

On resolution to Wu's conjecture on Cauchy function's exterior singularities

Theodore Yaotsu Wu

Received: 28 February 2011 / Revised: 1 March 2011 / Accepted: 1 March 2011

©The Chinese Society of Theoretical and Applied Mechanics and Springer-Verlag Berlin Heidelberg 2011

Abstract This is a series of studies on Wu's conjecture and on its resolution to be presented herein. Both are devoted to expound all the comprehensive properties of Cauchy's function $f(z)$ ($z = x + iy$) and its integral $J[f(z)] \equiv (2\pi i)^{-1} \oint_C f(t)(t-z)^{-1} dt$ taken along the unit circle as contour C , inside which (the open domain \mathcal{D}^+) $f(z)$ is regular but has singularities distributed in open domain \mathcal{D}^- outside C . Resolution is given to the inverse problem that the singularities of $f(z)$ can be determined in analytical form in terms of the values $f(t)$ of $f(z)$ numerically prescribed on C ($|t| = 1$), as so enunciated by Wu's conjecture. The case of a single singularity is solved using complex algebra and analysis to acquire the solution structure for a standard reference. Multiple singularities are resolved by reducing them to a single one by elimination in principle, for which purpose a general asymptotic method is developed here for resolution to the conjecture by induction, and essential singularities are treated with employing the generalized Hilbert transforms. These new methods are applicable to relevant problems in mathematics, engineering and technology in analogy with resolving the inverse problem presented here.

Keywords Cauchy function · Singularity distribution · Wu's conjecture · Resolution by induction

1 Introduction

The articles of Wu [1] on Wu's conjecture and its resolution by Wu [2] are revised for new publication. This series of studies are based on Cauchy's theorem and integral formula

Th. Y. Wu (✉)

California Institute of Technology, Pasadena, CA 91125 USA
Honorary Research Fellow, Institute of Mechanics,
Chinese Academy of Sciences
e-mail: tywu@caltech.edu

$$J[f(z)] \equiv \frac{1}{2\pi i} \oint_C \frac{f(t)}{t-z} dt = f(z),$$

$(z \in \mathcal{D}^+ \text{ — open domain inside } C),$ (1a)

$$J[f(z)] \equiv \frac{1}{2\pi i} \oint_C \frac{f(t)}{t-z} dt = 0,$$

$(z \in \mathcal{D}^- \text{ — open domain outside } C),$ (1b)

where Cauchy's function $f(z)$ is assumed to be analytic, regular $\forall z \in \mathcal{D}^+$ and continuous for $z = t$ on contour C ($|t| = 1$) taken in the positive (counter-clockwise) sense. Here, Eq. (1b) follows from Cauchy's integral theorem that $\oint_C g(t) dt = 0$ if $g(t)$ is regular within and on contour C (as is $g(t) = f(t)/(t-z) \forall z \in \mathcal{D}^-$), whereas Eq. (1a) is known as Cauchy's integral formula, or called Cauchy's functional relation, holding for z in open domain \mathcal{D}^+ (but literally not including z on boundary contour C).

The task of determining the value of Cauchy's integral $J[f(z)]$ for z situated right on C has been accomplished by Wu [1] with adopting a generalized condition that

$$f(z) \text{ be } C^n \quad \forall z \in \mathcal{D}^+ \text{ and in a neighborhood } \mathcal{N}_C$$

striding across contour C (n being arbitrary), (2)

where the corresponding function $f(z)$ is called the generalized Cauchy's function. This new condition enables the contour C to be indented exclusively about a generic point z_0 on C into an infinitesimal semi-circle C_ϵ^\pm onto the \mathcal{D}^\mp -side, of radius ϵ about z_0 (see Wu [1], Fig. 1) so that a point $z \in \mathcal{D}^\pm$ can reach z_0 without crossing the so deformed contour C , whilst with its remainder part $C - C_\epsilon^\pm$ kept intact. In the limit as $z \rightarrow z_0$ and $\epsilon \rightarrow 0$, $f(z) (\forall z \in \mathcal{D}^+) \rightarrow f^+(z_0)$, an undetermined limit, $f(z) (\forall z \in \mathcal{D}^-) \rightarrow 0$ by Eq. (1b), whereas the integral over $C - C_\epsilon^\pm$ assumes its Cauchy principal value. The final two limit equations thus yield three key relations (cf. Wu [1], Eq. (6)) as

$$\begin{aligned}
 \text{(I)} : f^+(z) &= f(z); \\
 \text{(II)} : f^-(z) &= 0; \\
 \text{(III)} : f(z) &= \frac{1}{\pi i} \mathcal{P} \oint_C \frac{f(t)}{t-z} dt, \quad z \in C,
 \end{aligned} \tag{3}$$

in which symbol \mathcal{P} denotes the Cauchy principal value of the integral whilst the suffix of z_0 is omitted for all z on C . Here, relation (I), $f^+(z) = f(z)$, proves the uniform continuity of $f(z)$ in the closed domain $\overline{\mathcal{D}^+} = [\mathcal{D}^+ + C]$, relation (II) is conjoint to (I), and relation (III) relates $f(z)$ for each $z \in C$ in terms of all the other values of $f(t)$ over C . Consequently, this further establishes the theorem that Cauchy's integral $J[f(z)]$ is uniformly convergent in the closed domain $\overline{\mathcal{D}^+} = [\mathcal{D}^+ + C]$ as all so proved [2].

The key relations (I)-(III) in Eq. (3) have bounteous prospects for applications and developments. They have been adapted by Wu [1] to derive, hence prove, the Hilbert-type integral transform formulas between the conjugate functions u and v of analytic function $f(z) = u(x, y) + iv(x, y)$ to hold for domains of four geometric forms in particular, namely one that circumventing the upper-half (or lower-half) of the z -plane, and another inside (or outside) the unit circle $|z| = 1$.

The relations (I)-(III) have also enabled Wu [2] to show that a unique relation exists between the numerical values of an analytic function $f(t) \forall |t| = 1$ over contour C and all the singularities of $f(z) \forall z \in \mathcal{D}^-$ outside C , which is illustrated by direct problems having the singularities prescribed ($\forall z \in \mathcal{D}^-$) to make its inverse problem appear more clarified in significance and structure.

This study is motivated by Wu's [1] conjecture on the inverse problem enunciated as follows.

The inverse problem.

The inverse problem is to adopt the values of $f(t)$ prescribed only in numerics $\forall t$ on contour C for a function $f(z)$ which is regular inside C to determine analytically all the exact singularity distributions of $f(z) \forall z \in \mathcal{D}^-$ outside C , whatever the singularity distribution.

