.

- [LaS94] Lamarre, P., Shoham, Y., Knowledge, Certainty, Belief, and Conditionalization. Pages 415–424 in Proceedings of the Fourth International Conference of Principles of Knowledge Representation and Reasoning. Eds. J. Doyle, E. Sandewall, P. Torasso. Morgan Kaufmann Publishers, 1994.
- [Leh95] Lehmann, D., Belief Revision, Revised. Pages 1534–1540 in Proceedings of the Eleventh International Joint Conference on Artificial Intelligence. Morgan Kaufmann Publishers, 1995.
- [Lev84] Levesque, H. J., Foundations of a Functional Approach to Knowledge Representation. *Artificial Intelligence* 23 (1984), 155–212.
- [Lew73] Lewis, D.K., Counterfactuals. Blackwell, 1973.
- [Lif90] Lifschitz, V., Frames in the Space of Situations. *Artificial Intelligence* 46 (1990), 365–376.
- [MaG91] Makinson, D., Gärdenfors, P., Relations Between the Logic of Theory Change and Nonmonotonic Logic. Pages 185–205 in *The Logic of Theory Change. Lecture Notes in Artificial Intelligence* 465. Eds. A. Fuhrmann and M. Morreau. Springer-Verlag 1991.
- [Mak97] Makinson, D., Screened revision. *Theoria* 63 (1997), 14–23.
- [McC87] McCarthy, J., Circumscription – A Form of Non-Monotonic Reasoning. Pages 145–152 in *Readings in Nonmonotonic Reasoning*. Ed. M.L. Ginsberg. Morgan Kaufmann Publishers, 1987.
- [Nii99] Niiniluoto, I., Belief revision and truthlikeness. An electronic essay, 1999. In *Spinning Ideas, Electronic Essays Dedicated to Peter Gärdenfors on His Fiftieth Birthday*. <http://www.lucs.lu.se/spinning/categories/dynamics/>
- [Nyk11] Nykänen, M., Eloranta, S., Niinivaara, O., Hakli, R., Cooperative Replies to Unbelievable Assertions: A Dialogue Protocol Based on Logical Interpolation. Pages 245–250 in Proceedings of the 3rd International Conference on Agents and Artificial Intelligence (ICAART 2011), volume 2 – Agents. SciTe Press, 2011.
- [Poo88] Poole, D., A Logical Framework for Default Reasoning. *Artificial Intelligence* 36 (1988), 1, 27–47.
- [PST96] Priest, G., Surendonk, T. J., Tanaka, K., An Error in Grove’s Proof. Technical Report TR-ARP-07-96, Australian National University, Dec 1996.

- [Rei87] Reiter, R., A Theory of Diagnosis from First Principles. *Artificial Intelligence* 32 (1987), 57–95.
- [Rei88] Reiter, R., On Integrity Constraints. Pages 97–111 in *Proceedings of the Second Conference on Theoretical Aspects of Reasoning About Knowledge*. Ed. M.Y. Vardi. Morgan Kaufmann Publishers, 1988.
- [Rev93] Revesz, P.Z., On the Semantics of Theory Change: Arbitration between Old and New Information. Pages 1–12 in *Proceedings of the 12th ACM SIGACT-SIGMOD-SIGART Symposium on Principles of Database Systems*. ACM Press, 1993.
- [RyS92] Ryan, M., Sadler, M., Valuation Systems and Consequence Relations. Pages 1–78 in *Handbook of Logic in Computer Science, Vol. 1*. Eds. S. Abramsky, D.M. Gabbay and T.S.E. Maibaum. Oxford University Press, 1992.
- [Sat88] Satoh, K., Nonmonotonic Reasoning by Minimal Belief Revision. Pages 455–462 in *Proceedings of the International Conference on Fifth Generation Computer Systems*. Ed. ICOT. Springer-Verlag, 1988.
- [Spo88] Spohn, W., Ordinal Conditional Functions: A Dynamic Theory of Epistemic States. Pages 105–134 in *Causation in Decision, Belief Change, and Statistics*. Eds. W.L. Harper and B. Skyrms. Kluwer Academic Publishers, 1988.
- [Spo99] Spohn, W., Ranking Functions, AGM Style. An electronic essay, 1999.  
In *Electronic Essays Dedicated to Peter Gärdenfors on His Fiftieth Birthday*, <http://www.lucs.lu.se/spinning/index.html> [12.02.2003].
- [ToA91] Torlone, R., Atzeni, P., Updating Deductive Databases with Functional Dependencies. Pages 278–291 in *Deductive and Object-Oriented Databases. Second International Conference DOOD'91, Proceedings*. Lecture Notes in Computer Science 566. Eds. C. Delobel, M. Kifer and Y. Masunaga. Springer-Verlag, 1991.
- [Web85] Weber, A., Updating Propositional Formulas. Universität Karlsruhe, 1985.



