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Abstract

This paper introduces a new regime switching vector autoregressive (VAR) model where the regime switching dynamics is described by a qualitative response (QR) variable. Parameters of the QR-VAR model can conveniently be estimated by the method of maximum likelihood and multiperiod forecasts can be constructed using a simulation-based forecasting method. The model is applied to provide a new characterization of the nonlinear dynamic relationship between the U.S. interest rates and business cycle measured in terms of the NBER recession and expansion periods. A strong bidirectional predictive linkage between the variables is found, and due to the predictability of the business cycle regimes, the QR-VAR model yields superior out-of-sample forecasts for the interest rate variables compared with the VAR model.

JEL Classification: C32, C53, E43, E44

Keywords: Probit model, recession, regime switching model, vector autoregression

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1 Introduction

In the econometric literature, various regime switching models with different regime switching mechanisms have been considered during the past few decades. In this study, the focus is on regime switching vector autoregressive (RS-VAR) models. The previous literature on RS-VAR models includes the models of Sola and Driffill (1994), Krolzig (1997), Ang and Bekaert (2002a,b), Guidolin and Timmermann (2006) and Henkel et al. (2011), among others. In these models, the employed regime switching mechanisms have typically been based on a latent state variable and possibly time-varying regime probabilities that have often been specified as logistic functions of lagged endogenous variables.

In this study, our aim is to present a new regime switching VAR model based on the novel idea that the regime is determined by an observed qualitative response (QR) variable modeled simultaneously within the model. The joint model is referred to the QR-VAR model. We restrict ourselves to the case where the qualitative variable is binary (i.e. two regimes), such as the state of the business cycle considered in our application.

When considering the nonlinear regime switching patterns in economic time series, we are often, eventually, interested in understanding which economic forces drive the regime switches. In contrast to the observable binary variable determining the regime, the former regime switching models involve unobserved regimes whose probabilities are determined within the model. The regimes are often interpreted to reflect, for example, business cycle fluctuations (see, e.g., Ang and Bekaert 2002a; Henkel et al. 2011), asset return regimes (Guidolin and Timmermann 2006) or policy changes (Sims and Zha 2006). However, many other latent factors than the ones assumed may also affect the extracted regimes and their probabilities. In the QR-VAR model, the regime switching mechanism is fully specified via an observable binary time series without a need to interpret regime switches themselves. The resulting conditional probabilities of the regimes can be constructed with a binary response model, simplifying parameter estimation carried out straightforwardly with the method of maximum likelihood (cf. difficulties reported in the parameter estimation of the previous models (see, e.g., Gray 1996; Ang and Bekaert 2002a,b)).

In addition to the regime switching perspective emphasized above, the QR-VAR model adds to the very scant literature on models where continuous real-valued and qualitative dependent time series are modeled simultaneously within one model. Dueker (2005) and Fornari and Lemke (2010) are two rare exceptions where the VAR model is augmented with a latent variable determining the values of the considered binary time series. The QR-VAR model differs from those previous models in various ways. In particular, Dueker (2005) and Fornari and Lemke (2010) do not allow a regime switching structure in their VAR models, and the latter also employ a commonly used static model for the binary variable. In line with the univariate models of Rydberg and Shephard (2003), Benjamin, Rigby and Stasinopoulos (2003), Kauppi and Saikkonen (2008) and Startz (2008), we employ a dynamic binary response model as a part of the QR-VAR model leading to the model specification where parameter estimation and forecasting is easier than in the dynamic model of Dueker (2005). Overall, the structure of the QR-VAR model has some similarities to the regime switching GARCH-in-mean model of Nyberg (2012).

In general, if the values of a binary variable, such as the state of the business cycle, are predictable, then so are the regime switches in the QR-VAR model. This should, in principle, lead to superior forecast performance compared with the single-regime VAR model (provided there are regime switches in the VAR process). The QR-VAR model is designed to produce dynamic iterative forecasts constructed sequentially for the binary and continuous variables. Simulation methods are needed to obtain multiperiod forecasts as closed-form forecasting formulae are generally not available. The examined simulation experiments show that the proposed Monte Carlo forecasting method is not, however, computationally burdensome.

We apply the QR-VAR model to explore the bidirectional linkages between the U.S. interest rates and the state of the business cycle. As an example, Ang and Piazzesi (2003), Bansal et al. (2004) and Huse (2011) have shown that macroeconomic factors measuring real economic activity can help to predict future movements in the yield curve. In contrast, Estrella and Mishkin (1998) and Rudebusch and Williams (2009), among others, have found that the term spread between the long-term and short-term interest rates is the main leading indicator of the future state of the business cycle. Interestingly, almost all previous studies have concentrated on these one-way linkages while, e.g., Estrella (2005) and Diebold, Rudebusch and Aruoba (2006) have been supportive for a bidirectional relationship. In the QR-VAR model, instead of using ex post observations of the U.S. business cycle regimes, the regimes are modeled simultaneously with the interest rate variables revealing hence partly the real-time expectations on the state of the business cycle. To the best of our knowledge, this type of simultaneous regime switching modeling approach has not been considered before in the literature.

Our empirical results provide several interesting insights. In particular, strong evidence of business cycle-specific effects in the bivariate system of the U.S. short-term interest rate and the term spread is obtained. The dynamics of the short rate are closely dependent on the expansion and recession periods of the U.S. economy whereas the lags of interest rate variables predict the state of the business cycle. Furthermore, and most importantly, due to the obtained predictability of business cycle turning points, the out-of-sample forecasts of the QR-VAR model outperform those of the single-regime VAR model for the term spread and, especially, the short-term interest rate.

The rest of the paper is organized as follows. Section 2 introduces the QR-VAR model. Parameter estimation and computation of forecasts, including the proposed Monte Carlo forecasting method, are considered in Section 3. The empirical results on the bidirectional linkages and feedback mechanisms between the interest rates and the state of the business cycle are reported in Section 4. Finally, Section 5 concludes.

2 QR-VAR Model

Consider the time series s_t and $\mathbf{y}_t$, $t = 1, 2, ..., T$, where s_t is a qualitative response variable and $\mathbf{y}_t = [y_{1t}, \ldots, y_{Kt}]'$ is a $K \times 1$ random vector of real-valued continuous variables. In this study, we concentrate on the case where s_t is binary taking values 0 or

1. For notational convenience, the variables are collected to the vector

$$\boldsymbol{z}_t = [s_t \quad \mathbf{y}_t^{'}]^{'}. \tag{1}$$

The novel idea is to construct a regime switching VAR model where the regimes are determined by the observable binary variable s_t. We refer this model to as the Qualitative Response-Vector AutoRegressive (QR-VAR) model.

The regime switching VAR model can be written as

$$\mathbf{y}_t = s_t\Big(\boldsymbol{w}_1 + \sum_{i=1}^{p_1} \boldsymbol{A}_{i,1}\mathbf{y}_{t-i} + \mathbf{e}_{1t}\Big) + \Big(1 - s_t\Big)\Big(\boldsymbol{w}_0 + \sum_{i=1}^{p_0} \boldsymbol{A}_{i,0}\mathbf{y}_{t-1} + \mathbf{e}_{0t}\Big), \tag{2}$$

where depending on whether s_t takes the value 0 or 1, $\mathbf{y}_t$ follows a different VAR model. In other words, if $s_t = 1$, we are in the regime 1 and otherwise ($s_t = 0$) in the regime 0. The intercepts $\boldsymbol{w}_j$, coefficient matrices $\boldsymbol{A}_{i,j}$, $i = 1, \ldots, p_j$, and the error terms $\mathbf{e}_{jt}$, $j = 0, 1$, are all regime-specific allowing for flexible and different dynamics in two regimes. Model (2) encompasses the conventional VAR(p) model when $p_0 = p_1$, $\mathbf{e}_{0t} = \mathbf{e}_{1t}$ and all the corresponding parameters are the same irrespective of the regime s_t.