For its general solution, the existence is asserted in terms of the following conjecture (cf. Wu [1]):

The conjecture. *Solution to this inverse problem is conjectured to exist, which may not be unique.*

Solutions to the inverse problem in multi-forms can be exemplified by the classical potential flow past a unit sphere with a solution consisted of a source-sink pair properly distributed over the front and rear hemisphere, or by a similar (but stronger) distribution over an interior confocal sphere, and ultimately by a dipole at the center. Resolution to this inverse problem is of vital importance. Urgent needs for the inverse problem to be resolved have ever been so accentuated and long-standing. Just for one cause, it dates back to the pioneering work of Sir George Gabriel Stokes [3] who developed the perturbation expansion theory in 1847, used for the very first time to study the nonlinear effects in water waves, followed by all inspired mathematicians dedicated

to the same cause, yet still striving for a conclusion on the convergence of the power series expansion. This is among a growing list of such pursuits for resolution of the inverse problem, as given in an expository review by Wu [1].

On the other hand, the issue on the exterior singularities has also been apprehended as a challenging subject of vital importance. Grant [4] considered the highest periodic water wave (with Stokes's 120° corner crest), formulated with the physical coordinates as a function $z = x + iy = z(f)$ in the lower half of the complex potential $f = \phi + i\psi$ plane continued analytically to the entire f -plane. He showed by analysis that the primary singularity of order 2/3, with $f^{2/3}$ in $z(f)$ at $f = 0$ of the Stokes crest is not a regular singularity as previously assumed, since it is joined by a secondary singularity with an irrational power of f , an essential discovery which is necessary for valid description of the wave. Grant contended that for lower waves, there exist several singularities of order $1/2 \notin \overline{\mathcal{D}^+}$ which coalesce at the wave crest to become the highest wave singular of order 2/3. This has been supported by the computational studies of Schwartz [5] who examined the singularity using the Padé approximation and Domb-Sykes plots, finding that the singularity varies continuously with the wave height from order 1/2 to 1/3 (for the complex velocity df/dz), and drawing a conjecture that the change involves coalescence of several singularities of order 1/2. However, Tanveer [6] examined the number of singularities, based on the argument principle, finding that there is just one singularity outside the flow field. Nevertheless, the basic mechanisms underlying singularities coalescing still remain to be fully expounded.

The foregoing two issues add to accentuate the value in resolving the conjecture on the inverse problem, which is the primary objective of this study. In outline, we classify function $f(z)$, being regular in domain \mathcal{D}^+ within contour C , to have (i) a single; (ii) double; or (iii) multiple singularities of the algebraic type, namely $f(z) = M(1 - z/z_1)^k$, lying at z_1 in domain \mathcal{D}^- outside C , with their base parameters (M, k, z_1) being solved as the inverse problem in terms of the values of $f(z)$ prescribed only in numerics on C . The problems (i) and (ii) are treated first, using complex algebra and analysis to expose the solution structure for useful reference. In Sect. 3, singularities appearing in complex conjugate pairs have their arguments determined for radial search of their locations and nature. In Sect. 4, two or more singularities of various kinds are analyzed by reducing them to one by singularity elimination in principle. In Sect. 5, a general method is developed for resolution to the conjecture by induction. In Sect. 6, the conjecture is resolved for the complement function regular in \mathcal{D}^- . And the essential singularities are expounded in Sect. 7 with discussions in Sect. 8 for conclusion.

2 Generalized Cauchy's function with a single point singularity

First, we consider the primary case when $f(z)$ (regular within

and on contour C) has only a single point singularity (of number $N_s = 1$) which can be represented generically as

$$f(z) = M(1 - z/z_1)^k, \quad |z_1| > 1, \tag{4}$$

where $k \in \mathbb{R}$, assumed real, so that it is a zero (or a pole) for k being a positive (or negative) integer, or an algebraic branch connecting z_1 and $z = \infty$ with a cut for k not being an integer, each lying at a point z_1 ($|z_1| > 1$) outside contour C which is taken to be the unit circle $|z| = 1$ for simplicity throughout. The base parameters (M, k, z_1) of the sole singularity are to be determined in terms of the values $f(t)$ of $f(z)$ prescribed only numerically for $t = e^{i\theta}$ ($-\pi < \theta \leq \pi$) $\in C$, say

$$f(t) = f(e^{i\theta}) = \tilde{f}(\theta) = \sum_{n=0}^N c_n e^{in\theta}, \tag{5}$$

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-in\theta} \tilde{f}(\theta) d\theta,$$

with $\tilde{f}(\theta)$ and N prescribed to a desired accuracy in numerics to satisfy condition equation (2). Numerically, $\tilde{f}(\theta)$ and the complex coefficients c_n are clearly equivalent, for they are merely a quadrature apart.

To proceed, we expand $f(z)$ of Eq. (4) into a power series in z , giving

$$f(z) = M \sum_{n=0}^N \gamma_n \left(-\frac{z}{z_1}\right)^n, \tag{6}$$

$$\gamma_n(k) = \binom{k}{n} = \frac{k(k-1)\cdots(k-n+1)}{n!}$$

$$= \frac{\Gamma(k+1)}{\Gamma(k-n+1)n!},$$

where $\Gamma(k)$ is the Gamma function, $\Gamma(k+1) = k\Gamma(k)$, $\Gamma(n+1) = n!$, here $\gamma_0 = 1$, $\gamma_1 = k$, $\gamma_2 = k(k-1)/2$.

Next, matching series equation (6) evaluated with $z = e^{i\theta} \in C$ termwise with series equation (5) readily links the numerical coefficients c_n with the algebraic ones γ_n , giving

$$f(e^{i\theta}) = M \sum_{n=0}^N \gamma_n (-e^{i\theta}/z_1)^n, \tag{7a}$$

$$c_0 = M\gamma_0 = M,$$

$$c_n = M\gamma_n (-z_1)^{-n}.$$

The two series equations (5) and (6) have the ratio of consecutive coefficients as, for $n = 0, 1, \dots$,

$$R_n(k, z_1) = \frac{c_{n+1}}{c_n} = -\frac{\gamma_{n+1}}{\gamma_n} \frac{1}{z_1} = \frac{n-k}{n+1} \frac{1}{z_1}, \tag{7b}$$

which shows that series equation (6) converges for $|z/z_1| < 1$ within the circle of convergence of radius $\rho = |z_1|$ on which the point singularity lies at $z = z_1$. Further, taking the ratio $R_{n+1}/R_n \equiv D_n(k)$ gives

$$D_n(k) = \frac{R_{n+1}}{R_n} = \left(1 - \frac{k+1}{n+2}\right) \left(1 - \frac{k+1}{n+1}\right)^{-1}, \tag{7c}$$

$$n = 1, 2, \dots,$$

in which the unknown location z_1 is eliminated. From Eq. (7) the base parameters readily result as

$$k = n + \left(1 - \frac{n+2}{n+1} D_n\right)^{-1} (\equiv k(n)),$$

$$z_1 = \frac{n-k}{(n+1)R_n} (\equiv z_1(n)), \tag{8}$$

$$M = c_0,$$

all being given in terms of the complex coefficients c_n ($n = 0, 1, \dots$) in numerics. Hence