- [Wil93] Williams, M.-A., Transmutations of Knowledge Systems. Thesis, University of Sydney, September 1993.
- [Win86] Winslett Wilkins, M., A Model-Theoretic Approach to Updating Logical Databases. Pages 224–234 in Proceedings of the Fifth ACM SIGACT-SIGMOD-SIGART Symposium on Principles of Database Systems. ACM Press, 1986.
- [Win88a] Winslett, M., A Framework for Comparison of Update Semantics. Pages 315–324 in Proceedings of the Seventh ACM SIGACT-SIGMOD-SIGART Symposium on Principles of Database Systems. ACM Press, 1988.
- [Win88b] Winslett, M., Reasoning About Action Using Possible Models Approach. Pages 89–93 in Proceedings AAAI -88 Seventh National Conference on Artificial Intelligence. Morgan Kaufmann Publishers, 1988.
- [Win90] Winslett, M., Updating Logical Databases. Cambridge University Press, 1990.
- [YiT07] Yi, J., Thielscher, M., Iterated belief revision, revised. *Artificial Intelligence* 171 (2007), 1–18.



# Appendix A

## Rationality criteria for knowledge base and its change

### Static rationality criteria on epistemic states

- (S1):  $T_B \models T_K$ .
- (S2):  $T_B$  is consistent.

### Refined postulates for belief-revision operator $\circ$

- (K0):  $(T \circ A)_K \equiv T_K$ .
- (K1):  $(T \circ A)_B \models (T \circ A)_K$ .
- (R0): If  $T_K \models \neg A$ , then  $(T \circ A)_B \equiv T_B$ .
- (R1): If  $T_K \not\models \neg A$ , then  $(T \circ A)_B \models A$ .
- (R2): If  $T_B \not\models \neg A$ , then  $(T \circ A)_B \equiv T_B \cup \{A\}$ .
- (R3):  $(T \circ A)_B$  is consistent.
- (RR1): If  $B \models A$  and  $T_K \not\models \neg B$ , then  $((T \circ A) \circ B)_B \equiv (T \circ B)_B$ .
- (RR2): If  $B \models \neg A$  and  $T_K \not\models \neg B$ , then  $((T \circ A) \circ B)_B \equiv (T \circ B)_B$ .
- (RR3): If  $(T \circ B)_B \models A$ , then  $((T \circ A) \circ B)_B \models A$ .
- (RR4): If  $(T \circ B)_B \not\models \neg A$ , then  $((T \circ A) \circ B)_B \not\models \neg A$ .
- (RR5): If  $T_B \not\models A$  and  $(T \circ B)_B \not\models \neg A$ , then  $((T \circ A) \circ B)_B \models A$ .

Redundant postulates:

- (R4): If  $A \equiv B$ , then  $(T \circ A)_B \equiv (T \circ B)_B$ .
- (R5):  $(T \circ A)_B \cup \{B\} \models (T \circ (A \wedge B))_B$ .
- (R6): If  $(T \circ A)_B \not\models \neg(A \wedge B)$ , then  $(T \circ (A \wedge B))_B \models (T \circ A)_B \cup \{B\}$ .