In model (2), the error terms $\mathbf{e}_{0t}$ and $\mathbf{e}_{1t}$ are assumed to follow multivariate normal distributions with zero means and possibly different covariance matrices $\boldsymbol{\Sigma}_0$ and $\boldsymbol{\Sigma}_1$ depending on the regime. Thus, we write

$$\mathbf{e}_{jt} = \boldsymbol{\Sigma}_j^{1/2}\mathbf{e}_t,\ j = 0, 1, \quad \mathbf{e}_t \sim \text{NID}(\mathbf{0}, \mathbf{I}_K), \tag{3}$$

and assume that $\mathbf{e}_t$ and Ω_{t-1} are independent with $\Omega_{t-1} = \{\boldsymbol{z}_{t-1}, \boldsymbol{z}_{t-2}, \ldots, \boldsymbol{z}_1\}$ denoting the information set containing the lags of $\mathbf{y}_t$ and s_t (see (1)) at time $t-1$. Furthermore, $\mathbf{e}_t$ and s_t are assumed to be independent conditional on Ω_{t-1}.

Throughout this paper, we assume that in (2) the contemporaneous value of s_t has an effect on $\mathbf{y}_t$, but not vice versa (cf. the model of Nyberg 2012). Although the main interest is in the regime switching VAR model (2), a model for the binary variable s_t is also needed, for example, in forecasting $\mathbf{y}_t$ (see Section 3.2). Conditional on the information set Ω_{t-1}, s_t follows a Bernoulli distribution

$$s_t|\Omega_{t-1} \sim B(p_t). \tag{4}$$

In this expression, p_t is the conditional expectation of s_t (denoted by $E_{t-1}(s_t)$) or equivalently the conditional probability of the outcome $s_t = 1$ (denoted by $P_{t-1}(s_t = 1)$)

$$p_t = E_{t-1}(s_t) = P_{t-1}(s_t = 1) = \Phi(\pi_t), \tag{5}$$

where $\Phi(\cdot)$ is a standard normal cumulative distribution function leading to the probit model and π_t is a linear function of variables included in the information set Ω_{t-1}. An alternative to the probit model, a logit model, is obtained by replacing $\Phi(\cdot)$ in (5) with the logistic function.

To complete the model for the binary variable s_t, we specify

$$\pi_t = \nu + a\pi_{t-1} + \mathbf{x}_{t-1}^{'}\boldsymbol{b}, \tag{6}$$

where $|a| < 1$ and ν is an intercept term. This model was suggested by Kauppi and Saikkonen (2008) in the context of univariate binary time series models (see also Rydberg and Shephard (2003) and Benjamin et al. (2003)). For simplicity, we restrict ourselves to the case where the predictors included in the vector $\mathbf{x}_{t-1}$ are the lagged values of $\mathbf{y}_t$. For example, if $K = 2$, then we can set $\mathbf{x}_{t-1} = [y_{1,t-k_1} \quad y_{2,t-k_2}]^{'}$ with k_1 and $k_2 \geq 1$. By recursive substitutions, it can be seen that π_t will depend on the whole lagged history of variables included in $\mathbf{x}_{t-1}$. In Section 4, we compare the autoregressive model (6) to the commonly used static alternative obtained by setting $a = 0$ in (6):

$$\pi_t = \nu + \mathbf{x}_{t-1}^{'}\boldsymbol{b}. \tag{7}$$

The univariate probit model is obtained when the predictors $\mathbf{x}_{t-1}$ are treated as exogenous variables. In the previous business cycle recession forecasting literature, dynamic univariate models, such as model (6), have been found to outperform the static model (7) (see, e.g., Kauppi and Saikkonen 2008; Nyberg 2010).

The expressions (2), (3), (5) and (6) define together the QR-VAR(p_0, p_1) model, where p_0 and p_1 denote the lag lengths of $\mathbf{y}_t$ in the regimes of model (2). Equation (2) shows the regime switching mechanism of the QR-VAR model but in forecast computation in

Section 3.2, we need the conditional expectation of $\mathbf{y}_t$ given Ω_{t-1}. This results in

$$\begin{aligned} E_{t-1}(\mathbf{y}_t) &= E_{t-1}\Big[s_t\Big(\boldsymbol{w}_1 + \sum_{i=1}^{p_1} \boldsymbol{A}_{i,1}\mathbf{y}_{t-i} + \mathbf{e}_{1t}\Big) + \Big(1 - s_t\Big)\Big(\boldsymbol{w}_0 + \sum_{i=1}^{p_0} \boldsymbol{A}_{i,0}\mathbf{y}_{t-1} + \mathbf{e}_{0t}\Big)\Big] \\ &= p_t\boldsymbol{\mu}_{1t} + \Big(1 - p_t\Big)\boldsymbol{\mu}_{0t}, \end{aligned} \tag{8}$$

where $\boldsymbol{\mu}_{jt} = \boldsymbol{w}_j + \sum_{i=1}^{p_j} \boldsymbol{A}_{i,j}\mathbf{y}_{t-i}$, $j = 0, 1$, and the law of iterated expectations and the assumptions made in (3) imply

$$E_{t-1}(s_t\mathbf{e}_{jt}) = E_{t-1}[E(s_t\mathbf{e}_{jt}|s_t, \Omega_{t-1})] = E_{t-1}[s_tE(\mathbf{e}_{jt}|s_t, \Omega_{t-1})] = \mathbf{0}, \quad j = 0, 1.$$

Thus, the conditional expectation of $\mathbf{y}_t$ is a weighted average of the conditional expectations of the VAR regimes where the weight $p_t = E_{t-1}(s_t)$ is given in (5). Furthermore, the conditional variance of $\mathbf{y}_t$ can be written as

$$\mathrm{Var}_{t-1}(\mathbf{y}_t) = p_t\boldsymbol{\Sigma}_1 + (1 - p_t)\boldsymbol{\Sigma}_0 + p_t(1 - p_t)(\boldsymbol{\mu}_{1t} - \boldsymbol{\mu}_{0t})(\boldsymbol{\mu}_{1t} - \boldsymbol{\mu}_{0t})'. \tag{9}$$

The conditional variance is hence nonconstant depending on the conditional probability p_t as well as the conditional means of the regimes of $\mathbf{y}_t$.

3 Estimation and Forecasting

3.1 ML Estimation

In the QR-VAR model, the parameters can conveniently be estimated by the method of maximum likelihood (ML). The difficulties in the estimation of many previously considered (univariate and multivariate) regime switching models are typically related to the determination of the (unobserved) regimes and their conditional probabilities (see, e.g., Gray 1996; Ang and Bekaert 2002a,b). In our approach, parameter estimation greatly simplifies because an observable binary time series determines the regime.