Theorem 1. *Primary resolution to the conjecture. If (i) function $f(z)$ is regular in domain \mathcal{D}^+ bounded by unit circle $C(z = e^{i\theta})$; (ii) $f(z)$ has a single point singularity $f(z) = M(1 - z/z_1)^k$ ($k \in \mathbb{R}, |z_1| > 1$), lying outside C ; (iii) $f(z)$ is numerically prescribed on C by series equation (5) with complex coefficients $\{c_n, n = 0, 1, \dots\}$, then this inverse problem has the solution (M, k, z_1) given by Eq. (8).*

In principle, if Eq. (8) is exact in numerics and if indeed $N_s = 1$ for only one singularity, then solution equation (8) should produce fixed values to (M, k, z_1) by sufficient orders of n , and independent of n aside from what maybe due to any round-off errors in determining the c_n 's. This implies that solution equation (8) can result from the leading four coefficients, c_0 to c_3 , with c_n ($n \geq 4$) for attesting $N_s = 1$ and accuracies.

Example 1. Illustration. We let series equation (5) be prescribed for a specific case with

$$c_0 = 3, \quad c_1 = 1.999\ 895, \quad c_2 = 1.333\ 415,$$

$$c_3 = 0.889\ 123, \quad c_4 = 0.592\ 593, \quad c_5 = 0.395\ 062,$$

for the five leading terms of Eq. (5) taken here for solution and verification. We then find, by Eq. (8)

$$R_1 = c_2/c_1 = 0.666\ 743, \quad R_2 = 0.666\ 801,$$

$$R_3 = 0.666\ 492, \quad R_4 = 0.666\ 669,$$

$$D_1 = R_2/R_1 = 0.999\ 912, \quad D_2 = 0.999\ 535,$$

$$D_3 = 1.000\ 26,$$

and by Eq. (8), $k(n)$ can be determined for $n = 1, 2, 3$ as

$$k(1) = -0.999\ 823,$$

$$k(2) = -1.002\ 791,$$

$$k(3) = -0.996\ 851.$$

As a result, it seems pertinent to have these $k(n)$'s represented by their rounded value of $\check{k} = -1$, with such a uniformly small errors as exhibited. Similarly, we may let each of $z_1(n) = 1/R_n$ assume the rounded value of $\check{z}_1 = 1.5$ with supporting small errors. We therefore obtain the final solution as

$$\check{k} = -1, \quad \check{z}_1 = 1.5, \quad \check{M} = c_0 = 3,$$

in resolving the conjecture here to produce $f(z) = 2(1.5-z)^{-1}$ as a sole simple pole.

Example 2. Error estimate. We have just seen that $N_s = 1$ is well attested for this case, with a confirming feature of consistency in verifying the sole singularity, unknown a priori. For error estimate, we assign a small error ϵ_n to each c_n to give the rounded coefficients \check{c}_n and

$$\begin{aligned} \check{R}_n &= \check{c}_{n+1}/\check{c}_n = (c_{n+1} + \epsilon_{n+1})/(c_n + \epsilon_n), \\ \longrightarrow \check{k}(n) &= k + \kappa(\epsilon_n, \epsilon_{n+1}, \epsilon_{n+2}), \end{aligned} \tag{9}$$

so that κ , the error to k , being by Eq. (7c) an algebraic function of three specific ϵ 's, vanishes with the ϵ 's instead of being nonlinearly magnified. A similar conclusion can be drawn for (z_1, M) . We therefore can draw from the rounded coefficients \check{c}_n 's their specific errors as

$$\begin{aligned} \check{c}_n &= c_n + \epsilon_n, \longrightarrow \epsilon_1 = 1.05 \times 10^{-4}, \\ \epsilon_2 &= -8.2 \times 10^{-5}, \quad \epsilon_3 = 2.35 \times 10^{-4}, \dots, \end{aligned}$$

with similar error estimates for $|\check{D}_n - D_n|, \ni \check{D}_n - D_n \rightarrow 0 \forall \epsilon_n \rightarrow 0$. Here we point out that the final solution can be used to re-generate the Fourier coefficients for an ultimate verification of the errors.

Example 3. Logarithmic singularity. For a sole logarithmic singularity generically given by

$$\begin{aligned} f_\ell(z) &= M \lg(1 - z/z_1) = - \sum_{n=1}^N \frac{M}{n} \left(\frac{z}{z_1}\right)^n \\ \longrightarrow c_n &= -\frac{M}{n} z_1^{-n}, \quad R_n = \frac{c_{n+1}}{c_n} = \frac{n}{n+1} z_1^{-1}, \end{aligned} \tag{10}$$

for $n = 1, 2, \dots$, suggesting, by comparison with Eq. (7b), that $k = 0$, thus implying that if k of Eq. (8) should result in $k = 0$, then whether $f(z)$ is a logarithmic function can be ascertained accordingly.

Thus, we see that the primary cases of Eqs. (4) and (10) can well exemplify all the others in this group. Being primary, they are the base to which more general cases can be reduced or referenced.

2.1 The Domb–Sykes plot

Reflecting on the conspicuous structure of $R_n(k, z_1)$ exhibited by Eq. (7b), the Domb–Sykes scheme gives the graphical plot of $\eta(\xi) = c_{n+1}/c_n$ versus $\xi = 1/n$ according to

$$\eta(\xi) = \frac{c_{n+1}}{c_n} = R(\xi, k, z_1), \quad \xi = \frac{1}{n}, \quad z_1 \in R, \tag{11}$$

in which z_1 is taken to be real for simplicity (or can be attained by a rotation of the coordinates, with $\arg z_1$ given by that of c_{n+1}/c_n). This graph, called the Domb–Sykes plot [7], can effectively provide the index k and location z_1 in $R(\xi, k, z_1)$ as $\xi \rightarrow 0$ ($n \rightarrow \infty$). In fact, writing $R(\xi, k, z_1)$ in Eq. (7b) as

$$\begin{aligned} R(\xi, k, z_1) &= \frac{1 - k\xi}{1 + \xi} \frac{1}{z_1} \longrightarrow R(0, k, z_1) = \frac{1}{z_1}, \\ \frac{\partial}{\partial \xi} R(0, k, z_1) &= -\frac{1+k}{z_1}. \end{aligned} \tag{12}$$

This yields k and z_1 by plotting a graph of $R(\xi, k, z_1) = c_{n+1}/c_n$ versus $\xi = 1/n$ with $n = N, N + 1, \dots, \forall N \gg 1$ to determine the slope and the η -intercept of $\eta = c_{n+1}/c_n$. Similarly, the Domb–Sykes plot can serve as a powerful tool over a broad scope involving multi-singularities, as shown in Sect. 5.