## Postulates for belief-contraction operator $\bullet$

- (K0):  $(T \bullet A)_K \equiv T_K.$
- (K1):  $(T \bullet A)_B \models (T \bullet A)_K.$
- (C0): If  $T_K \models A$ , then  $(T \bullet A)_B \equiv T_B.$
- (C1): If  $T_K \not\models A$ , then  $(T \bullet A)_B \not\models A.$
- (C2): If  $T_B \not\models A$ , then  $(T \bullet A)_B \equiv T_B.$
- (C3):  $T_B \models (T \bullet A)_B.$
- (CR1): If  $B \models A$  and  $T_K \not\models \neg B$ , then  $((T \bullet A) \circ B)_B \equiv (T \circ B)_B.$
- (CR2): If  $B \models \neg A$  and  $T_K \not\models \neg B$ , then  $((T \bullet A) \circ B)_B \equiv (T \circ B)_B.$
- (CR3): If  $(T \circ B)_B \models \neg A$ , then  $((T \bullet A) \circ B)_B \models \neg A.$
- (CR4): If  $(T \circ B)_B \not\models A$ , then  $((T \bullet A) \circ B)_B \not\models A.$

Redundant postulates:

- (C4): If  $A \equiv B$ , then  $(T \bullet A)_B \equiv (T \bullet B)_B.$
- (C5):  $(T \bullet A)_B \cup \{A\} \models T_B.$

## Rationality criteria for knowledge-expansion operator $\oplus$

- (K1):  $(T \oplus A)_B \models (T \oplus A)_K.$
- (K2): If  $T_K \not\models \neg A$ , then  $(T \oplus A)_K \equiv T_K \cup \{A\}.$
- (R2): If  $T_B \not\models \neg A$ , then  $(T \oplus A)_B \equiv T_B \cup \{A\}.$
- (R3):  $(T \oplus A)_B$  is consistent.
- (RR1): If  $B \models A$  and  $T_K \not\models \neg B$ , then  $((T \oplus A) \oplus B)_B \equiv (T \oplus B)_B.$

## Rationality criteria for competing-evidence operator $*$

- (K0):  $(T * A)_K \equiv T_K.$
- (K1):  $(T * A)_B \models (T * A)_K.$
- (R0): If  $T_K \models \neg A$ , then  $(T * A)_B \equiv T_B.$
- (R2): If  $T_B \not\models \neg A$ , then  $(T * A)_B \equiv T_B \cup \{A\}.$
- (R3):  $(T * A)_B$  is consistent.
- (NP1):  $((T * A) * B)_B \equiv ((T * B) * A)_B,$
- (NR1): If  $B \models A$  and  $T_K \not\models \neg B$ , then  $((T * A) \circ B)_B \equiv (T \circ B)_B.$
- (NR2): If  $B \models \neg A$  and  $T_K \not\models \neg B$ , then  $((T * A) \circ B)_B \equiv (T \circ B)_B.$
- (NR3): If  $(T \circ B)_B \not\models \neg A$ , then  $((T * A) \circ B)_B \models A.$

### Postulates for belief-update operator $\diamond$

- (K0):  $(T \diamond A)_K \equiv T_K.$
- (K1):  $(T \diamond A)_B \models (T \diamond A)_K.$
- (U0): If  $T_K \models \neg A$ , then  $(T \diamond A)_B \equiv T_B.$
- (U1): If  $T_K \not\models \neg A$ , then  $(T \diamond A)_B \models A.$
- (U2): If  $T_B \models A$ , then  $(T \diamond A)_B \equiv T_B.$
- (U3):  $(T \diamond A)_B$  is consistent.
- (U4): If  $T_K \equiv T'_{K'}$ ,  $T_B \equiv T'_{B'}$  and  $A \equiv B$ , then  $(T \diamond A)_B \equiv (T' \diamond B)_{B'}$ .
- (U5):  $(T \diamond A)_B \cup \{B\} \models (T \diamond (A \wedge B))_B.$
- (U6): If  $(T \diamond A)_B \models B$  and  $(T \diamond B)_B \models A$ , then  $(T \diamond A)_B \equiv (T \diamond B)_B.$
- (U7): If  $T_B$  is complete, then  $(T \diamond A)_B \cup (T \diamond B)_B \models (T \diamond (A \vee B))_B.$
- (U8): If  $T_K \equiv T'_K \equiv T''_K$  and  $T_B \equiv (T'_B \vee T''_B)$ , then  
 $(T \diamond A)_B \equiv (T' \diamond A)_B \vee (T'' \diamond A)_B.$
- (U9): If  $(T \diamond (A \vee B))_B \not\models \neg A$  and  $A$  is complete,  
then  $A \models (T \diamond (A \vee B))_B.$