Conditional on the information set Ω_{t-1}, the density function of $\boldsymbol{z}_t$ (see (1)) is characterized by

$$g_{t-1}(\boldsymbol{z}_t; \boldsymbol{\theta}) = f(\mathbf{y}_t|s_t, \Omega_{t-1}; \boldsymbol{\theta})P(s_t|\Omega_{t-1}; \boldsymbol{\theta}), \tag{10}$$

where $f(\mathbf{y}_t|s_t, \Omega_{t-1}; \boldsymbol{\theta})$ is the conditional density function of the random vector $\mathbf{y}_t$ conditional on the value of the binary variable s_t and $P(s_t|\Omega_{t-1}; \boldsymbol{\theta})$ is the conditional probability mass function of s_t. The vector of parameters $\boldsymbol{\theta}$ contains all the parameters of the model. Assume that $\boldsymbol{\theta} = (\boldsymbol{\theta}_1^{'} \quad \boldsymbol{\theta}_2^{'})^{'}$ where $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$ contain the parameters related to the regime switching VAR model (2) and to the model for the binary variable, respectively. The density function (10) can therefore be written as

$$g_{t-1}(\boldsymbol{z}_t; \boldsymbol{\theta}) = f(\mathbf{y}_t|s_t, \Omega_{t-1}; \boldsymbol{\theta}_1) P(s_t|\Omega_{t-1}; \boldsymbol{\theta}_2). \tag{11}$$

Under the normality assumption of $\mathbf{e}_{jt}, j = 0, 1$ (see (3)), the conditional density function of model (2) is

$$f(\mathbf{y}_t|s_t, \Omega_{t-1}; \boldsymbol{\theta}_1) = (2\pi)^{-K/2} \det(\boldsymbol{\Sigma}_{s_t})^{-1/2} \exp\Big(-\frac{1}{2}\mathbf{e}_{s_t,t}^{'} \boldsymbol{\Sigma}_{s_t}^{-1} \mathbf{e}_{s_t,t}\Big), \quad s_t = 0, 1. \tag{12}$$

In the case of binary variable s_t, the conditional probability mass function is

$$P(s_t|\Omega_{t-1}; \boldsymbol{\theta}_2) = \Big(\Phi(\pi_t)\Big)^{s_t} \Big(1 - \Phi(\pi_t)\Big)^{1-s_t}, \quad s_t = 0, 1, \tag{13}$$

where π_t is specified as in (6) or (7).

Assume that we have observed the time series $\mathbf{y}_t$ and s_t, $t = 1, 2, ..., T$, with the initial values treated as fixed constants. Based on the conditional density function (11) of $\boldsymbol{z}_t$, the log-likelihood function over the whole sample, given the initial values, is

$$l_T(\boldsymbol{\theta}) = \sum_{t=1}^{T} l_t(\boldsymbol{\theta}) = \sum_{t=1}^{T} \log f(\mathbf{y}_t|s_t, \Omega_{t-1}; \boldsymbol{\theta}_1) + \sum_{t=1}^{T} \log P(s_t|\Omega_{t-1}; \boldsymbol{\theta}_2), \tag{14}$$

where the two factors of $g_{t-1}(\boldsymbol{z}_t; \boldsymbol{\theta})$ in (11) are defined in (12) and (13). Thus, $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$ can be estimated separately and the maximum likelihood estimate $\widehat{\boldsymbol{\theta}}$ is obtained by maximizing (14) by numerical methods.

At the moment, no formal proof of the stationarity conditions or the consistency and asymptotic normality of the maximum likelihood estimator $\widehat{\boldsymbol{\theta}}$ is available for the QR-VAR model. Nevertheless, under some regularity conditions, such as the stationarity of $\mathbf{y}_t$, s_t and π_t, it is reasonable to assume that the ML estimator $\widehat{\boldsymbol{\theta}}$ is asymptotically normally distributed and that a consistent estimator of the asymptotic covariance matrix can be based on the Hessian of the log-likelihood function. Standard errors of the parameter estimators as well as the conventional likelihood-based statistical tests, such as the Wald and the likelihood ratio (LR) tests, for the components of the parameter vector $\boldsymbol{\theta}$ can then be obtained in the usual way.

3.2 Computing Forecasts

After an adequate description of the joint dynamics of the variables s_t and $\mathbf{y}_t$ has been obtained, the QR-VAR model can be used to forecast future values of the time series. An advantage of the QR-VAR model over the forecast horizon-specific univariate binary response models (see, e.g., Estrella and Mishkin 1998; Kauppi and Saikkonen 2008; Nyberg 2010) is that this leads to the dynamic iterative multiperiod forecasting approach (cf. the conventional VAR model and the models of Dueker (2005) and Fornari and Lemke (2010)) without a need to specify a new model for every forecast horizon h.

Based on the information set at time T, the optimal h-period-ahead forecast of $\boldsymbol{z}_{T+h}$ (in the mean-square sense) is the conditional expectation

$$E_T(\boldsymbol{z}_{T+h}) = E(\boldsymbol{z}_{T+h}|\Omega_T) = \Big[E_T(s_{T+h}) \quad E_T(\mathbf{y}_{T+h})\Big]', \tag{15}$$

where the information set Ω_T includes the history of the time series $\boldsymbol{z}_t$ up to time T. Due to the recursive structure of the QR-VAR model, forecasts for the binary variable s_t are constructed first.

The one-period forecast of s_{T+1} (cf. (5)) is given by

$$p_{T+1} = E_T(s_{T+1}) = P_T(s_{T+1} = 1) = \Phi(\pi_{T+1}). \tag{16}$$

In the case of model (6), the linear function $\pi_{T+1} = \nu + a\pi_T + \mathbf{y}_T'\boldsymbol{b}$ depends only on information available at time T and, thus, the forecast (16) can be constructed straightforwardly. Following (8), the one-period forecast of $\mathbf{y}_{T+1}$ is the conditional expectation

$$E_T(\mathbf{y}_{T+1}) = p_{T+1}\,\boldsymbol{\mu}_{1,T+1} + \Big(1 - p_{T+1}\Big)\boldsymbol{\mu}_{0,T+1}, \tag{17}$$

where $\boldsymbol{\mu}_{j,T+1} = \boldsymbol{w}_j + \sum_{i=1}^{p_j} \boldsymbol{A}_{i,j}\mathbf{y}_{T-i+1}$, $j = 0, 1$ and p_{T+1} is the one-period forecast of s_{T+1} given in (16).

When the forecast horizon is longer than one period ($h > 1$), forecast computation becomes much more complicated. As an example, let us consider two-period forecasts ($h = 2$). As in (16), the forecast of s_{T+2} is the conditional expectation

$$p_{T+2} = E_T(s_{T+2}) = P_T(s_{T+2} = 1) = E_T\Big(\Phi(\pi_{T+2})\Big), \tag{18}$$

where $\pi_{T+2} = \nu + a\pi_{T+1} + \mathbf{y}_{T+1}^{'}\boldsymbol{b} = \nu + a^2\pi_T + a\Big(\nu + \mathbf{y}_T^{'}\boldsymbol{b}\Big) + \mathbf{y}_{T+1}^{'}\boldsymbol{b}$. Thus, it depends nonlinearly, via the function $\Phi(\cdot)$, on the value $\mathbf{y}_{T+1}$ which is unknown at time T. In particular, the conditional expectation (18) is not, in general, equal to the conditional probability of outcome $s_{T+2} = 1$ evaluated at the expected value of $\mathbf{y}_{T+1}$ given in (17). Decomposing $\mathbf{y}_{T+1}$ into an expected component $E_T(\mathbf{y}_{T+1})$ and the innovation $\mathbf{y}_{T+1} - E_T(\mathbf{y}_{T+1}) \overset{\text{def}}{=} \mathbf{e}_{j,T+1}^{+}$, the conditional expectation (18) can be expressed as

$$p_{T+2} \;=\; \int_{-\infty}^{\infty} \Phi\Big(\nu + a^2\pi_T + a(\nu + \mathbf{y}_T^{'}\boldsymbol{b}) + (E_T(\mathbf{y}_{T+1}) + (\mathbf{e}_{j,T+1}^{+})^{'}\boldsymbol{b}\Big)\varphi(\mathbf{e}_{j,T+1}^{+})\, d\mathbf{e}_{j,T+1}^{+},$$

where $\varphi(\mathbf{e}_{j,T+1}^{+})$ is the density function of $\mathbf{e}_{j,T+1}^{+}$. As this density function is intractable and the integral above does not have a closed form solution, we cannot construct the forecast for s_{t+2} using an explicit formula (cf. the one-period forecast (16)).