3 Resolution for function with complex conjugate pair of singularities

A case of special interest is for function $f(z)$ having a complex conjugate pair of singularities. In this case, the series for $f(z)$, in virtue of Schwarz's symmetry, has its coefficients all real. Here it is apt to first consider a $f(z)$ having a conjugate pair of logarithmic singularities at $z_1 = e^{i\alpha}$ and \bar{z}_1

$$f(z) = \lg[(1 - e^{-i\alpha}z)(1 - e^{i\alpha}z)]^{-1/2} = \sum_{n=1}^{\infty} a_n z^n = s(z), \tag{13}$$

$$a_n = \frac{1}{n} \cos n\alpha,$$

$s(z)$ being the series function defined by the series ($\forall |z| < 1$) versus the sum function $f(z)$ for all z . In case when the series function $s(z)$ is the only data available for locating the singularities of $f(z)$ while $f(z)$ is still undetermined, this particular series in Eq. (13) actually can be useful, as follows.

3.1 Sign pattern criteria of series

For series equation (13), there are certain definite relations between α (hence the $\arg z_1$) and the resulting sign pattern displayed in the series. In this particular case, the sign of a_n will depend on the sign of $\cos(n\alpha)$ and will result in an inter-lacing sign pattern with first N_0 terms of positive a_n 's, then N_1 terms negative, and again N_2 terms positive, \dots , so that

$$\begin{aligned} -\frac{\pi}{2} < N_0\alpha < \frac{\pi}{2}, \quad \frac{\pi}{2} < (N_0 + N_1)\alpha < \frac{3\pi}{2}, \dots, \\ \left(j - \frac{1}{2}\right)\pi < (N_0 + \dots + N_j)\alpha < \left(j + \frac{1}{2}\right)\pi, \end{aligned} \tag{14}$$

for $j = 1, 2, \dots$. In the limit, we actually have just delineated a proof of a theorem by Li [8] in extending the pioneering work on this subject by Van Dyke [9].

Theorem 2. *Li's Theorem on Location of a conjugate pair of singularities in series. If the power series $\sum a_n z^n$, with $\lim |a_n|^{1/n} = 1$, has in turn N_0 terms positive, N_1 terms negative, N_2 terms positive, \dots , then the series has a conjugate pair of singularities at $z_1 = e^{i\alpha}$ and $\bar{z}_1 = e^{-i\alpha}$, with*

$$\alpha = \lim_{j \rightarrow \infty} j\pi \left/ \left(\sum_{k=0}^j N_k \right) \right. \tag{15}$$

In particular, Eq. (15) reduces for $f(z) = -\lg(1 - z) = \sum_{n=1} z^n/n$ to $\alpha = 0$ with $N_1 = \infty$. Likewise, for $f(z) = -\lg(1 + z) = \sum_{n=1} (-z)^n/n$, Eq. (15) gives $\alpha = \pi$ with

$N_1 = N_2 = \dots = 1$, hence $j\pi \left(\sum_{k=1}^j N_k \right) = j\pi/j = \pi$ for $j \geq 2$, thus showing the singularity located at $z = -1$ just as that for $f(z) = -\lg(1 + z)$. From these two basic cases, we can also infer others that, e.g. $\alpha = \pi/N$ if the signs of a_n ’s interlace in N -tuples for $N = 2n + 1$ ($n = 1, 2, \dots$).

Furthermore, Li’s Theorem can be generalized to hold for products of multiple algebraic and logarithmic conjugate functions. For instance, for $f(z) = (z - z_1)^{-k}(z - \bar{z}_1)^{-k}(z_1 = e^{i\alpha})$, we have

$$g(z) = \lg f(z) = -k \lg[(1 - z_1 z)(1 - \bar{z}_1 z)] = 2k \sum_{n=1}^{\infty} n^{-1} z^n \cos n\alpha.$$

With the product of the conjugate functions reduced to the sum of their logarithms, the singularities of $g(z)$ are therefore resolved, just as above by Li’s Theorem, and these singularities are of course also those of $f(z) = \exp[g(z)]$. The same logical argument holds for other types of such functions.

4 Resolution to the conjecture for $f(z)$ having multiple singularities

For more general cases, we consider next the case of $N_s \geq 2$ for singularities of two kinds, one being those of equal order and arbitrary locations whereas the other for those of arbitrary orders and locations, to begin first with the case of $N_s = 2$ for subsequent extension to multiple singularities.

4.1 Resolution for $f(z)$ having two singularities of equal order

We now consider $f(z)$ having two singularities ($N_s = 2$) of equal order given by

$$f(z) = M_1(1 - z/z_1)^k + M_2(1 - z/z_2)^k = \sum_{n=0}^{\infty} (M_1 z_1^{-n} + M_2 z_2^{-n}) \gamma_n(k) (-z)^n, \tag{16}$$

where $\gamma_n(k)$ is given in Eq. (6). Similar to Eqs. (5)–(8), matching series equation (16) for $z = e^{i\theta}$ termwise with Eq. (5) gives

$$(M_1 z_1^{-n} + M_2 z_2^{-n}) \gamma_n = (-1)^n c_n, \quad n = 0, 1, \dots \tag{17}$$

These are the set of algebraic equations for solving (M_1, M_2, k, z_1, z_2) and verification for this case.

First, we eliminate the terms with M_2 by operation for $[(k - n)/(n + 1)/z_2]c_n + c_{n+1}$, giving

$$M_1(z_2^{-1} - z_1^{-1}) \gamma_{n+1} z_1^{-n} = (-1)^n \tilde{c}_{n+1}, \tag{18a}$$

$$\tilde{c}_{n+1} = \frac{k - n}{(n + 1)z_2} c_n + c_{n+1}.$$

Next, M_1 can be eliminated by taking the ratio $\tilde{c}_{n+1}/\tilde{c}_n$ ($n = 1, 2, \dots$), yielding

$$\frac{\tilde{c}_{n+1}}{\tilde{c}_n} = -\frac{\gamma_{n+1}}{\gamma_n z_1} = \frac{n - k}{n + 1} \frac{1}{z_1} = \frac{n}{n + 1} H_n(k, z_2), \tag{18b}$$

$$H_n(k, z) = \frac{(n + 1)R_n z - (n - k)}{nz - (n - 1 - k)/R_{n-1}},$$

which can be written in symmetry between the two singularities in equality as

$$G_n(k, z_1) = G_n(k, z_2), \quad n = 1, 2, \dots \tag{18c}$$

$$G_n(k, z) = H_n(k, z)/z,$$

Finally, expanding out Eq. (18b) yields the basic equation for this case as

$$(n + 1)R_n z_1 z_2 - (n - k)[(z_1 + z_2) - (n - k - 1)/(nR_{n-1})] = 0, \quad n = 1, 2, \dots, \tag{19}$$

which also exhibits the symmetry between the two singularities interchangeable in designation.