### Postulates for belief-erasure operator $\blacklozenge$

- (K0):  $T_K \equiv (T \blacklozenge A)_K.$
- (K1):  $(T \blacklozenge A)_B \models (T \blacklozenge A)_K.$
- (E0): If  $T_K \models A$ , then  $(T \blacklozenge A)_B \equiv T_B.$
- (E1): If  $T_K \not\models A$ , then  $(T \blacklozenge A)_B \not\models A.$
- (E2): If  $T_B \models \neg A$ , then  $(T \blacklozenge A)_B \equiv T_B.$
- (E3):  $T_B \models (T \blacklozenge A)_B.$
- (E4): If  $T_K \equiv T'_{K'}$ ,  $T_B \equiv T'_{B'}$  and  $A \equiv B$ , then  $(T \blacklozenge A)_B \equiv (T' \blacklozenge B)_{B'}$ .
- (E5):  $(T \blacklozenge A)_B \cup \{A\} \models T_B.$
- (E8): If  $T_K \equiv T'_K \equiv T''_K$  and  $T_B \equiv (T'_B \vee T''_B)$ ,  
then  $(T \blacklozenge A)_B \equiv (T' \blacklozenge A)_B \vee (T'' \blacklozenge A)_B.$



TIETOJENKÄSITTELYTIETEEN LAITOS  
PL 68 (Gustaf Hällströmin katu 2 b)  
00014 Helsingin yliopisto

DEPARTMENT OF COMPUTER SCIENCE  
P.O. Box 68 (Gustaf Hällströmin katu 2 b)  
FIN-00014 University of Helsinki, FINLAND

JULKAISUSARJA A

SERIES OF PUBLICATIONS A

Reports may be ordered from: Kumpula Science Library, P.O. Box 64, FIN-00014 University of Helsinki, FINLAND.

- A-2007-1 P. Sarolahti: TCP Performance in Heterogeneous Wireless Networks. 171 pp. (Ph.D. Thesis)
- A-2007-2 M. Raento: Exploring privacy for ubiquitous computing: Tools, methods and experiments. 61+150 pp. (Ph.D. Thesis)
- A-2007-3 L. Aunimo: Methods for Answer Extraction in Textual Question Answering. 127+18 pp. (Ph.D. Thesis)
- A-2007-4 T. Roos: Statistical and Information-Theoretic Methods for Data Analysis. 82+75 pp. (Ph.D. Thesis)
- A-2007-5 S. Leggio: A Decentralized Session Management Framework for Heterogeneous Ad-Hoc and Fixed Networks. 230 pp. (Ph.D. Thesis)
- A-2007-6 O. Riva: Middleware for Mobile Sensing Applications in Urban Environments. 195 pp. (Ph.D. Thesis)
- A-2007-7 K. Palin: Computational Methods for Locating and Analyzing Conserved Gene Regulatory DNA Elements. 130 pp. (Ph.D. Thesis)
- A-2008-1 I. Autio: Modeling Efficient Classification as a Process of Confidence Assessment and Delegation. 212 pp. (Ph.D. Thesis)
- A-2008-2 J. Kangasharju: XML Messaging for Mobile Devices. 24+255 pp. (Ph.D. Thesis).
- A-2008-3 N. Haiminen: Mining Sequential Data – in Search of Segmental Structures. 60+78 pp. (Ph.D. Thesis)
- A-2008-4 J. Korhonen: IP Mobility in Wireless Operator Networks. 186 pp. (Ph.D. Thesis)
- A-2008-5 J.T. Lindgren: Learning nonlinear visual processing from natural images. 100+64 pp. (Ph.D. Thesis)
- A-2009-1 K. Hätönen: Data mining for telecommunications network log analysis. 153 pp. (Ph.D. Thesis)
- A-2009-2 T. Silander: The Most Probable Bayesian Network and Beyond. 50+59 pp. (Ph.D. Thesis)
- A-2009-3 K. Laasonen: Mining Cell Transition Data. 148 pp. (Ph.D. Thesis)
- A-2009-4 P. Miettinen: Matrix Decomposition Methods for Data Mining: Computational Complexity and Algorithms. 164+6 pp. (Ph.D. Thesis)
- A-2009-5 J. Suomela: Optimisation Problems in Wireless Sensor Networks: Local Algorithms and Local Graphs. 106+96 pp. (Ph.D. Thesis)
- A-2009-6 U. Köster: A Probabilistic Approach to the Primary Visual Cortex. 168 pp. (Ph.D. Thesis)
- A-2009-7 P. Nurmi: Identifying Meaningful Places. 83 pp. (Ph.D. Thesis)
- A-2009-8 J. Makkonen: Semantic Classes in Topic Detection and Tracking. 155 pp. (Ph.D. Thesis)
- A-2009-9 P. Rastas: Computational Techniques for Haplotype Inference and for Local Alignment Significance. 64+50 pp. (Ph.D. Thesis)
- A-2009-10 T. Mononen: Computing the Stochastic Complexity of Simple Probabilistic Graphical Models. 60+46 pp. (Ph.D. Thesis)
- A-2009-11 P. Kontkanen: Computationally Efficient Methods for MDL-Optimal Density Estimation and Data Clustering. 75+64 pp. (Ph.D. Thesis)