The two-period forecast of $\mathbf{y}_{T+2}$ can be expressed as

$$\begin{aligned} E_T(\mathbf{y}_{T+2}) \;=\;& E_T\Big[s_{T+2}\Big(\boldsymbol{w}_1 + \boldsymbol{A}_{1,1}\mathbf{y}_{T+1} + \ldots + \boldsymbol{A}_{p_1,1}\mathbf{y}_{T-p_1+2} + \mathbf{e}_{1,T+2}\Big) + \\ & (1 - s_{T+2})\Big(\boldsymbol{w}_0 + \boldsymbol{A}_{1,0}\mathbf{y}_{T+1} + \ldots + \boldsymbol{A}_{p_0,0}\mathbf{y}_{T-p_0+2} + \mathbf{e}_{0,T+2}\Big)\Big]. \end{aligned} \tag{19}$$

In comparison to (17), as $E_T(s_{T+2}\mathbf{y}_{T+1}) \neq E_T(s_{T+2})E_T(\mathbf{y}_{T+1})$, we cannot take the conditional expectations of s_{T+2} and the VAR regimes separately. The situation is similar when the forecast horizon h lengthens. Thus, the expressions (18) and (19) demonstrate that there are no closed-form forecasting formulae (cf. the conventional VAR model) to construct multiperiod forecasts for $\mathbf{y}_{T+h}$, $h \geq 2$, and we have to resort to simulation-based forecasting techniques. The Monte Carlo forecasting procedure described below is, however, quite easy to implement and computationally feasible. It has some similarities to the forecasting methods employed for other (mainly univariate) regime switching models (see, e.g., Teräsvirta et al. 2010, chap. 14).

The essential idea is to simulate recursively a large number of independent realizations of the variables $s_{T+1}, \mathbf{y}_{T+1}, s_{T+2}, \mathbf{y}_{T+2}, \ldots$ Forecasts of s_{T+h} and $\mathbf{y}_{T+h}$ for a given forecast horizon h are then obtained as averages of the independently simulated realizations $s_{T+h}^{(i)}$ and $\mathbf{y}_{T+h}^{(i)}, i = 1, \ldots, N$. The forecast horizon h varies between 1 and $\bar{h}$ with $\bar{h}$ the maximum forecast horizon considered. Furthermore, for $h \geq 2$, let $\underline{\boldsymbol{z}}_{T+h-1}^{(i)}$ (cf. (1)) signify the vector containing the ith the simulated realizations $s_{T+1}^{(i)}, \mathbf{y}_{T+1}^{(i)}, \ldots, s_{T+h-1}^{(i)}, \mathbf{y}_{T+h-1}^{(i)}$ up to the forecast horizon $h-1$. Throughout it is assumed that the unknown values of the parameters, which in practice are replaced by their estimates, are known.

The forecast recursion for forecast horizons $h = 1, 2, \ldots, \bar{h}$ proceeds as follows:

Step 1: Initialize $\pi_T^{(i)} \equiv \pi_T$ and $\mathbf{y}_{T-j}^{(i)} \equiv \mathbf{y}_{T-j}, j \geq 0$. Start the recursion with one-period forecast horizon i.e. set $h = 1$ in Steps 2–5.

Step 2: Compute $\left(\pi_{T+h}^{(i)} \middle| \Omega_T, \underline{\boldsymbol{z}}_{T+h-1}^{(i)}\right) = \nu + a\pi_{T+h-1}^{(i)} + \mathbf{x}_{T+h-1}^{(i)}\boldsymbol{b}$, where, e.g., if $K = 2$ then $\mathbf{x}_{T+h-1}^{(i)} = \left[y_{1,T+h-k_1}^{(i)} \;\; y_{2,T+h-k_2}^{(i)}\right]'$ for some k_1 and k_2.

Step 3: Draw $\left(s_{T+h}^{(i)} \middle| \Omega_T, \underline{\boldsymbol{z}}_{T+h-1}^{(i)}\right) \sim B(\Phi(\pi_{T+h}^{(i)}))$, where $B(\cdot)$ denotes the Bernoulli distribution and $\pi_{T+h}^{(i)}$ is given in Step 2.

Step 4: Draw $(\mathbf{e}_{j,T+h}^{(i)} | s_{T+h}^{(i)} = j) \sim N(\mathbf{0}, \boldsymbol{\Sigma}_j)$, $j = 0, 1$.

Step 5: Compute $\left(\mathbf{y}_{T+h}^{(i)} \middle| \Omega_T, \underline{\boldsymbol{z}}_{T+h-1}^{(i)}, s_{t+h}^{(i)}\right) = s_{T+h}^{(i)}\Big(\boldsymbol{w}_1 + \boldsymbol{A}_{1,1}\mathbf{y}_{T+h-1}^{(i)} + \ldots$ $+\boldsymbol{A}_{p_1,1}\mathbf{y}_{T+h-p_1}^{(i)}\Big) + (1 - s_{T+h}^{(i)})\Big(\boldsymbol{w}_0 + \boldsymbol{A}_{1,0}\mathbf{y}_{T+h-1}^{(i)} + \ldots + \boldsymbol{A}_{p_0,0}\mathbf{y}_{T+h-p_0}^{(i)}\Big) + \mathbf{e}_{j,T+h}^{(i)}$.

Step 6: Go to Step 2 and repeat Steps 3–5 starting from $h = 2$ up to $h = \bar{h}$.

Step 7: Repeat Steps 2–6 independently N times $(i = 1, \ldots, N)$.

The idea in the above recursion is first to use the horizon $h = 1$ to obtain realizations $\pi_{T+1}^{(1)}, s_{T+1}^{(1)}, \mathbf{y}_{T+1}^{(1)}$. Next, the recursion is repeated for $h = 2$, conditional on $\underline{\boldsymbol{z}}_{T+h-1}^{(i)}$, to obtain $\pi_{T+2}^{(2)}, s_{T+2}^{(2)}, \mathbf{y}_{T+2}^{(2)}$. This is continued up to $h = \bar{h}$. Finally, forecasts for s_{T+h} and $\mathbf{y}_{T+h}$, $h = 1, \ldots, \bar{h}$, are obtained by computing the averages (cf. equation (15))

$$\widehat{p}_{T+h} = E_T(s_{T+h}) = P_T(s_{T+h} = 1) = \frac{1}{N}\sum_{i=1}^{N} s_{T+h}^{(i)} \tag{20}$$

and

$$\widehat{\mathbf{y}}_{T+h} = E_T(\mathbf{y}_{T+h}) = \frac{1}{N}\sum_{i=1}^{N} \mathbf{y}_{T+h}^{(i)}, \tag{21}$$

where N is large. Note that the one-period forecasts ($h = 1$) obtained with (20) and (21) will be asymptotically equivalent to (16) and (17) but the above forecast recursion should accommodate also this horizon to start the recursion. Similarly, the conditional variance of $\mathbf{y}_{T+1}$, given Ω_T, can be obtained with expression (9) but in the case of multiperiod forecasts we have to resort to the simulation method described above. In addition to point forecasts, the expressions (20) and (21) can straightforwardly be used to construct possibly asymmetric interval and density forecasts.