To resolve the three unknown parameters (k, z_1, z_2) in Eq. (19), we can take four leading equations of (19) to eliminate $(z_1 + z_2)$ and $z_1 z_2$ in two steps, yielding one equation for k ($n = 1, 2, \dots$) as

$$\frac{K(-1, k)}{R_{n-1}} \left[\frac{R_{n+1}}{K(1, k)} - \frac{R_{n+2}}{K(2, k)} \right] + \frac{K(1, k)}{R_{n+1}} \frac{R_n}{K(0, k)} + \frac{K(0, k)}{R_n} \frac{R_{n+2}}{K(2, k)} = 2, \tag{20a}$$

$$K(j, k) = \frac{n + j - k}{n + 1 + j},$$

which is a cubic equation for k (after dismissing the complex conjugate roots by confirmation). With k so determined, $(z_1 z_2)$ can be deduced by eliminating $z_1 + z_2$ from Eq. (19), giving for $n = 1, 2, \dots$,

$$z_1 z_2 = \left[\frac{n - k}{(n + 1)R_n} - \frac{n - k - 1}{R_{n-1}} \right] \left[\frac{n + 1}{n - k} R_n - \frac{n + 2}{(n + 1) - k} R_{n+1} \right] (\equiv A(n, k)), \tag{20b}$$

for $n = 1$ say, with which $(z_1 + z_2) (= 2B(n, k))$, say then follows from Eq. (19) for $n = 1$. Finally, z_1 and z_2 are obtained by combining $z_1 z_2 = A(n, k)$ and $z_1 + z_2 = 2B(n, k)$ to give the quadratic equation

$$z_1^2 - 2Bz_1 + A = 0 \implies z_1 = B + \sqrt{B^2 - A}, \tag{20c}$$

$$z_2 = B - \sqrt{B^2 - A}.$$

The solution is then completed with M_1 and M_2 found from Eq. (18a) for $n = 0$ and verified for its feature of consistency using Eq. (16) for $n \geq 5$. Our resolution to the conjecture is

then accomplished for this case of $f(z)$ having two singularities of equal order. When there are more than two singularities of equal order, this method can again provide a scheme for solution by induction.

4.2 Resolution for $f(z)$ having two individual singularities

For $f(z)$ having two singularities of arbitrary orders and locations, we have

$$f(z) = M_1(1 - z/z_1)^{k_1} + M_2(1 - z/z_2)^{k_2} = \sum_{n=0}^{\infty} [M_1 z_1^{-n} \gamma_n(k_1) + M_2 z_2^{-n} \gamma_n(k_2)] (-z)^n. \tag{21}$$

Again, matching series equation (21) with series in Eq. (5) yields for this case for $n = 0, 1, \dots$,

$$M_1 z_1^{-n} \gamma_n(k_1) + M_2 z_2^{-n} \gamma_n(k_2) = (-1)^n c_n. \tag{22a}$$

These are the set of equations for solving $(M_i, k_i, z_i, i = 1, 2)$ and for verifications. Elimination of M_1 and M_2 can be carried out in close analogy with that for Eq. (16), yielding the basic equation as

$$\frac{z_1 - z_2}{(k_2 - n + 1)z_1 - (k_1 - n + 1)z_2} = 1 + \frac{nz_1}{k_1 - n + 1} \cdot \frac{(k_2 - n) + (n + 1)R_n z_2}{nz_2 - (n - k_2 - 1)/R_{n-1}}, \tag{22b}$$

which reduces to Eq. (19) for $k_1 = k_2 = k$. Invoking the symmetry that Eq. (22b) is invariant if (z_1, k_1) and (z_2, k_2) are interchanged between the two independent singularities yields

$$G_n(k_1, z_1) = G_n(k_2, z_2), \quad n = 1, 2, \dots, \tag{22c}$$

$$G_n(k, z) = H_n(k, z)/z,$$

$H(k, z)$ being given in Eq. (18b). Now, the four unknowns, k_1, k_2, z_1, z_2 can be solved by taking four equations of (22b) or (22c), for $n = 1, 2, 3, 4$, with the rest serving for verifying the solution.

Example 4. Here we have the coefficients in numerics for series equation (5) as

$$\begin{aligned} c_1 &= 4, & c_2 &= 1.5, & c_3 &= 0.583\,333, \\ c_4 &= 0.236\,111, & c_5 &= 0.099\,537, & c_6 &= 0.043\,596, \\ \dots, & \longrightarrow \\ R_1 &= 0.375, & R_2 &= 0.388\,889, & R_3 &= 0.404\,762, \\ R_4 &= 0.421\,569, & R_5 &= 0.437\,985, & \dots \end{aligned}$$

Not knowing the number of singularities, we first test for $N_s = 1$. Then, like in Example 1, we find

$$\begin{aligned} k(1) &= -1.074\,074, \\ k(2) &= -0.781\,818, \\ k(3) &= -0.572\,727, \dots \end{aligned}$$

where the lack of any consistency indicates this case being not in the single singularity group.

So next, we test out the group of two singularities for this $f(z)$. By substituting the above R_n 's in the four leading equations of (22b) to solve for (k_1, k_2, z_1, z_2) , adopting successive elimination or any other adequate algorithms, we obtain, after some algebra, the final solution as

$$k_1 = k_2 = -1, \quad z_1 = 2, \quad z_2 = 3, \quad M_1 = -2, \quad M_2 = -9,$$

in which M_1, M_2 follow from Eq. (22a) for $n = 0$ and $n = 1$, with the consistency exhibited as

$$\begin{aligned} G_1(z_1, k) &= G_1(z_2, k) = -\frac{1}{3}, \\ G_2(z_1, k) &= G_2(z_2, k) = -\frac{1}{2}, \\ G_3(z_1, k) &= G_3(z_2, k) = -\frac{2}{3}, \end{aligned}$$

followed by $G_4(z_1, k) = G_4(z_2, k) = -5/6$, each with a relative error of 10^{-8} , and likewise for $n \geq 5$.

Example 5. As the two singularities in Example 4 turn out to be of equal order, we can use the same coefficients c_n for a comparative study. Substituting the R_n 's of Example 4 in Eq. (20a) yields $k = -1$, with the other two complex conjugate roots dismissed by verification. With $k = -1$, Eq. (20b) for $n = 1$ produces $z_1 z_2 = 6$ and then Eq. (19) gives $z_1 + z_2 = 5$. Therefore by Eq. (20c), we obtain the solution

$$k = -1, \quad z_1 = 2, \quad z_2 = 3,$$

just as in Example 4, but the algebra involved here is noticeably simpler than in Example 4.