- A-2010-1 M. Lukk: Construction of a global map of human gene expression - the process, tools and analysis. 120 pp. (Ph.D. Thesis)
- A-2010-2 W. Hämäläinen: Efficient search for statistically significant dependency rules in binary data. 163 pp. (Ph.D. Thesis)
- A-2010-3 J. Kollin: Computational Methods for Detecting Large-Scale Chromosome Rearrangements in SNP Data. 197 pp. (Ph.D. Thesis)
- A-2010-4 E. Pitkänen: Computational Methods for Reconstruction and Analysis of Genome-Scale Metabolic Networks. 115+88 pp. (Ph.D. Thesis)
- A-2010-5 A. Lukyanenko: Multi-User Resource-Sharing Problem for the Internet. 168 pp. (Ph.D. Thesis)
- A-2010-6 L. Daniel: Cross-layer Assisted TCP Algorithms for Vertical Handoff. 84+72 pp. (Ph.D. Thesis)
- A-2011-1 A. Tripathi: Data Fusion and Matching by Maximizing Statistical Dependencies. 89+109 pp. (Ph.D. Thesis)
- A-2011-2 E. Junttila: Patterns in Permuted Binary Matrices. 155 pp. (Ph.D. Thesis)
- A-2011-3 P. Hintsanen: Simulation and Graph Mining Tools for Improving Gene Mapping Efficiency. 136 pp. (Ph.D. Thesis)
- A-2011-4 M. Ikonen: Lean Thinking in Software Development: Impacts of Kanban on Projects. 104+90 pp. (Ph.D. Thesis)
- A-2012-1 P. Parviainen: Algorithms for Exact Structure Discovery in Bayesian Networks. 132 pp. (Ph.D. Thesis)
- A-2012-2 J. Wessman: Mixture Model Clustering in the Analysis of Complex Diseases. 118 pp. (Ph.D. Thesis)
- A-2012-3 P. Pöyhönen: Access Selection Methods in Cooperative Multi-operator Environments to Improve End-user and Operator Satisfaction. 211 pp. (Ph.D. Thesis)
- A-2012-4 S. Ruohomaa: The Effect of Reputation on Trust Decisions in Inter-enterprise Collaborations. 214+44 pp. (Ph.D. Thesis)
- A-2012-5 J. Sirén: Compressed Full-Text Indexes for Highly Repetitive Collections. 97+63 pp. (Ph.D. Thesis)
- A-2012-6 F. Zhou: Methods for Network Abstraction. 48+71 pp. (Ph.D. Thesis)
- A-2012-7 N. Välimäki: Applications of Compressed Data Structures on Sequences and Structured Data. 73+94 pp. (Ph.D. Thesis)
- A-2012-8 S. Varjonen: Secure Connectivity With Persistent Identities. 139 pp. (Ph.D. Thesis)
- A-2012-9 M. Heinonen: Computational Methods for Small Molecules. 110+68 pp. (Ph.D. Thesis)
- A-2013-1 M. Timonen: Term Weighting in Short Documents for Document Categorization, Keyword Extraction and Query Expansion. 53+62 pp. (Ph.D. Thesis)
- A-2013-2 H. Wettig: Probabilistic, Information-Theoretic Models for Etymological Alignment. 130+62 pp. (Ph.D. Thesis)
- A-2013-3 T. Ruokolainen: A Model-Driven Approach to Service Ecosystem Engineering. 232 pp. (Ph.D. Thesis)
- A-2013-4 A. Hyttinen: Discovering Causal Relations in the Presence of Latent Confounders. 107+138 pp. (Ph.D. Thesis)