The accuracy of the proposed MC forecasting method depends on the choice of the number of replications N. For a good approximation, N should be large enough. On the other hand, the larger the number of replications the more computationally burdensome the method is although simulation in Steps 3–4 is straightforward and not time consuming. The simulation results obtained in the Appendix suggest that the proposed method is accurate even for relative small values of N (such as N=10 000).

4 Linkages between U.S. Interest Rates and Business Cycle

4.1 Background and Data Set

In this section, we examine the bidirectional predictive linkages between the U.S. interest rates and the state of the business cycle measured in terms of recession and expansion periods. We are, in particular, interested in whether superior forecasts can be obtained with the QR-VAR model over the single-regime VAR and univariate probit models.

We consider a monthly U.S. data set from January 1972 to December 2010. The starting point of the sample (i.e. the beginning of the 1970s) is consistent with many previous studies (see, e.g., Ang and Bekaert 2002a,b; Huse 2011). The state of the economy s_t is determined by the National Bureau of Economic Research (NBER) business cycle turning points where $s_t = 1$ indicates a recession and $s_t = 0$ denotes an expansion. The term spread (TS_t) is the difference between the long-term (10-year government bond) and the short-term i_t (three-month Treasury Bill rate) interest rates. The source of all data is the Federal Reserve Bank of St. Louis databank (FRED).

Following the expectations hypothesis of the term structure of interest rates (hereafter EH), the dynamics of the interest rates can be considered by using a bivariate model of $\mathbf{y}_t$ containing the term spread (TS_t) and the first-difference of the short rate (Δi_t) (see, e.g, Campbell and Shiller 1991; Sola and Driffill 1994). Although most of the empirical studies have rejected the EH, we are interested in knowing whether the term spread predicts the changes in the short rate (see, e.g., Ang and Bekeart 2002a; Bansal et al. 2004) when the business cycle regime is taken into account. The short-term interest rate is of particular interest in our analysis as it is a fundamental building block of many macroeconomic and financial models (see, e.g., the term structure (yield curve) models of Ang and Piazzesi (2003), Bansal et al. (2004), Diebold et al. (2006) and Huse (2011) incorporating macroeconomic variables or constructed factors). Furthermore, Filardo (1994), Sola and Driffill (1994) and Ang and Bekaert (2002a,b), among others, have examined econometric regime switching models for the short rate where the obtained regime probabilities are often interpreted to describe regimes in real economic activity.

Based on the structure of the QR-VAR model, the lags of $\mathbf{y}_t$ (i.e., the lags of the term spread and short rate) are used to predict the state of the business cycle s_t. Much of the previous research lends support, especially, to the term spread being the main leading indicator of future real activity (see, e.g., Estrella and Mishkin 1998; Estrella 2005; Rudebusch and Williams 2009). Ang, Piazzesi and Wei (2006) and Wright (2006) find that the short rate has also some additional predictive power.

Figure 1 lends support to the regime switching approach as the U.S. interest rate dynamics appears to be closely dependent on the state of the U.S. economy. The short rate has typically been increasing (decreasing) during the expansion (recession) periods while during the recessions (expansions) the yield curve is generally upward (downward) sloping. All of the recession periods are preceded by a low, or even negative, value of the term spread, explaining why it has been found a useful leading indicator of the recession periods. Recession periods have also been characterized by a high short rate compared with its recent past just before the beginning of recession.

4.2 Estimation and Model Selection Results

In this section, we report the estimation results of the QR-VAR model and examine the possible two-way linkage between the variables s_t and $\mathbf{y}_t$. A subsample period up to 1992:12 is used to select the models which are subsequently employed in out-of-sample forecasting in Section 4.3 for the period 1993:1–2010:12. Due to the recursive structure of the QR-VAR model, a model for the U.S. business cycle is specified first and treated independently of the regime switching VAR component (2).

Table 1 (Panel A) presents the model selection results of the autoregressive (6) and static (7) models where the term spread is employed as a single leading indicator of the business cycle. Following the findings of Kauppi and Saikkonen (2008) and Nyberg (2010, 2012), instead of the first lag ($k_1 = 1$) of the term spread (i.e. TS_{t-1}), an optimal selection in terms of in-sample predictive power appears to be three (TS_{t-3}) or four in model (6) and nine in the static model (7). Overall, with the exception of the longest lags, model (6) clearly outperforms the static model (7) including the case of TS_{t-9}.

Panel B of Table 1 shows that the first difference of the short rate (Δi_t), and especially its first lag (Δi_{t-1}), have substantial additional predictive power over and above the term spread (TS_{t-3}). In accordance with the findings of Ang et al. (2006) and Wright (2006), the level of the short rate has some predictive power in the static model (7). However, the level of the short rate is throughout an inferior predictor compared with its first difference and the autoregressive model (6) generally outperforms the static model (7), yielding the best in-sample predictions. Thus, we continue our analysis with model (6) with the term spread (TS_{t-3}) and the first difference of the short rate (Δi_{t-1}) as the predictors of the state of the economy (i.e. $\mathbf{x}_{t-1} = [TS_{t-3} \quad \Delta i_{t-1}]'$). As a robustness check, we also estimated the models with data from the full sample period (1972:1–2010:12). The results were essentially the same strengthening the selection of TS_{t-3}.

The detailed estimation results of model (6) based on the entire sample period are presented in Table 2. Due to the negative and statistically significant coefficients, a low value of the term spread and decreasing short rate increase the probability of recession. The values of the statistical goodness-of-fit measures, such as the pseudo-R^2 of Estrella (1998), and the probability of recession ($s_t = 1$) depicted in Figure 2 show that the selected model predicts the state of the U.S. business cycle accurately. The recession probability is high during the recessions and close to zero in the expansion periods except for a few short exceptions. Hence, the model matches the U.S. business cycle regimes accurately which has not always been the case in the previous alternative regime switching models. In fact, the obtained transition probabilities for the unobserved regimes have not been found to necessarily describe business cycle recession and expansion periods. Instead, Filardo (1994) and Henkel et al. (2011), among others, interpret the transition probabilities to describe low and high growth rate regimes in the real GDP which describe more general contraction and expansion periods in real activity than the NBER business cycle periods.

Next we turn our interest to the estimation results of the regime switching VAR model (2). At first, it is worth recalling that the VAR part of the QR-VAR model does not have an effect on s_t. Thus, the results of Table 2 apply to any specification of model (2).

So far, we have assumed that the lag lengths p_0 and p_1 in the QR-VAR(p_0, p_1) model are known. In the previous research on the RS-VAR models, Ang and Bekaert (2002a,b) and Henkel et al. (2011) have restricted themselves to the parsimonious first-order models ($p_0 = p_1 = 1$). This is also a reasonable benchmark in this study. According to our estimation sample period 1972:1–1992:12, the Schwarz information criterion favors the QR-VAR(1,1) and VAR(1) models while the Akaike criterion suggests the maximum sixth-order models. A sequential testing procedure, where the Likelihood ratio (LR) test is applied sequentially when the order of the model increases until the first non-rejection, selects the QR-VAR(4,4) and VAR(3) models. To gain efficiency, the order of the recession regime can be reduced to three (the p-value 0.590 in the LR test). Irrespective of the selected QR-VAR or VAR models, there is some evidence of remaining autocorrelation in the equation of the short rate and conditional heteroskedasticity in both variables, but among the examined specifications, the QR-VAR(4,3) model seems the best selection also in terms of the diagnostic checks.