When more than two singularities are encountered, this case can be adapted to provide a mathematical scheme for resolution by induction, i.e. by eliminating the magnitude of one singularity at a time till all the magnitude variables are eliminated for resolving the conjecture. However, the computation along this approach is still not simple, hence a more general approach is in order next.

5 Resolution to the conjecture involving multiple singularities by induction

For this general case, we first consider $f(z)$ having two singularities at z_1 and z_2 , both real and positive ($1 < z_1 < z_2$), lying outside the open domain \mathcal{D}^+ bounded by the unit circle C , a special case which is apt for illustrating the underlying basic principle more easily, and can be generalized to functions having more singularities situated more freely.

To have the singularities of $f(z)$ determined by induction, we first consider the direct problem.

5.1 The direct problem

The direct problem is to have the singularities all given in closed analytical form as a basis for developing a general

method to determine the singularities by induction so that the method can readily be adapted to resolving the inverse problem as asserted by the conjecture.

Thus, we take, parallel to Eq. (21), here with $1 < z_1 < z_2$,

$$f(z) = f_1(z) + f_2(z) = \sum_{j=1}^2 M_j(1 - z/z_j)^{k_j}$$

$$= \sum_{n=0}^{\infty} \sum_j M_j \gamma_n(k_j) z_j^{-n} (-z)^n = \sum_n \tilde{c}_n z^n, \tag{23}$$

$$\tilde{c}_n = (-1)^n M_1 \gamma_n(k_1) z_1^{-n} \left[1 + m \frac{\gamma_n(k_2)}{\gamma_n(k_1)} \lambda^n \right],$$

$$m = M_2/M_1, \quad 0 < \lambda = z_1/z_2 < 1,$$

where \tilde{c}_n derives its value from the analytic formulas given for this direct problem to give

$$R_n = \frac{\tilde{c}_{n+1}}{\tilde{c}_n} = \frac{1 - k_1/n}{1 + 1/n} \cdot \frac{Q_n}{z_1}, \tag{24}$$

$$Q_n = \frac{1 + m[\gamma_{n+1}(k_2)/\gamma_{n+1}(k_1)]\lambda^{n+1}}{1 + m[\gamma_n(k_2)/\gamma_n(k_1)]\lambda^n}.$$

For $0 < \lambda < 1$, $\lambda^n \rightarrow 0$ and $Q_n \rightarrow 1$ as $n \rightarrow \infty$. Hence, for given $\lambda < 1$, $\exists N \ni \forall n \geq N, |1 - Q_n| < \epsilon$. Thus, as $\epsilon \rightarrow 0$, R_n is reduced to that for the singularity which is closer to the origin and can then be determined in this asymptotic limit when the effects due to the singularity farther away fall off.

Domb–Sykes plot. Proceeding for this asymptotic solution, we make the Domb–Sykes plot by Eqs. (11) and (12) of $R_n(k_1, z_1, Q_n(\lambda))$ of Eq. (24) versus $1/n$ as a function $R_n(1/n) = R_n(\xi)$. The basic principle and the main features of the solution can be illustrated below.

Example 6. Function $f(z)$ having two logarithmic singularities. Of basic interest is that

$$f(z) = \lg[(1 - z/z_1)(1 - z/z_2)^2]$$

$$= \lg(1 - z/z_1) + 2 \lg(1 - z/z_2),$$

$$z_1 = 1.1, \quad z_1 < z_2. \tag{25}$$

Here, we have $f(z) = f_1(z) + f_2(z), k_1 = k_2 = 0, m = 2, z_1 = 1.1, \lambda = z_1/z_2 (0 < \lambda < 1)$, hence

$$R_n = \frac{n}{n + 1} \cdot \frac{Q_n}{z_1}, \quad n = 1, 2, \dots$$

$$Q_n = \frac{1 + 2\lambda^{n+1}}{1 + 2\lambda^n}, \tag{26}$$

This function $R_n(1/n) = R_n(\xi)$ has been computed by employing Mathematica 7 for three values of $\lambda = 0.125, 0.4, 0.8$, over the range $n = 1, 2, \dots, 100$, with the resulting $R(1/n)$ plotted in three lines in sequence of points for each n as shown in Fig. 1, with the “♦” indicating the points for $\lambda = 0.125$, the “■” for $\lambda = 0.4$, and the “●” for $\lambda = 0.8$. Here, for $n \geq N \approx 20$, the three lines of dotted points apparently merge into one straight line, ending at $1/n = 0.01$

(or $n = 100$). This line is further extrapolated and fitted by $R = (p_0 + p_1/n)/(1 + q_1/n)$ to reach the R -axis intercept at $R(0) = 0.9091$, of slope $R'(0)/R(0) = -1$ which gives $k = 0$, thus implying $f_1(z)$ logarithmic and further z_1 being located at $z_1 = 1/R(0) = 1.1$, with almost no error. These are the two important data found from the plot. It is of interest to note that for $n \leq N \approx 20$, the three dotted curves have the “♦” line (for $z_2 = 8z_1$) staying the closest to the asymptotic line, $L_a : R_a = R(0) + R'(0)/n$, whereas the “■” line (for $z_2 = 2.5z_1$) and the “●” line (for $z_2 = 1.25z_1$) bifurcate increasingly more from the straight L_a line with decreasing radial distances of z_2 . This signifies that the interaction between two point singularities increases with decreasing distances apart. This is another key feature demonstrated by the Domb–Sykes plot.

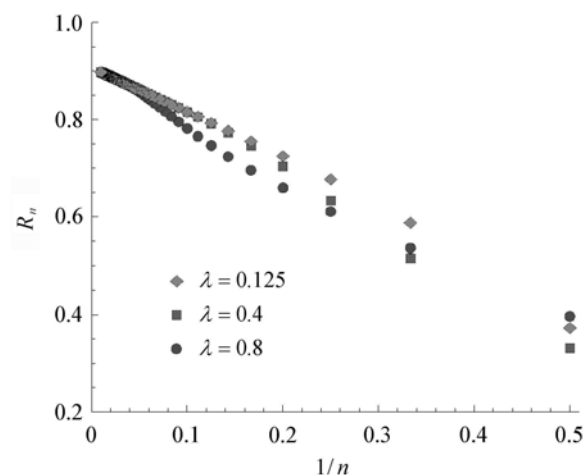


Fig. 1 Domb–Sykes plot of $R_n(1/n)$ for three values of singularity radial-distance ratio $\lambda = z_1/z_2, n = 1, 2, \dots, 100$

At this point, we note that the number of singularities, $N_s = 2$, and $k = 0$ being both known in this direct problem, we can directly solve for the other singularity by taking the first two R_1, R_2 to deduce from their formulas to attain $m = 2$ and the values of z_2 for the three cases, and their magnitudes M_2 from c_0 to complete the solution to this direct problem.