In Table 3, we present, for simplicity, the estimation results of the parsimonious QR-VAR(1,1) and VAR(1) models. The results of the QR-VAR(4,3) and VAR(4) models are available upon request. In the QR-VAR(1,1) model, the parameter estimates, especially the intercept terms, are different across the business cycle regimes and from the ones of the VAR(1) model. In line with Figure 1, the intercept term for the first-difference of the short rate is negative in the recession regime. Overall, irrespective of the lag length selection (results not reported), the QR-VAR model outperforms the VAR model as we can strongly reject the hypothesis of equal parameter coefficients in the expansion and recession regimes at all traditional significance levels. Thus, there appears to exist a bidirectional in-sample predictive linkage between the variables: The lags of the term spread and short rate predict the state of the business cycle (see Table 2). On the other hand, the VAR dynamics are strongly dependent on the business cycle regime. The estimated covariance matrices $\mathbf{\Sigma}_0$ and $\mathbf{\Sigma}_1$ are also different in two business cycle regimes. In particular, the diagonal elements are clearly higher in the recession regime implying higher volatility. Ang and Bekaert (2002b) found similar evidence in their RS-VAR model where they interpreted the regimes as high and low inflation regimes.

Following the lines of the EH, the estimation results show that in the QR-VAR(4,3) model the lags of the term spread are useful predictors of the short rate in both business cycle regimes (the p-values in the LR tests (H_0: no predictive power) were smaller than the 5% significance level). Similarly the changes of the short rate help to predict the term spread irrespective of the regime. These results hold also for other lag length selections p_0 and p_1, except the QR-VAR(1,1) model. In contrast to these findings, Ang and Bekaert (2002a) find that the lagged term spread predicts the short rate only in the high variance (recession) regime of their RS-VAR model (sample period 1972–1996) while the short rate is a useful predictor for the term spread only in the low variance (expansion) regime. In their another RS-VAR model, Ang and Bekaert (2002b) show that the lagged short rate and term spread have predictive power to each other only in the low inflation regime.

4.3 Out-of-Sample Forecasting

In this section, the MC forecasting method introduced in Section 3.2 is used to construct out-of-sample forecasts for the period 1993:1–2010:12. Forecasts are computed using an expansive window approach where the estimation sample period increases in each time when the parameters are re-estimated until the end of the sample. Because the state of the business cycle is uncertain in real time, parameters are re-estimated only when a complete business cycle from trough month to the next trough has been completed. Therefore, the out-of-sample forecasting period starts after the announcement of the business cycle trough for March 1991 made by the NBER in December 1992. Based on the Appendix, the number of simulated realizations N in the MC forecasting method is fixed to 10 000.

In Table 4, following the previous literature on the RS-VAR models, we report the results of the first-order QR-VAR(1,1) model along with the QR-VAR(4,3) model. The relative MSFE and QPS statistics are obtained relative to the VAR(1) and VAR(4) models and the univariate autoregressive probit (6) model. The VAR(4) model is used as a single-regime counterpart of the QR-VAR(4,3) model instead of the VAR(3) model (suggested by the sequential model selection procedure) as the former leads to inferior out-of-sample forecast performance compared with the VAR(4). The forecast evaluation for the short rate is executed for its level which is of interest in many applications and can easily be computed from the forecasts of the first-difference of i_t. Under the hypothesis of no business cycle-specific regimes the QR-VAR model nests the VAR model as a special case. Thus, the test of Clark and West (2007) is used to test the equal predictive performance between the QR-VAR and VAR models. The QR-VAR and univariate forecast horizon-specific models for the binary variable are not (generally) nested and, thus, the Diebold-Mariano (1995) test is employed in that case.

Many interesting findings emerge. Let us first consider forecasts for the short rate which are of most interest in this analysis. It can be seen that the QR-VAR(1,1) and QR-VAR(4,3) models clearly outperform their corresponding single-regime VAR(1) and VAR(4) models. Depending on the forecast horizon, the relative differences in the forecast accuracy typically range from 5% to even 20%. The first-order model seems to yield better forecasts than the QR-VAR(4,3) model. Based on the test of Clark and West (2007), the differences between the QR-VAR and VAR models are statistically significant at all conventional significance levels showing the superior predictive performance of the QR-VAR model.

The results for the term spread are basically the same as for the short rate. In this case, the QR-VAR(4,3) model produces somewhat better forecasts than the QR-VAR(1,1) model. However, in both cases, the QR-VAR models outperform the VAR models by a clear margin. The relative MSFEs are throughout below unity and the p-values of the Clark and West (2007) test are essentially zero.

As in Kauppi and Saikkonen (2008) and Nyberg (2010), the univariate autoregressive probit model (6) yields good forecasts for the state of the U.S. business cycle when the forecast horizon is relatively short. However, as expected and consistent with simulation results presented in the Appendix, when the forecast horizon lengthens towards the maximum 12-month horizon, the dynamic iterative forecasting approach employed in the QR-VAR model outperforms the forecast horizon-specific univariate model. According to the Diebold-Mariano (1995) test the differences are not, however, statistically significant. All in all, in possible future applications, such as construction of impulse response functions within the QR-VAR model (cf. Dueker 2005; Fornari and Lemke 2010), the dynamic iterative forecasting approach proposed in this study seems more appropriate.

As a whole, we conclude that superior forecasts for the interest rate variables can be obtained by allowing for the business cycle-specific regimes in the QR-VAR model. In the previous studies the relative differences between the single-regime and regime switching models have typically been smaller than in this study (see, e.g., Filardo 1994; Ang and Bekaert 2002a). In this respect, the QR-VAR model turns out to perform really well.

5 Conclusions

Regime switching models provide an attractive class of econometric models to capture regime changes in the stochastic behavior of interest rates. In this study, we suggested a new regime switching VAR model which can also be seen as a joint model between real-valued continuous variables and qualitative dependent variables. The QR-VAR model is easier to estimate than some previously considered multivariate regime switching models where the latent regimes are determined within the econometric model. Although a simulation-based forecasting method is required to construct multiperiod forecasts in the QR-VAR model the proposed MC method is not computationally burdensome.

The QR-VAR model is applied to describe the joint regime switching dynamics of the interest rates and the state of the business cycle where the latter explicitly determines the regime. The empirical results show that in the QR-VAR model there is a strong bidirectional linkage between the U.S. business cycle measured in terms of the NBER expansion and recession periods and the bivariate system of the U.S. term spread and the changes in the U.S. short-term interest rate. The results can be interpreted as positive evidence for a reduced-form model for the short rate incorporating business cycle shifts as the term spread and the short rate help to predict the future business cycle regimes while the state of the business cycle has also feedback effects back to them. Most importantly, the ability of the QR-VAR model to forecast business cycle turning points leads to superior out-of-sample forecast performance for the interest rate variables compared with the conventional single-regime VAR model.

The QR-VAR model can be extended various ways. One possibility is to replace the binary variable with other qualitative response variable, such as a multinomial variable allowing for more than two regimes. Another interesting extension could be to use the QR-VAR model in structural macroeconomic analysis. The impulse response functions implied by the QR-VAR model may lead to different conclusions than the VAR or, e.g., Markov switching models employed in the previous literature. To facilitate impulse response analysis, forecasts for the future values of the variables are required and, therefore, the proposed simulation-based iterative forecasting method is also of interest.

Appendix: Monte Carlo Forecasting Experiment

As discussed in Section 3.2, a simulation-based forecasting procedure is generally required to construct multiperiod forecasts in the QR-VAR model. In the proposed MC simulation method, the essential task is to specify the number of simulation replications N that affects the approximation error coming from the numerical integration. Thus, we consider a small-scale MC simulation experiment in order to specify the number of replications N and illustrate the properties of the forecasting method. The data generating process (DGP) is based on the QR-VAR(1,1) model presented in Tables 2–3.

We simulate 5 000 realizations of length $T+12$ observations from the above-mentioned DGP. Using the first T observations in each realization, we estimate the univariate probit model (6) and the VAR model along with the true QR-VAR model. Forecasts are computed for the forecast horizons from 1 to 12 periods. The mean-squared forecast errors (MSFE) and the QPS statistics (see Diebold and Rudebusch 1989) for the continuous and binary dependent variables are constructed, respectively. We experiment with two sample sizes (T=200 and T=500) and three choices of N (1 000, 10 000 and 50 000).

Table 5 presents the MSFE and QPS statistics of the QR-VAR model for different forecast horizons. The accuracy of forecasts for the binary variable appears to increase with the sample size T while this effect is not so clear for the continuous variables. As far as the number of replications is concerned, there is a slight improvement when N increases from 1 000 to 10 000, but basically no changes when N increases from 10 000 to 50 000. Thus, in conclusion, N=10 000 appears to be a sufficient selection.

The relative MSFE and QPS statistics in Table 6 are obtained by dividing the MSFE and the QPS statistics of the QR-VAR model reported in Table 5 by those of the corresponding VAR(1) and univariate probit (6) models. Most of entries are below unity for the variables y_{1t} and y_{2t} indicating the superiority of the true QR-VAR specification over the VAR model. The relative MSFEs in Table 5 are essentially the same with different selections of N. The relative QPS statistics for the binary variable show that the QR-VAR model designed to construct dynamic iterative multiperiod forecasts outperforms the forecast horizon-specific univariate model when the forecast horizon lengthens. As pointed out in Section 3.2, the one-period forecasts from the QR-VAR and the univariate autoregressive probit models are asymptotically equal.

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Tables and Figures

Table 1: Model selection results for the state of the business cycle s_t.

TS_{t-k1}	Δi_{t-k2}	i_{t-k2}	Autoregressive model (6)				Static model (7)			
$k1$	$k2$	$k2$	psR^2	QPS	AIC	BIC	psR^2	QPS	AIC	BIC
Panel A: Term spread as a single predictor										
1			0.435	0.180	68.406	73.627	0.061	0.290	111.951	115.432
2			0.459	0.173	65.524	70.745	0.110	0.273	106.086	109.567
3			0.474	0.170	63.729	68.950	0.179	0.249	97.781	101.261
4			0.475	0.171	63.668	68.889	0.224	0.239	92.523	96.004
5			0.470	0.173	64.252	69.473	0.247	0.232	89.775	93.255
6			0.464	0.175	64.904	70.125	0.286	0.216	85.116	88.596
7			0.453	0.177	66.219	71.440	0.317	0.205	81.427	84.908
8			0.438	0.181	68.035	73.256	0.337	0.200	78.979	82.460
9			0.420	0.186	70.163	75.384	0.354	0.192	77.028	80.508
10			0.396	0.194	72.953	78.174	0.341	0.198	78.515	81.996
11			0.376	0.202	75.362	80.583	0.338	0.204	78.905	82.386
12			0.355	0.208	77.839	83.060	0.335	0.210	79.258	82.739
Panel B: The term spread and the short-term interest rate as predictors										
3	1		0.532	0.146	57.865	64.826	0.212	0.243	94.935	100.156
3		9	0.482	0.168	63.865	70.826	0.356	0.210	77.710	82.931
9	1		0.531	0.141	58.038	64.999	0.464	0.163	64.935	70.156
9		4	0.451	0.177	67.502	74.463	0.431	0.181	68.804	74.025

Notes: Different lags of the predictors are denoted by $k1$ and $k2$. The pseudo-R^2 of Estrella (1998) (denoted by psR^2) and the QPS statistic (see Diebold and Rudebusch 1989) are the counterparts of the coefficient of determination and the mean-square prediction error used in linear models. AIC and BIC are the Akaike and Schwarz information criteria, respectively. In Panel B, only the best models in terms of the psR^2 and QPS are presented.

Table 2: Estimation results of the autoregressive binary response model (6).

π_t	ν	π_{t-1}	TS_{t-3}	Δi_{t-1}
	0.066	0.935	-0.119	-0.319
	(0.014)	(0.009)	(0.015)	(0.074)
psR^2	0.419		QPS	0.152
AIC	110.701		BIC	118.946
CR$_{50\%}$	0.893		CR$_{25\%}$	0.849

Notes: The estimated coefficients are based on the full sample period (1972:1–2010:12). TS_{t-3} and Δi_{t-1} denotes the employed lags of the term spread and the first difference of the short rate. The standard errors of the estimated coefficients are given in the parentheses. CR$_{50\%}$ and CR$_{25\%}$ denote the percentages of correct recession and expansion signal forecasts when the 50% and 25% thresholds are used to construct signal forecasts from the probability of recession (see (5)). See also the notes to Table 1.

Table 3: Estimation results of the QR-VAR(1,1) and VAR(1) models.

	QR-VAR(1,1)						VAR(1)		
	Expansion ($s_t = 0$)			Recession ($s_t = 1$)					
	0.974	-0.251		0.779	-0.307			0.934	-0.274
$\mathbf{A}_{1,0}$	(0.012)	(0.043)	$\mathbf{A}_{1,1}$	(0.054)	(0.083)		$\mathbf{A}_1$	(0.014)	(0.036)
	0.010	0.315		0.183	0.363			0.048	0.347
	(0.013)	(0.049)		(0.076)	(0.117)			(0.017)	(0.045)
$\boldsymbol{w}_0$	0.023	0.005	$\boldsymbol{w}_1$	0.418	-0.411		$\boldsymbol{w}$	0.114	-0.091
	(0.026)	(0.030)		(0.098)	(0.138)			(0.029)	(0.036)
	0.088	-0.064		0.377	-0.438			0.146	-0.136
$\boldsymbol{\Sigma}_0$	(0.006)	(0.006)	$\boldsymbol{\Sigma}_1$	(0.063)	(0.081)		$\boldsymbol{\Sigma}$	(0.010)	(0.011)
	-0.064	0.113		-0.438	0.750			-0.136	0.227
	(0.006)	(0.008)		(0.081)	(0.125)			(0.011)	(0.015)
logL	605.529						logL	496.243	
AIC	-587.529						AIC	-487.243	
BIC	-550.426						BIC	-468.692	

Notes: In the QR-VAR model, the reported values of the log-likelihood function (logL) and the Akaike and Schwarz information criteria (AIC and BIC) are based only on the VAR part of the model. The whole sample period (1972:1–2010:12) is used in these estimation results although in model selection only the subsample period (1972:1–1992:12) is employed.

Table 4: Out-of-sample forecasts.