5.2 The inverse problem

Returning to resolve the conjecture, we now take up the inverse problem which has only the Fourier coefficients c_n of Eq. (5) prescribed in numerics, ordinarily by the solution to a problem solved by numerical scheme, but here the c_n ’s are taken to assume the value \tilde{c}_n of the above direct problem for a single case with $\lambda = 0.4$, say. With $c_n = \tilde{c}_n$ known, we plot the coefficient ratio $R_n = c_{n+1}/c_n$ to attain in this case again the same result as that shown in Fig. 1, giving $(k_1 = 0, z_1 = 1.1)$ for the first singularity $f_1(z)$ (lying nearest to the unit circle C) as

$$f_1(z) = M_1 \lg(1 - z/z_1) = c'_0 - \sum_{n=1} c'_n z^n,$$

$$c'_0 = 0, \quad c'_n = M_1 z_1^{-n}/n, n \geq 1, \tag{27}$$

where $c'_0 = c_0 = 0$, $M_1 = 1$ by deduction from $c'_n = c_n$ asymptotically, and hence $f_1(z)$ is all known.

To this end, it is vital to note that the dotted line (here with “■”) being curved indicates that $f(z)$ has more singularities, possibly not just one more, to complete resolving the inverse problem.

5.3 Resolving the inverse problem by induction

To continue, we rid $f(z)$ of $f_1(z)$, giving

$$f_2(z) = f(z) - f_1(z) = \sum_0 c''_n z^n, \quad n = 1, 2, \dots \tag{28}$$

$$c''_n = c_n - c'_n \longrightarrow R''_n = c''_{n+1}/c''_n,$$

With c''_n now all known, we can carry out the Domb–Sykes plot for $R''_n(1/n)$ to determine the primary parameters (k_2, z_2) for $f_2(z)$. In this particular case for $f(z)$ of Eq. (25), we find from the $R''_n(1/n)$ plot that the entire dotted line is straight, of slope $dR''/d(1/n) = -1$, and the R'' -intercept at $1/z_2 = 0.363\ 64$ for $z_2 = 2.75 = 2.5z_1$, as the last singularity of $f(z)$, thus completing this inverse problem.

In general, for generic cases, the $R''(1/n)$ line may appear again curved, in which case we repeat the analogous plot for $R^{(3)}(1/n)$ for $f_3(z), \dots$, until $R^{(\ell)}(1/n)$ for $f_\ell(z)$ bearing out a straight line in the plot for the very last singularity $f_\ell(z)$ of $f(z)$ to complete the inverse problem. This then constitutes the general method of resolving the conjecture by induction.

6 The complement conjecture for the complement function $F(z)$

The complement function, denoted by $F(z)$, arising with the roles of domain \mathcal{D}^+ and \mathcal{D}^- interchanged, is hence defined as being regular of order $C^n \forall z \in \mathcal{D}^-$ and in a neighborhood striding across contour C (cf. Wu [1]). Being so defined, it starts with its integral theorem that

$$J^- [F(z)] = \oint_{C^-} F(z) dz = - \oint_C F(z) dz = 0, \tag{29}$$

where contour C^- is on the unit circle $|z| = 1$, taken in clockwise direction in its own positive sense. By analogy, the complement function must have a singularity distribution inside contour C (unless $F(z) = \text{const.}$ on C), for which fact we have the analogous conjecture as follows.

The complement conjecture If the complement function $F(z)$ has on contour $C(\forall t = e^{i\theta})$ one of its conjugate functions of $F(e^{i\theta}) = \hat{U}(\theta) + i\hat{V}(\theta)$, say \hat{V} numerically prescribed, equal in value to $\hat{v}(\theta)$ of $f(e^{i\theta}) = \hat{u}(\theta) + i\hat{v}(\theta)$ given numerically on C ($f(z)$ being regular $\forall z \in \mathcal{D}^+$), then the new conjecture asserts that the exact singularity distribution of $F(z) \forall z \in \mathcal{D}^+$ exists inside C and can be determined analyt-

ically.

For resolution, we note that invoking $\hat{V}(\theta) = \hat{v}(\theta)$ dictates that on contour $C(\forall t = e^{i\theta})$,

$$F(e^{i\theta}) = \hat{F}(\theta) = \hat{U}(\theta) + i\hat{V}(\theta) = -\overline{f(e^{i\theta})} = -\hat{u}(\theta) + i\hat{v}(\theta), \tag{30a}$$

$$\longrightarrow \hat{U}(\theta) = -\hat{u}(\theta), \quad \hat{V}(\theta) = \hat{v}(\theta), \quad -\pi < \theta \leq \pi, \tag{30b}$$

where $\overline{f(t)}$ denotes the complex conjugate of $f(t)$. This therefore implies that on C ($\forall t = e^{i\theta}$),

$$-F(t) = \overline{f(t)} = \overline{f}(e^{-i\theta}) = \overline{M} \sum_{n=0} (-\bar{z}_1)^{-n} \gamma_n(k) e^{-in\theta} (= \sum_{n=0} \bar{c}_n e^{-in\theta}), \quad -\pi < \theta \leq \pi, \tag{31a}$$

where the first step follows from Eq. (7a) whereas the second from Eq. (5). This series is convergent and can be continued analytically to all $z = re^{i\theta}$ over the entire z -plane, thus giving

$$-F(z) = \sum \bar{c}_n (re^{i\theta})^{-n} = \sum \bar{c}_n z^{-n} = \overline{L} \sum_{n=0} (-\bar{z}_1)^{-n} \gamma_n(k) z^{-n} = \overline{M} [1 - (z\bar{z}_1)^{-1}]^k, \tag{31b}$$

where $F(z)$ and $f(z)$ have their imaginary parts equal on C . This $F(z)$ has a singularity $(z - 1/\bar{z}_1)^k$ at $z = 1/\bar{z}_1$ plus another singularity z^{-k} at $z = 0$ (in reflection to that at $z = \infty$ for $f(z)$) which can be made single-valued when k is not an integer with a cut between $z = 0$ and $z = 1/\bar{z}_1$ lying within C .

Concluding on this extension, we note that with $f(z)$ and $F(z)$ having their imaginary parts equal and real parts opposite in sign, they further satisfy Cauchy’s integral formulas (1) and, for $F(z)$

$$J^- [F(z)] \equiv \frac{-1}{2\pi i} \oint_C \frac{F(t)}{t - z} dt = 0, \quad z \in \overline{\mathcal{D}^+} = \mathcal{D}^+ + C, \tag{32a}$$

$$J^- [F(z)] \equiv \frac{-1}{2\pi i} \oint_C \frac{F(t)}{t - z} dt = F(z), \quad z \in \overline{\mathcal{D}^-} = \mathcal{D}^- + C. \tag{32b}$$

These integral formulas have been shown to hold by their correspondence theorems given in Wu [1].