Model	Forecast horizon (months)					
	1	2	3	6	9	12
	MSFE, term spread (TS_t)					
QR-VAR(1,1)	0.059	0.151	0.239	0.515	0.761	0.982
VAR(1)	0.062	0.166	0.269	0.623	0.929	1.158
relative MSFE	0.944***	0.909***	0.868***	0.826***	0.819***	0.848***
QR-VAR(4,3)	0.056	0.154	0.235	0.483	0.732	0.948
VAR(4)	0.061	0.172	0.267	0.592	0.894	1.138
relative MSFE	0.920***	0.892***	0.879***	0.817***	0.818***	0.832***
	MSFE, short rate (level, i_t)					
QR-VAR(1,1)	0.033	0.094	0.166	0.493	0.891	1.375
VAR(1)	0.036	0.112	0.207	0.615	1.086	1.616
relative MSFE	0.907***	0.842***	0.801***	0.801***	0.821***	0.851***
QR-VAR(4,3)	0.042	0.113	0.194	0.536	1.040	1.701
VAR(4)	0.051	0.143	0.236	0.628	1.154	1.770
relative MSFE	0.821***	0.788***	0.824***	0.853***	0.901***	0.960***
	QPS, business cycle (s_t)					
Univariate model (see (6))	0.187	0.185	0.187	0.186	0.185	0.186
QR-VAR(1,1)	0.188	0.192	0.198	0.190	0.177	0.177
relative QPS	1.001	1.039	1.054	1.025	0.957	0.950
QR-VAR(4,3)	0.188	0.189	0.192	0.182	0.171	0.171
relative QPS	1.003	1.025	1.023	0.977	0.921	0.921

Notes: The entries are the MSFE and QPS statistics of different models. Relative MSFEs (QPS) are obtained as dividing the MSFE (QPS) of the QR-VAR model by the MSFE (QPS) of the VAR (univariate probit) model. The number of simulation replications in the MC forecasting procedure is N=10 000. In the table, $*, **, ***$ denote the 10%, 5% and 1% level of significance in the test of Clark and West (2007) for equal predictive accuracy between the QR-VAR and the VAR model.

Table 5: MSFE and QPS statistics of the QR-VAR(1,1) model where the DGP is the QR-VAR(1,1) given in Tables 2 and 3.

	MSFE, y_{1t}			MSFE, y_{2t}			QPS, s_t		
N	1 000	10 000	50 000	1 000	10 000	50 000	1 000	10 000	50 000
Forecast horizon					$T = 200$				
1	0.158	0.158	0.158	0.261	0.261	0.261	0.207	0.206	0.206
2	0.400	0.399	0.399	0.277	0.276	0.276	0.200	0.199	0.199
3	0.671	0.668	0.668	0.265	0.263	0.263	0.218	0.217	0.217
6	1.234	1.228	1.228	0.270	0.269	0.269	0.240	0.238	0.238
9	1.535	1.530	1.529	0.278	0.276	0.276	0.257	0.257	0.257
12	1.770	1.765	1.764	0.293	0.292	0.292	0.267	0.265	0.265
					$T = 500$				
1	0.171	0.170	0.170	0.253	0.251	0.251	0.204	0.203	0.203
2	0.444	0.442	0.442	0.277	0.276	0.276	0.205	0.204	0.204
3	0.739	0.735	0.735	0.263	0.262	0.262	0.210	0.209	0.208
6	1.455	1.447	1.447	0.264	0.262	0.262	0.219	0.217	0.217
9	2.070	2.061	2.060	0.265	0.264	0.264	0.225	0.224	0.224
12	2.723	2.710	2.709	0.283	0.282	0.282	0.236	0.235	0.235

Notes: The entries are based on 5 000 realizations. The sample size is 200 or 500 observations (T=200 or T=500) and the number of simulation replications in forecast computation is denoted by N where N=1 000, 10 000 or 50 000. In simulations from the DGP, following the business cycle periods determined by the NBER, an additional censoring rule is imposed guaranteeing that the sequences of zeros and ones of the values of s_t are at least six-period long.

Table 6: The relative MSFE and QPS statistics of the QR-VAR(1,1) relative to the VAR(1) model and the univariate autoregressive probit model (6).

		Forecast horizon					
T		1	2	3	6	9	12
$T = 200$	MSFE, y_{1t}	0.971	0.930	0.884	0.773	0.704	0.641
	MSFE, y_{2t}	0.986	0.987	0.983	0.994	1.020	1.058
	QPS, s_t	1.000	0.939	0.940	0.906	0.941	0.915
$T = 500$	MSFE, y_{1t}	0.964	0.947	0.931	0.899	0.850	0.832
	MSFE, y_{2t}	0.997	0.985	0.978	1.006	0.997	0.998
	QPS, s_t	1.000	1.007	1.000	0.961	0.938	0.950

Notes: The number of simulated realizations is 5 000 and the number of replications in the forecast computation of the QR-VAR model is N=10 000. See also the notes to Table 5.

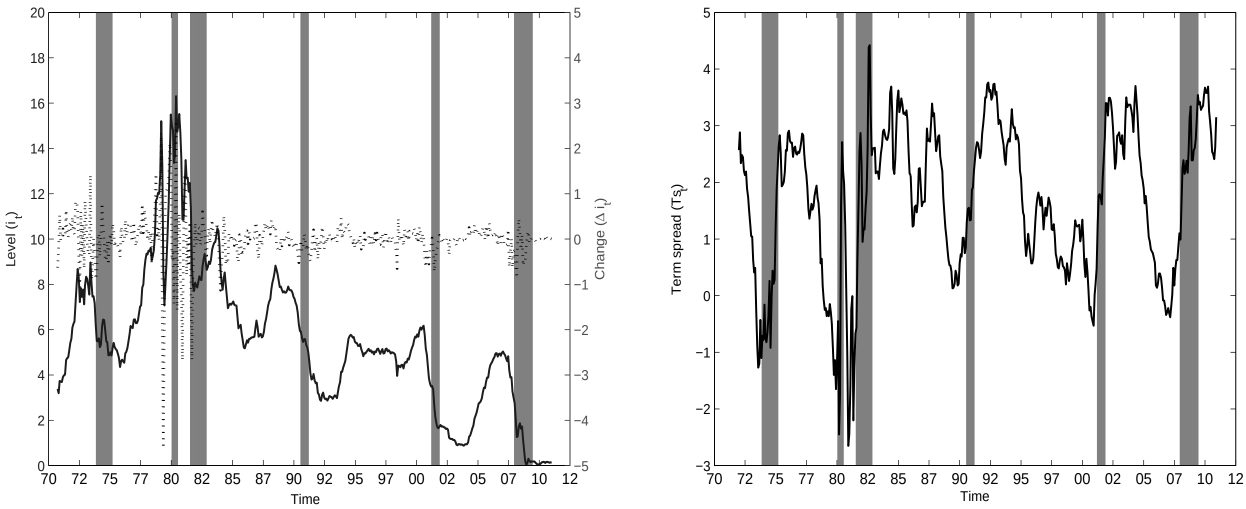

Figure 1: In the left panel, the U.S. short-term interest rate (i_t) and its first difference (Δi_t, dashed line) are depicted with the U.S. recession ($s_t = 1$, shaded areas) and expansion periods. The right panel shows the U.S. term spread (TS_t).

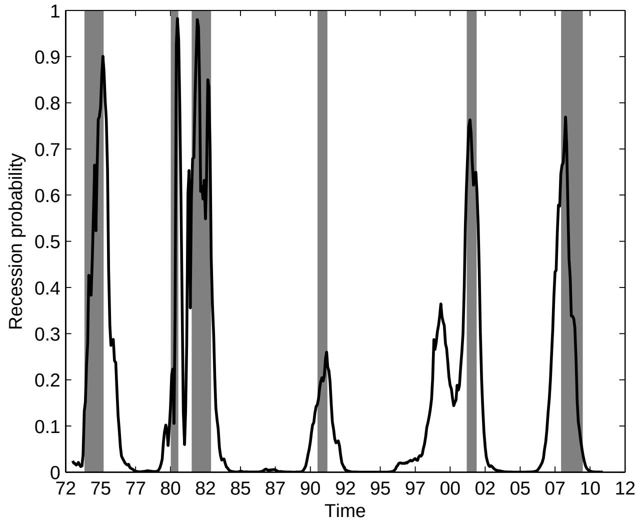

Figure 2: Estimated conditional recession probability ($s_t = 1$) of the model presented in Table 2.