7 Resolution to the conjecture involving essential singularities

Here, we first cite two Hilbert-type integral transforms given by Wu [1]. Applying the key relations in Eq. (3) to Cauchy function $f(z) = u(x, y) + iv(x, y)$, regular in the upper half z -plane for $\text{Im } z \geq 0$ (vanishing as $|z| \rightarrow \infty$ uniformly in $0 \leq \arg z \leq \pi$), produces the Hilbert transform formula

$$\begin{aligned}
 u(\xi) &= H[v(x)] = \frac{1}{\pi} \mathcal{P} \int_{-\infty}^{\infty} \frac{v(x)}{x - \xi} dx, \\
 v(x) &= H^{-1}[u(\xi)] = \frac{-1}{\pi} \mathcal{P} \int_{-\infty}^{\infty} \frac{u(\xi)}{\xi - x} d\xi.
 \end{aligned}
 \tag{33}$$

Similarly, applying Eq. (3) to complement function $F(z) = U(x, y) + iV(x, y)$, regular in the lower half z -plane, produces the complement Hilbert transform formula

$$\begin{aligned}
 U(\xi) &= \overline{H}[V(x)] = \frac{-1}{\pi} \mathcal{P} \int_{-\infty}^{\infty} \frac{V(x)}{x - \xi} dx, \\
 V(x) &= \overline{H}^{-1}[U(\xi)] = \frac{1}{\pi} \mathcal{P} \int_{-\infty}^{\infty} \frac{U(\xi)}{\xi - x} d\xi.
 \end{aligned}
 \tag{34}$$

For $v(x) = V(x) = \sin x$, we obtain by using Eqs. (33) and (34) the two transforms as

$$\begin{aligned}
 u(\xi) &= \frac{1}{\pi} \mathcal{P} \int_{-\infty}^{\infty} \frac{\sin x}{x - \xi} dx = \cos(\xi), \\
 U(\xi) &= -\frac{1}{\pi} \mathcal{P} \int_{-\infty}^{\infty} \frac{\sin x}{x - \xi} dx = -\cos(\xi).
 \end{aligned}
 \tag{35}$$

The resultant $f(x) = u(x) + iv(x)$, $F(x) = U(x) + iV(x)$ can be continued analytically to the entire z -plane, yielding

$$\begin{aligned}
 f(x) &= e^{ix} \quad F(x) = -e^{-ix} \\
 \rightarrow f(z) &= e^{iz}, \quad F(z) = -e^{-iz}, \\
 \forall z \in 0 \leq |z| < \infty.
 \end{aligned}
 \tag{36}$$

Therefore, an equal value of one conjugate function, e.g. $v(x) = V(x) = \sin x$, produces an analytic function $f(z)$ regular in \mathcal{D}^+ and $F(z)$ regular in \mathcal{D}^- , whilst both having an essential singularity in each of their complement domain, respectively. This rather typical comparative study can well exemplify more general ones, e.g. $f(z)$ (or $F(z)$) being regular in $|z| \leq 1$ (or $|z| \geq 1$); $f(z) = \sum_{n=1}^{\infty} \lg(1 - z^2/n^2)$ so that $g(z) = \exp(f(z))$ has an isolated essential singularity at $z = \infty$.

8 Discussion and conclusion

In this study we have shown methods developed for resolution to the conjecture on the inverse problem for function $f(z)$ to have arbitrary number of singularities lying outside its domain of regularity. The principle of these methods takes the approach, by and large, to reduce multiple singularities to one for determining its nature, at least its argument, by means of complex algebra and analysis. This approach also includes the general method developed in Sect. 5 for resolution to the conjecture by induction, which suits especially well for singularities located at distinct radial distances so that the one nearest to the origin can be dealt with by applying the Domb–Sykes plot for an asymptotic solution, with the remaining singularities to be resolved likewise one at a time. The success of these methods and the high accuracy of the general results can be ascribed to the central role played

by the power series in complex form composed of orthogonal terms. These methods can be employed for applications and further development to all pertinent problems in mathematics, engineering, and technology.

There may exist various areas of mathematics where challenges still prevailing which can be investigated in analogy with resolution to Wu’s conjecture. The long and rich history of the fully nonlinear theory of dispersive water waves can serve as a splendid representative for other disciplines where challenging basic hindrances remain to be surmounted. In all respects, new grasp of the exterior singularities outside the solution domain of the primary variables may furnish crucial data base to ascertain the certitude of solution validity and profound comprehension of the target phenomena.

In concluding, it is reasonable to say that this is just a beginning. New issues may arise to require answers. The exterior singularities of holonomic functions may present a variety of new interests, e.g. coalescence of singularities with changes in parameters, productions of essential singularities, etc. We may indeed expect to see more advances accumulating to enrich more of this new field.

Acknowledgments I am indebted to Prof. Jia-Chun Li for his interesting comments on articles [1, 2] that have been a splendid stimulus leading me to bring forth this final version. I also wish to thank Prof. Joe Keller, Prof. Lu Ting and Prof. John C.K. Chu for their interest in resolution to the conjecture enunciated in Ref. [2], and I am especially grateful to Prof. Thomas Y. Hou for his very careful scrutiny of the present article with stimulating queries. I would also like to thank Yue Yang for his assistance in preparing the figure, and I am further highly appreciative for the gracious encouragement from Dr. Chinhua S. Wu, F. Baiyueh Wu and the American-Chinese Scholarship Foundation.

References

- 1 Wu, Th.Y.: On the generalized Cauchy function and new conjecture on its exterior singularities. *Acta Mech. Sin.* **27**(2), 135–151 (2011)
- 2 Wu, Th.Y.: On resolution to Wu’s conjecture. 2009. arXiv: 0909.0298v1
- 3 Stokes, G.G.: On the theory of oscillatory waves. *Trans. Cambridge Phil. Soc.* **8**, 441–455 (1847)
- 4 Grant, M.A.: The singularity at the crest of a finite amplitude progressive Stokes wave. *J. Fluid Mech.* **59**(2), 257–262 (1973)
- 5 Schwartz, L.M.: Computer extension and analytic continuation of Stokes expansion of gravity waves. *J. Fluid Mech.* **62**(3), 553–578 (1974)
- 6 Tanveer, S.: Singularities in water waves and Rayleigh–Taylor instability. *Proc. R. Soc. Lond.* **A 435**, 137–158 (1991)
- 7 Domb, C., Sykes, M.F.: On the susceptibility of a ferromagnetic above the Curies point. *Proc. R. Soc. Lond.* **A 240**, 214–218 (1957)
- 8 Li, J.: Singularity criteria for perturbation series. *Scientia Sinica* **A 25**(6), 593–600 (1982)
- 9 Van Dyke, M.D.: Analysis and improvement of perturbation series. *Q. J. Mech. Appl. Math.* **27**, 423–440 (1974)